1.9 Applications of Linear Systems

In this section we will discuss some brief applications of linear systems. These are but a small sample of the wide variety of real-world problems to which our study of linear systems is applicable.

Network Analysis

The concept of a *network* appears in a variety of applications. Loosely stated, a ***network*** is a set of ***branches*** through which something "flows." For example, the branches might be electrical wires through which electricity flows, pipes through which water or oil flows, traffic lanes through which vehicular traffic flows, or economic linkages through which money flows, to name a few possibilities.

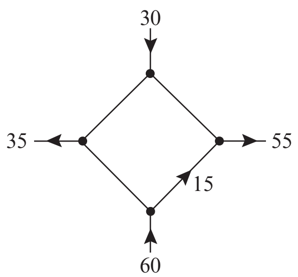
In most networks, the branches meet at points, called ***nodes*** or ***junctions***, where the flow divides. For example, in an electrical network, nodes occur where three or more wires join, in a traffic network they occur at street intersections, and in a financial network they occur at banking centers where incoming money is distributed to individuals or other institutions.

In the study of networks, there is generally some numerical measure of the rate at which the medium flows through a branch. For example, the flow rate of electricity is often measured in amperes, the flow rate of water or oil in gallons per minute, the flow rate of traffic in vehicles per hour, and the flow rate of European currency in millions of Euros per day. We will restrict our attention to networks in which there is ***flow conservation*** at each node, by which we mean that *the rate of flow into any node is equal to the rate of flow out of that node.* This ensures that the flow medium does not build up at the nodes and block the free movement of the medium through the network.

A common problem in network analysis is to use known flow rates in certain branches to find the flow rates in all of the branches. Here is an example.

▶ EXAMPLE 1 Network Analysis Using Linear Systems

Figure 1.9.1 shows a network with four nodes in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.



▲ Figure 1.9.1

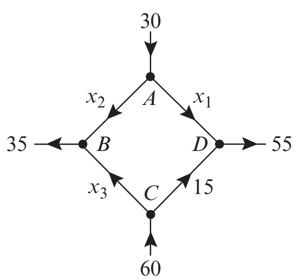
Solution As illustrated in Figure 1.9.2, we have assigned arbitrary directions to the unknown flow rates x_1, x_2, and x_3. We need not be concerned if some of the directions are incorrect, since an incorrect direction will be signaled by a negative value for the flow rate when we solve for the unknowns.

It follows from the conservation of flow at node A that

$$x_1 + x_2 = 30$$

Similarly, at the other nodes we have

$$\begin{aligned} x_2 + x_3 &= 35 \quad \text{(node } B\text{)} \\ x_3 + 15 &= 60 \quad \text{(node } C\text{)} \\ x_1 + 15 &= 55 \quad \text{(node } D\text{)} \end{aligned}$$



▲ Figure 1.9.2

These four conditions produce the linear system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & & & = & 30 \\ & & x_2 & + & x_3 & = & 35 \\ & & & & x_3 & = & 45 \\ x_1 & & & & & = & 40 \end{array}$$

which we can now try to solve for the unknown flow rates. In this particular case the system is sufficiently simple that it can be solved by inspection (work from the bottom up). We leave it for you to confirm that the solution is

$$x_1 = 40, \quad x_2 = -10, \quad x_3 = 45$$

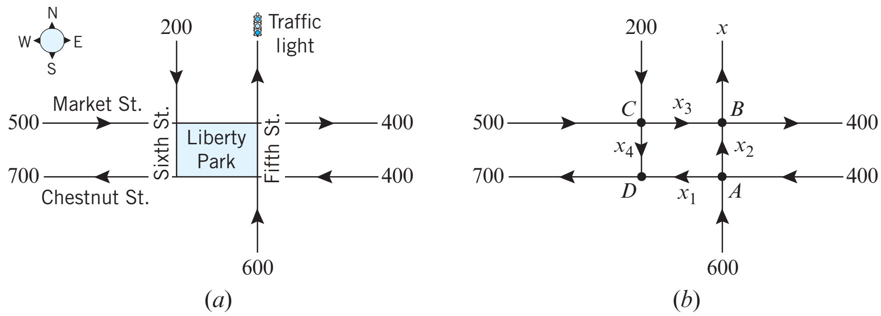
The fact that x_2 is negative tells us that the direction assigned to that flow in Figure 1.9.2 is incorrect; that is, the flow in that branch is *into* node A.

