

## LECTURE 2: RECTANGULAR CHEBYSHEV MATRICES

HOW TO GET HIGHER- ORDER SPATIAL DISCRETIZATION?

SIMPLEST (BAD) IDEA: REPLACE 2ND ORDER CENTERED DIFFERENCE OPERATOR WITH HIGHER ORDER

PROBLEM: WIDER STENCIL MAKES IMPLEMENTING BOUNDARY CONDITIONS EVEN HARDER

BETTER: SPECTRAL COLLOCATION

LET  $-1 \leq x_0 \leq \dots \leq x_N \leq 1$  BE COLLOCATION POINTS

$$u_j = u(x_j) \quad j=1, \dots, N$$

$I_N^{\vec{u}} =$  DEGREE - N INTERPOLATING POLYNOMIAL THROUGH  $(x_j, u_j) \quad j=0, \dots, N$

LET  $\vec{u} = (u_0, \dots, u_N)^T$   
DEFINE A MATRIX

$$D^{(m)} \text{ BY } (D^{(m)} \vec{u})_j = u^{(m)}(x) \Big|_{x=x_j}$$

PROBLEM: OBVIOUS CHOICE

$$x_j = -1 + \frac{2j}{N} \text{ LEADS TO RUNGE PHENOMENON}$$

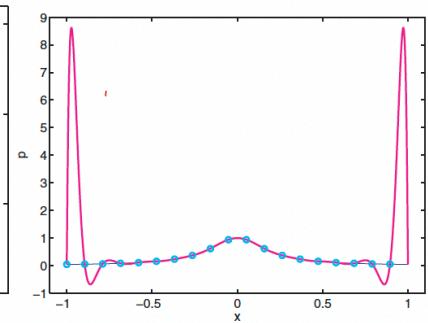
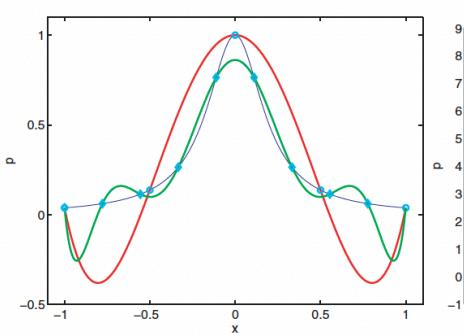
$$u(x)$$

FAMOUS EXAMPLE:

$$u = \frac{1}{1+25x^2}$$

DATA & INTERPOLATING  
POLYNOMIALS

$$I_4, I_9, I_{19}$$



REASON  $\| u(x) - I_N u(x) \|_{\infty} \leq \left\| \prod_{i=0}^N (x - x_i) \right\|_{\infty} \frac{\| u^{(N+1)} \|_{\infty}}{(N+1)!}$

IF  $u$  ANALYTIC  
THE FIRST FACTOR  $\prod_{i=0}^N (x - x_i)$  IS THE PROBLEM  
AND GROWS FAST FOR LARGE  $N$  AND UNIFORMLY-SPACED POINTS

THE FIX: CHOOSE  $x_0, \dots, x_N$  TO MINIMIZE FIRST  
FACTOR

THIS IS ACCOMPLISHED USING THE CHEBYSHEV POLYNOMIALS  
(TCHEBYCHEV À FRANCE)

RECALL THE CHEBYSHEV POLYNOMIAL ARE

$$T_n(x) = \cos(n \cdot \cos^{-1} x)$$

$$T_0 = 1$$

$$T_1 = x$$

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x) \Rightarrow$$

$\cos((n+1)t) = \cos nt \cos t - \sin nt \sin t$
$\cos((n-1)t) = \cos nt \cos t + \sin nt \sin t$
$\cos(nt) + \cos((n-1)t) = 2\cos nt \cos t$
$t = \cos^{-1} x$