▶ EXAMPLE 2 Design of Traffic Patterns

The network in Figure 1.9.3 shows a proposed plan for the traffic flow around a new park that will house the Liberty Bell in Philadelphia, Pennsylvania. The plan calls for a computerized traffic light at the north exit on Fifth Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the complex. All streets are one-way.

(a) How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the complex is the same as the average number of vehicles flowing out?

(b) Assuming that the traffic light has been set to balance the total flow in and out of the complex, what can you say about the average number of vehicles per hour that will flow along the streets that border the complex?



▶ Figure 1.9.3

Solution (a) If, as indicated in Figure 1.9.3b, we let x denote the number of vehicles per hour that the traffic light must let through, then the total number of vehicles per hour that flow in and out of the complex will be

Flowing in: $500 + 400 + 600 + 200 = 1700$

Flowing out: $x + 700 + 400$

Equating the flows in and out shows that the traffic light should let $x = 600$ vehicles per hour pass through.

Solution (b) To avoid traffic congestion, the flow in must equal the flow out at each intersection. For this to happen, the following conditions must be satisfied:

Intersection	Flow In		Flow Out
A	$400 + 600$	$=$	$x_1 + x_2$
B	$x_2 + x_3$	$=$	$400 + x$
C	$500 + 200$	$=$	$x_3 + x_4$
D	$x_1 + x_4$	$=$	700

Thus, with $x = 600$, as computed in part (a), we obtain the following linear system:

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & & & & & = & 1000 \\ & & x_2 & + & x_3 & & & = & 1000 \\ & & & & x_3 & + & x_4 & = & 700 \\ x_1 & & & & & + & x_4 & = & 700 \end{array}$$

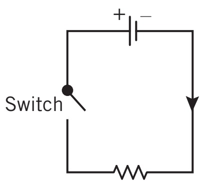
We leave it for you to show that the system has infinitely many solutions and that these are given by the parametric equations

$$x_1 = 700 - t, \quad x_2 = 300 + t, \quad x_3 = 700 - t, \quad x_4 = t \tag{1}$$

However, the parameter t is not completely arbitrary here, since there are physical constraints to be considered. For example, the average flow rates must be nonnegative since we have assumed the streets to be one-way, and a negative flow rate would indicate a flow in the wrong direction. This being the case, we see from (1) that t can be any real number that satisfies $0 \leq t \leq 700$, which implies that the average flow rates along the streets will fall in the ranges

$$0 \leq x_1 \leq 700, \quad 300 \leq x_2 \leq 1000, \quad 0 \leq x_3 \leq 700, \quad 0 \leq x_4 \leq 700$$ ◀

Electrical Circuits



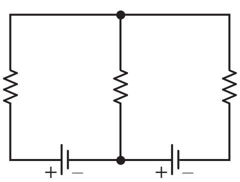
▲ Figure 1.9.4

Next we will show how network analysis can be used to analyze electrical circuits consisting of batteries and resistors. A ***battery*** is a source of electric energy, and a ***resistor***, such as a lightbulb, is an element that dissipates electric energy. Figure 1.9.4 shows a schematic diagram of a circuit with one battery (represented by the symbol $\dashv\vdash$), one resistor (represented by the symbol $-\!\wedge\!\!\!\wedge\!\!\!\wedge\!-$), and a switch. The battery has a ***positive pole*** (+) and a ***negative pole*** (−). When the switch is closed, electrical current is considered to flow from the positive pole of the battery, through the resistor, and back to the negative pole (indicated by the arrowhead in the figure).

Electrical current, which is a flow of electrons through wires, behaves much like the flow of water through pipes. A battery acts like a pump that creates "electrical pressure" to increase the flow rate of electrons, and a resistor acts like a restriction in a pipe that reduces the flow rate of electrons. The technical term for electrical pressure is ***electrical potential***; it is commonly measured in ***volts*** (V). The degree to which a resistor reduces the electrical potential is called its ***resistance*** and is commonly measured in ***ohms*** (Ω). The rate of flow of electrons in a wire is called ***current*** and is commonly measured in ***amperes*** (also called ***amps***) (A). The precise effect of a resistor is given by the following law:

Ohm's Law If a current of I amperes passes through a resistor with a resistance of R ohms, then there is a resulting drop of E volts in electrical potential that is the product of the current and resistance; that is,

$$E = IR$$

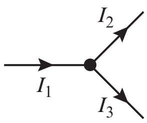


▲ Figure 1.9.5

A typical electrical network will have multiple batteries and resistors joined by some configuration of wires. A point at which three or more wires in a network are joined is called a ***node*** (or ***junction point***). A ***branch*** is a wire connecting two nodes, and a ***closed loop*** is a succession of connected branches that begin and end at the same node. For example, the electrical network in Figure 1.9.5 has two nodes and three closed loops—two inner loops and one outer loop. As current flows through an electrical network, it undergoes increases and decreases in electrical potential, called ***voltage rises*** and ***voltage drops***, respectively. The behavior of the current at the nodes and around closed loops is governed by two fundamental laws:

> **Kirchhoff's Current Law** The sum of the currents flowing into any node is equal to the sum of the currents flowing out.

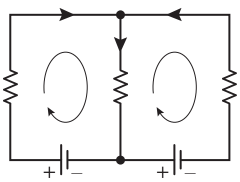
> **Kirchhoff's Voltage Law** In one traversal of any closed loop, the sum of the voltage rises equals the sum of the voltage drops.



▲ Figure 1.9.6

Kirchhoff's current law is a restatement of the principle of flow conservation at a node that was stated for general networks. Thus, for example, the currents at the top node in Figure 1.9.6 satisfy the equation $I_1 = I_2 + I_3$.

In circuits with multiple loops and batteries there is usually no way to tell in advance which way the currents are flowing, so the usual procedure in circuit analysis is to assign *arbitrary* directions to the current flows in the branches and let the mathematical computations determine whether the assignments are correct. In addition to assigning directions to the current flows, Kirchhoff's voltage law requires a direction of travel for each closed loop. The choice is arbitrary, but for consistency we will always take this direction to be *clockwise* (Figure 1.9.7).We also make the following conventions:



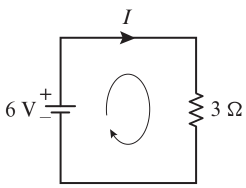
Clockwise closed-loop convention with arbitrary direction assignments to currents in the branches

▲ Figure 1.9.7

- A voltage drop occurs at a resistor if the direction assigned to the current through the resistor is the same as the direction assigned to the loop, and a voltage rise occurs at a resistor if the direction assigned to the current through the resistor is the opposite to that assigned to the loop.
- A voltage rise occurs at a battery if the direction assigned to the loop is from $-$ to $+$ through the battery, and a voltage drop occurs at a battery if the direction assigned to the loop is from $+$ to $-$ through the battery.

If you follow these conventions when calculating currents, then those currents whose directions were assigned correctly will have positive values and those whose directions were assigned incorrectly will have negative values.

▶ EXAMPLE 3 A Circuit with One Closed Loop



▲ Figure 1.9.8

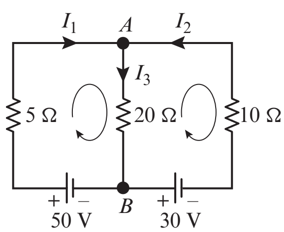
Determine the current I in the circuit shown in Figure 1.9.8.