$T_n$  IS A DEGREE  $n$  POLYNOMIAL  
WITH LEADING COEFFICIENT  $2^{n-1}$

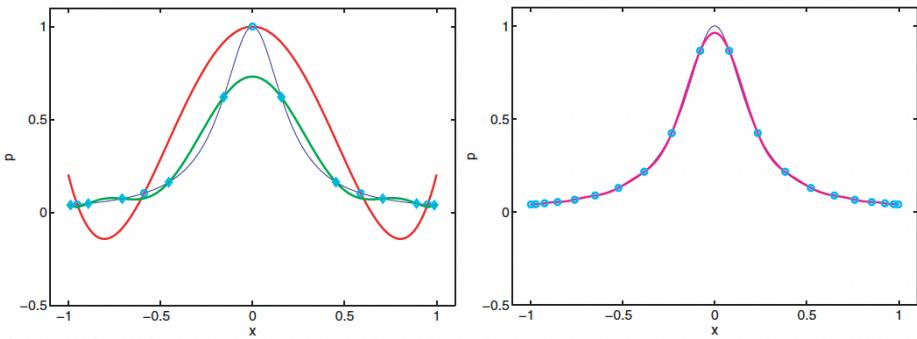
$T_n$  HAS  $n$  ROOTS AT THE CHEBYSHEV POINTS  
OF THE 1ST KIND  $x_k = \cos \frac{(2k-1)\pi}{2N}$ ,  $k=1, \dots, N$

$T_n$  HAS  $n+1$  EXTREMA AT 2ND KIND CHEBYSHEV POINTS  
 $\tilde{x}_k = \cos \frac{k\pi}{N}$ ,  $k=0, \dots, N$ ,  $T_n(\tilde{x}_k) = \pm 1$

USING 1ST-KIND CHEBYSHEV POINTS

$$\left| \prod_{i=0}^N (x - x_i) \right| < \frac{1}{2^N}$$

THEN APPROXIMATION IS SPECTRAL =  $O(N^{-p})$   $\forall p$

(a)  $n = 4, 9.$ (b)  $n = 19.$ 

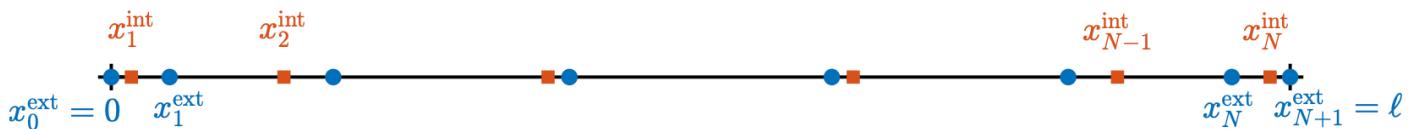
WE WANT TO MIMIC OUR APPROACH FROM THE CENTERED DIFFERENCES. WE WANT AN  $N \times (N+2)$  MATRIX THAT MAPS LINEARLY-VARYING VECTORS TO ZERO

FOLLOW A PAPER BY DRISCOLL + HALE

TWO GRIDS

$$x_k^{\text{ext}} = \frac{\ell}{2} \cos \frac{k\pi}{N+1} \quad k=0, \dots, N+1 \quad \text{2ND-KIND CHEBYSHEV POINTS}$$

$$x_k^{\text{int}} = \frac{\ell}{2} \cos \frac{(2k-1)\pi}{N} \quad k=1, \dots, N \quad \text{1ST KIND CHEBYSHEV POINTS}$$



STRATEGY: DEFINE THE  $(N+2) \times (N+2)$  COLLOCATION 2<sup>ND</sup> DERIVATIVE MATRIX  $D^{(2)}$ , DEFINED ON  $\tilde{X}^{\text{ext}}$   
 RESAMPLE THIS TO  $\tilde{X}^{\text{int}}$  BY EVALUATING THE CHEBYSHEV INTERPOLANT AT THESE POINTS