Solution Since the direction assigned to the current through the resistor is the same as the direction of the loop, there is a voltage drop at the resistor. By Ohm's law this voltage drop is $E = IR = 3I$. Also, since the direction assigned to the loop is from $-$ to $+$ through the battery, there is a voltage rise of 6 volts at the battery. Thus, it follows from Kirchhoff's voltage law that

$$3I = 6$$

from which we conclude that the current is $I = 2$ A. Since I is positive, the direction assigned to the current flow is correct.

▶ EXAMPLE 4 A Circuit with Three Closed Loops



▲ Figure 1.9.9

Determine the currents I_1, I_2, and I_3 in the circuit shown in Figure 1.9.9.

Solution Using the assigned directions for the currents, Kirchhoff's current law provides one equation for each node:

Node	Current In		Current Out
A	$I_1 + I_2$	$=$	I_3
B	I_3	$=$	$I_1 + I_2$

However, these equations are really the same, since both can be expressed as

$$I_1 + I_2 - I_3 = 0 \tag{2}$$

To find unique values for the currents we will need two more equations, which we will obtain from Kirchhoff's voltage law. We can see from the network diagram that there are three closed loops, a left inner loop containing the 50 V battery, a right inner loop containing the 30 V battery, and an outer loop that contains both batteries. Thus, Kirchhoff's voltage law will actually produce three equations. With a clockwise traversal of the loops, the voltage rises and drops in these loops are as follows:

	Voltage Rises	**Voltage Drops**
Left Inside Loop	50	$5I_1 + 20I_3$
Right Inside Loop	$30 + 10I_2 + 20I_3$	0
Outside Loop	$30 + 50 + 10I_2$	$5I_1$

These conditions can be rewritten as

$$\begin{aligned} 5I_1 \phantom{{}-10I_2} + 20I_3 &= 50 \\ 10I_2 + 20I_3 &= -30 \\ 5I_1 - 10I_2 \phantom{{}+20I_3} &= 80 \end{aligned} \tag{3}$$

However, the last equation is superfluous, since it is the difference of the first two. Thus, if we combine (2) and the first two equations in (3), we obtain the following linear system of three equations in the three unknown currents:

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 5I_1 \phantom{{}+I_2} + 20I_3 &= 50 \\ 10I_2 + 20I_3 &= -30 \end{aligned}$$

We leave it for you to show that the solution of this system in amps is $I_1 = 6$, $I_2 = -5$, and $I_3 = 1$. The fact that I_2 is negative tells us that the direction of this current is opposite to that indicated in Figure 1.9.9. ◀

Balancing Chemical Equations

Chemical compounds are represented by ***chemical formulas*** that describe the atomic makeup of their molecules. For example, water is composed of two hydrogen atoms and one oxygen atom, so its chemical formula is H_2O; and stable oxygen is composed of two oxygen atoms, so its chemical formula is O_2.

When chemical compounds are combined under the right conditions, the atoms in their molecules rearrange to form new compounds. For example, when methane burns,

Gustav Kirchhoff (1824–1887)

Historical Note The German physicist Gustav Kirchhoff was a student of Gauss. His work on Kirchhoff's laws, announced in 1854, was a major advance in the calculation of currents, voltages, and resistances of electrical circuits. Kirchhoff was severely disabled and spent most of his life on crutches or in a wheelchair.

[*Image: ullstein bild - histopics/akg-im*]

Thus, it follows from (14) that the augmented matrix for the linear system in the unknowns a_0, a_1, a_2, and a_3 is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & y_1 \\ 1 & x_2 & x_2^2 & x_2^3 & y_2 \\ 1 & x_3 & x_3^2 & x_3^3 & y_3 \\ 1 & x_4 & x_4^2 & x_4^3 & y_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix}$$

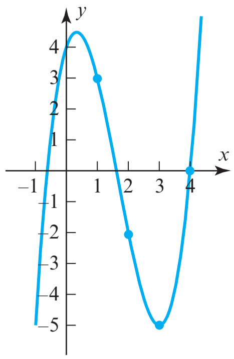
We leave it for you to confirm that the reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

from which it follows that $a_0 = 4, a_1 = 3, a_2 = -5, a_3 = 1$. Thus, the interpolating polynomial is

$$p(x) = 4 + 3x - 5x^2 + x^3$$

The graph of this polynomial and the given points are shown in Figure 1.9.12. ◀



▲ Figure 1.9.12

Remark Later we will give a more efficient method for finding interpolating polynomials that is better suited for problems in which the number of data points is large.

CALCULUS AND CALCULATING UTILITY REQUIRED

▶ EXAMPLE 7 Approximate Integration

There is no way to evaluate the integral

$$\int_0^1 \sin\left(\frac{\pi x^2}{2}\right) dx$$

directly since there is no way to express an antiderivative of the integrand in terms of elementary functions. This integral could be approximated by Simpson's rule or some comparable method, but an alternative approach is to approximate the integrand by an interpolating polynomial and integrate the approximating polynomial. For example, let us consider the five points

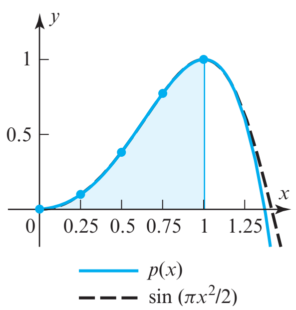
$$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1$$

that divide the interval [0, 1] into four equally spaced subintervals (Figure 1.9.13). The values of

$$f(x) = \sin\left(\frac{\pi x^2}{2}\right)$$

at these points are approximately

$$f(0) = 0, \quad f(0.25) = 0.098017, \quad f(0.5) = 0.382683,$$
$$f(0.75) = 0.77301, \quad f(1) = 1$$



▲ Figure 1.9.13

The interpolating polynomial is (verify)

$$p(x) = 0.098796x + 0.762356x^2 + 2.14429x^3 - 2.00544x^4 \tag{15}$$

and

$$\int_0^1 p(x)\,dx \approx 0.438501 \tag{16}$$

As shown in Figure 1.9.13, the graphs of f and p match very closely over the interval [0, 1], so the approximation is quite good. ◀

Exercise Set 1.9

1. The accompanying figure shows a network in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.

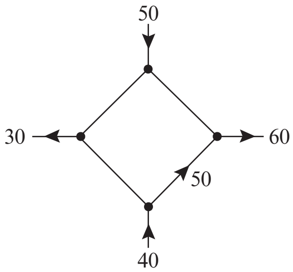


Figure Ex-1

2. The accompanying figure shows known flow rates of hydrocarbons into and out of a network of pipes at an oil refinery.
   (a) Set up a linear system whose solution provides the unknown flow rates.
   (b) Solve the system for the unknown flow rates.
   (c) Find the flow rates and directions of flow if $x_4 = 50$ and $x_6 = 0$.

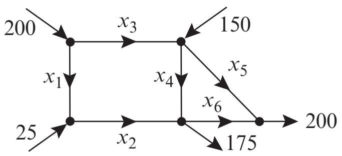


Figure Ex-2

3. The accompanying figure shows a network of one-way streets with traffic flowing in the directions indicated. The flow rates along the streets are measured as the average number of vehicles per hour.
   (a) Set up a linear system whose solution provides the unknown flow rates.
   (b) Solve the system for the unknown flow rates.
   (c) If the flow along the road from A to B must be reduced for construction, what is the minimum flow that is required to keep traffic flowing on all roads?

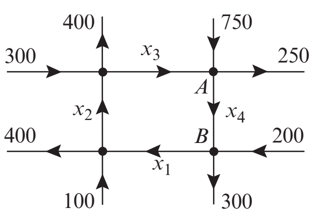


Figure Ex-3

4. The accompanying figure shows a network of one-way streets with traffic flowing in the directions indicated. The flow rates along the streets are measured as the average number of vehicles per hour.
   (a) Set up a linear system whose solution provides the unknown flow rates.
   (b) Solve the system for the unknown flow rates.
   (c) Is it possible to close the road from A to B for construction and keep traffic flowing on the other streets? Explain.