RECALL, GIVEN POINTS  $x_0, \dots, x_N$

DEFINE  $L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^N \frac{x - x_i}{x_i - x_j}$  THE LAGRANGE POLYNOMIALS OF DEGREE N

$$\text{Then } L_j(x_i) = \delta_{ij}$$

SO GIVEN DATA  $y_0, \dots, y_N$

$$P_N(x) = \sum_{j=0}^N y_j L_j(x)$$

INTERPOLATES THE POINTS  $(x_j, y_j)$   $j=0, \dots, N$

$$\text{Let } \psi(x) = \prod_{\substack{i=0 \\ i \neq j}}^N (x - x_i) \Rightarrow L_j(x) = \frac{w_j \psi(x)}{x - x_j}$$

$$w_j = \frac{1}{\prod_{\substack{i=0 \\ i \neq j}}^N (x_i - x_j)}$$

$$\begin{aligned} P_N(x) &= \sum \frac{w_j y_j \psi(x)}{x - x_j} \\ &= \psi(x) \sum \frac{w_j y_j}{x - x_j} \end{aligned}$$

Letting  $y_j = 1$ ,  $j=0, \dots, N$

$$\text{WE GET } 1 = \psi(x) \sum \frac{w_j}{x - x_j} \Rightarrow \psi = \frac{1}{\sum \frac{w_j}{x - x_j}}$$

So INTERPOLATE FROM  $x^{ext}$  to  $x^{int}$  USING  
 $w_k = \prod_{\substack{j=0 \\ k \neq j}}^{N+1} (x_k^{ext} - x_j^{ext})$   $k=0, \dots, N+1$

$$p_{N+1}(x) = \frac{\sum_{k=0}^{N+1} \frac{w_k}{x-x_k^{ext}} y_k}{\sum_{k=0}^{N+1} \frac{w_k}{x-x_k^{ext}}}$$

EVALUATING THIS AT  $\vec{x}^{int} = [x_1^{int} \dots x_N^{int}]^T$

WE FIND  $p_{N+1}(\vec{x}^{int}) = P_{int} \cdot p_{N+1}(\vec{x}^{ext})$

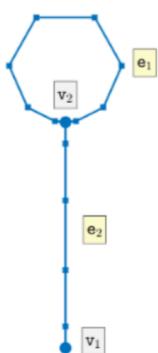
$$(P_{int})_{jk} = \frac{w_k}{x_j^{int} - x_k^{ext}} \left( \sum_{l=0}^{N+1} \frac{w_l}{x_l^{int} - x_k^{ext}} \right)^{-1} \quad \{ N \times (N+2) \}$$

DEFINE OUR 2ND DERIVATIVE MATRIX BY

$L_{int} = P_{int} D^{(2)}$   
 + 2 BOUNDARY CONDITIONS

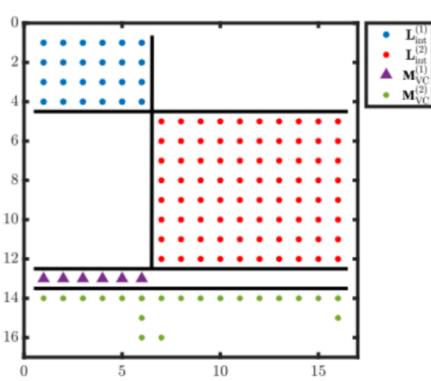
To SOLVE  $u'' = f$

DISCRETIZED GRAPH



$$L_{VC} = \begin{bmatrix} L_{int} \\ M_{VC} \end{bmatrix}$$

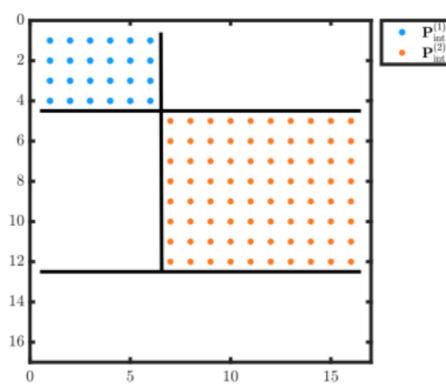
$$P_0 = \begin{bmatrix} P_{int} \\ O_2 \otimes I_{(N+2) \times (N+2)} \end{bmatrix}$$



(a)

(b)

(c)



BIGGEST TAKEAWAY:  
 ON EACH EDGE k DEFINE GRIDS  
 $\vec{u} = \frac{\vec{P} + \vec{F}}{2}$   
 $\vec{u}_{Bc} = \vec{g}$   
 $\vec{x}^{\text{ext}} \text{ OF } N_k+2 \text{ POINTS}$   
 $\vec{x}^{\text{int}} \text{ OF } N_k \text{ POINTS}$

(1) DATA GIVEN ON  $\vec{x}^{\text{ext}}$  BUT  
 ALL EDGE-DEFINED EQNS EVALUATED ON  $\vec{x}^{\text{int}}$ .

(2) SIMULTANEOUSLY, THE UNKNOWN IN ANY EQUATION WE SET UP  
 MUST ALSO SOLVE THE DISCRETIZED VERTEX CONDITIONS.

(SEMI) INFINITE EDGES

CONSIDER NLS ON A HALF LINE

$$\Delta\psi + \psi'' + 2\psi^3 = 0 \quad 0 < x < \infty$$

$$\psi'(0) + \alpha\psi(0) = 0$$

$$\lim_{x \rightarrow \infty} \psi(x) = 0 \quad (\text{AT WHAT RATE?})$$

HOW DO WE COMPUTE THIS ACCURATELY?

FIRST ATTEMPT: TRUNCATE  $[0, \infty)$  TO  $[0, L]$

APPLY DIRICHLET B.C AT  $x=L$

WHAT KIND OF DISCRETIZATION?

START WITH UNIFORM, 2ND ORDER CENTERED DIFF.

$$\text{let } h = \frac{L}{N}, \quad x_k = kh, \quad k=0, \dots, N$$

$$\begin{aligned}\psi_k &\approx \psi(x_k) \\ \psi_N &= 0\end{aligned}$$

NOW THERE ARE TWO SOURCES OF ERROR  
 WHICH WE WANT TO BALANCE

① DISCRETIZATION ERROR  $\epsilon_{\text{disc}} \propto h^2 \max_{0 \leq x \leq L} |\psi''(x)| = \frac{L^2}{N^2} \|\psi''\|_2$

② TRUNCATION ERROR  $\epsilon_{\text{trunc}} \propto \psi_{\text{exact}}^{(L)}$

- IF  $\psi(x)$  DECAYS SLOWLY, THEN MUST TAKE  $L$  LARGE, THEN MUST TAKE  $N$  LARGE ENOUGH TO MAKE  $\epsilon_{\text{disc}}$  COMPARABLE
- LARGE  $N \Rightarrow$  LARGE MATRICES, SLOW COMPUTATION
- WASTEFUL BECAUSE THE VALUES OF  $\psi_k$  NEAR  $x=L$  CONTRIBUTE LITTLE TO APPROXIMATION OR ERROR

ONE SOLUTION

Let  $s \in [0, 1]$ , DISCRETIZE  $h = \frac{1}{N}$ ,  $s_k = kh$   
 $x = f(s)$   $f(0) = 0, f'(0) > 0$  ON  $[0, 1]$   
 $x_k = f(s_k)$   
 $L = f(1)$

COMMON CHOICE  $f(s) = \sinh Ms$   
 $L = \sinh M$

$$\frac{d}{dx} = \frac{\frac{ds}{dx}}{\frac{ds}{ds}} = \frac{1}{f'(s)} \frac{d}{ds} = g(s) \frac{d}{ds} \Rightarrow \frac{d}{dx} = \frac{1}{M \cosh Ms} \frac{d}{ds}$$

FIND

$$\frac{d^2}{dx^2} = \frac{\sech^2 Ms}{M^2} \frac{d^2}{ds^2}$$

$$\frac{d^2}{dx^2} = g \cdot g' \frac{d}{ds} + g^2 \frac{d^2}{ds^2}$$

$$-\frac{1}{M} \operatorname{sech}^2 Ms \tanh Ms \frac{d}{ds}$$

SOME THEORY TO SHOW HOW TO CHOOSE  $M$



SPACING NEARLY UNIFORM FOR  $k$  small

SPARSE FOR  $k$  LARGE

FOR CHGBYSHEV, THIS ISN'T ENOUGH TO COUNTERACT  
CLUSTERING NEAR RIGHT ENDPOINT

$$x = \sinh M((1 - \sqrt{1-s}))$$

KEEPS CLUSTERING NEAR  $x=s=0$

SPREADS OUT POINTS NEAR  $s=1$ ,  $x = \sinh M$