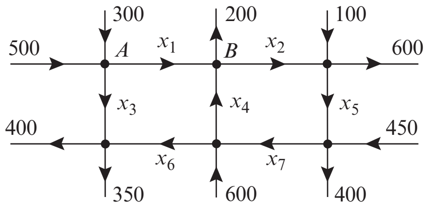
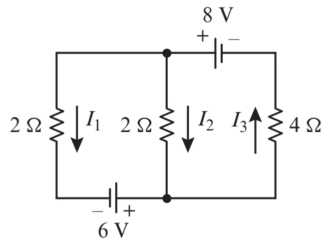


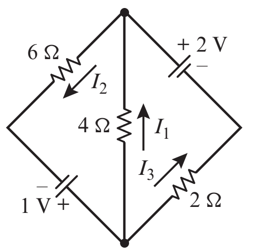
Figure Ex-4

In Exercises 5–8, analyze the given electrical circuits by finding the unknown currents.

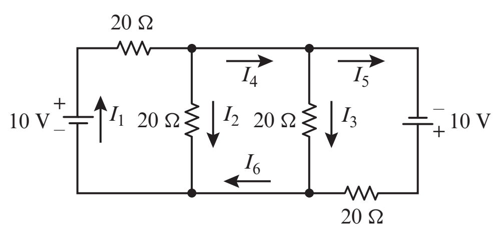
5.



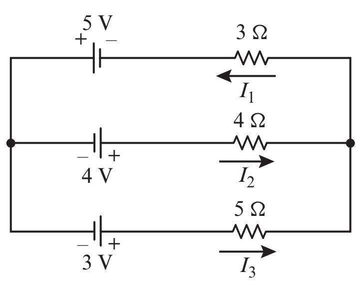
6.



7.



8.



In Exercises 9–12, write a balanced equation for the given chemical reaction.

9. $C_3H_8 + O_2 \rightarrow CO_2 + H_2O$ (propane combustion)

10. $C_6H_{12}O_6 \rightarrow CO_2 + C_2H_5OH$ (fermentation of sugar)

11. $CH_3COF + H_2O \rightarrow CH_3COOH + HF$

12. $CO_2 + H_2O \rightarrow C_6H_{12}O_6 + O_2$ (photosynthesis)

13. Find the quadratic polynomial whose graph passes through the points (1, 1), (2, 2), and (3, 5).

14. Find the quadratic polynomial whose graph passes through the points (0, 0), (−1, 1), and (1, 1).

15. Find the cubic polynomial whose graph passes through the points (−1, −1), (0, 1), (1, 3), (4, −1).

16. The accompanying figure shows the graph of a cubic polynomial. Find the polynomial.

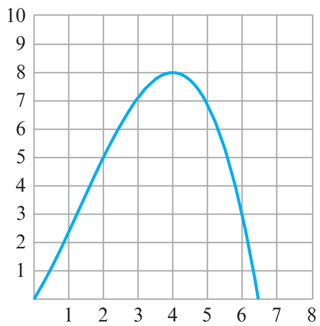


Figure Ex-16

17. (a) Find an equation that represents the family of all second-degree polynomials that pass through the points (0, 1) and (1, 2). [*Hint:* The equation will involve one arbitrary parameter that produces the members of the family when varied.]

(b) By hand, or with the help of a graphing utility, sketch four curves in the family.

18. In this section we have selected only a few applications of linear systems. Using the Internet as a search tool, try to find some more real-world applications of such systems. Select one that is of interest to you, and write a paragraph about it.

True-False Exercises

TF. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) In any network, the sum of the flows out of a node must equal the sum of the flows into a node.

(b) When a current passes through a resistor, there is an increase in the electrical potential in a circuit.

(c) Kirchhoff's current law states that the sum of the currents flowing into a node equals the sum of the currents flowing out of the node.

(d) A chemical equation is called balanced if the total number of atoms on each side of the equation is the same.

(e) Given any n points in the xy-plane, there is a unique polynomial of degree $n - 1$ or less whose graph passes through those points.

Working with Technology

T1. The following table shows the lifting force on an aircraft wing measured in a wind tunnel at various wind velocities. Model the data with an interpolating polynomial of degree 5, and use that polynomial to estimate the lifting force at 2000 ft/s.

Velocity (100 ft/s)	1	2	4	8	16	32
Lifting Force (100 lb)	0	3.12	15.86	33.7	81.5	123.0

T2. (***Calculus required***) Use the method of Example 7 to approximate the integral

$$\int_0^1 e^{x^2}\,dx$$

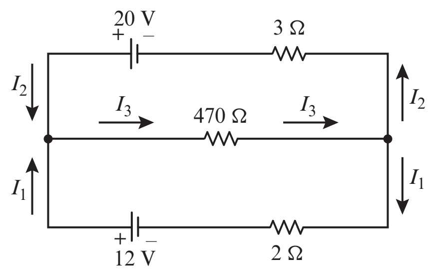
by subdividing the interval of integration into five equal parts and using an interpolating polynomial to approximate the integrand. Compare your answer to that obtained using the numerical integration capability of your technology utility.

T3. Use the method of Example 5 to balance the chemical equation

$$Fe_2O_3 + Al \rightarrow Al_2O_3 + Fe$$

$$(Fe = \text{iron},\ Al = \text{aluminum},\ O = \text{oxygen})$$

T4. Determine the currents in the accompanying circuit.



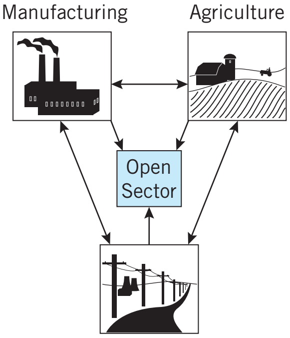
1.10 Leontief Input-Output Models

In 1973 the economist Wassily Leontief was awarded the Nobel prize for his work on economic modeling in which he used matrix methods to study the relationships among different sectors in an economy. In this section we will discuss some of the ideas developed by Leontief.

Inputs and Outputs in an Economy

One way to analyze an economy is to divide it into ***sectors*** and study how the sectors interact with one another. For example, a simple economy might be divided into three sectors—manufacturing, agriculture, and utilities. Typically, a sector will produce certain ***outputs*** but will require ***inputs*** from the other sectors and itself. For example, the agricultural sector may produce wheat as an output but will require inputs of farm machinery from the manufacturing sector, electrical power from the utilities sector, and food from its own sector to feed its workers. Thus, we can imagine an economy to be a network in which inputs and outputs flow in and out of the sectors; the study of such flows is called ***input-output analysis***. Inputs and outputs are commonly measured in monetary units (dollars or millions of dollars, for example) but other units of measurement are also possible.

The flows between sectors of a real economy are not always obvious. For example, in World War II the United States had a demand for 50,000 new airplanes that required the construction of many new aluminum manufacturing plants. This produced an unexpectedly large demand for certain copper electrical components, which in turn produced a copper shortage. The problem was eventually resolved by using silver borrowed from Fort Knox as a copper substitute. In all likelihood modern input-output analysis would have anticipated the copper shortage.

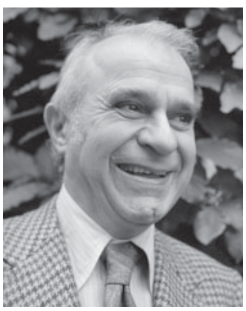


▲ Figure 1.10.1

Most sectors of an economy will produce outputs, but there may exist sectors that consume outputs without producing anything themselves (the consumer market, for example). Those sectors that do not produce outputs are called ***open sectors***. Economies with no open sectors are called ***closed economies***, and economies with one or more open sectors are called ***open economies*** (Figure 1.10.1). In this section we will be concerned with economies with one open sector, and our primary goal will be to determine the output levels that are required for the productive sectors to sustain themselves and satisfy the demand of the open sector.

Leontief Model of an Open Economy

Let us consider a simple open economy with one open sector and three product-producing sectors: manufacturing, agriculture, and utilities. Assume that inputs and outputs are measured in dollars and that the inputs required by the productive sectors to produce one dollar's worth of output are in accordance with Table 1.



Wassily Leontief (1906–1999)

Historical Note It is somewhat ironic that it was the Russian-born Wassily Leontief who won the Nobel prize in 1973 for pioneering the modern methods for analyzing free-market economies. Leontief was a precocious student who entered the University of Leningrad at age 15. Bothered by the intellectual restrictions of the Soviet system, he was put in jail for anti-Communist activities, after which he headed for the University of Berlin, receiving his Ph.D. there in 1928. He came to the United States in 1931, where he held professorships at Harvard and then New York University.

[*Image:* © *Bettmann/CORBIS*]

Table 1

Input Required per Dollar Output

Provider	Manufacturing	Agriculture	Utilities
Manufacturing	\$ 0.50	\$ 0.10	\$ 0.10
Agriculture	\$ 0.20	\$ 0.50	\$ 0.30
Utilities	\$ 0.10	\$ 0.30	\$ 0.40

Usually, one would suppress the labeling and express this matrix as

$$C = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.4 \end{bmatrix} \tag{1}$$

This is called the ***consumption matrix*** (or sometimes the ***technology matrix***) for the economy. The column vectors

$$\mathbf{c}_1 = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0.1 \\ 0.5 \\ 0.3 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.4 \end{bmatrix}$$

What is the economic significance of the row sums of the consumption matrix?

in C list the inputs required by the manufacturing, agricultural, and utilities sectors, respectively, to produce \$1.00 worth of output. These are called the ***consumption vectors*** of the sectors. For example, $\mathbf{c}_1$ tells us that to produce \$1.00 worth of output the manufacturing sector needs \$0.50 worth of manufacturing output, \$0.20 worth of agricultural output, and \$0.10 worth of utilities output.

Continuing with the above example, suppose that the open sector wants the economy to supply it manufactured goods, agricultural products, and utilities with dollar values:

d_1 dollars of manufactured goods
d_2 dollars of agricultural products
d_3 dollars of utilities

The column vector $\mathbf{d}$ that has these numbers as successive components is called the ***outside demand vector***. Since the product-producing sectors consume some of their own output, the dollar value of their output must cover their own needs plus the outside demand. Suppose that the dollar values required to do this are

x_1 dollars of manufactured goods
x_2 dollars of agricultural products
x_3 dollars of utilities

The column vector $\mathbf{x}$ that has these numbers as successive components is called the ***production vector*** for the economy. For the economy with consumption matrix (1), that portion of the production vector $\mathbf{x}$ that will be consumed by the three productive sectors is

$$\underbrace{x_1 \begin{bmatrix} 0.5 \\ 0.2 \\ 0.1 \end{bmatrix}}_{\text{Fractions consumed by manufacturing}} + \underbrace{x_2 \begin{bmatrix} 0.1 \\ 0.5 \\ 0.3 \end{bmatrix}}_{\text{Fractions consumed by agriculture}} + \underbrace{x_3 \begin{bmatrix} 0.1 \\ 0.3 \\ 0.4 \end{bmatrix}}_{\text{Fractions consumed by utilities}} = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C\mathbf{x}$$

The vector $C\mathbf{x}$ is called the ***intermediate demand vector*** for the economy. Once the intermediate demand is met, the portion of the production that is left to satisfy the outside demand is $\mathbf{x} - C\mathbf{x}$. Thus, if the outside demand vector is $\mathbf{d}$, then $\mathbf{x}$ must satisfy the equation

$$\underbrace{\mathbf{x}}_{\boxed{\text{Amount produced}}} - \underbrace{C\mathbf{x}}_{\boxed{\text{Intermediate demand}}} = \underbrace{\mathbf{d}}_{\boxed{\text{Outside demand}}}$$

which we will find convenient to rewrite as

$$(I - C)\mathbf{x} = \mathbf{d} \tag{2}$$

The matrix $I - C$ is called the ***Leontief matrix*** and (2) is called the ***Leontief equation***.

EXAMPLE 1 Satisfying Outside Demand

Consider the economy described in Table 1. Suppose that the open sector has a demand for \$7900 worth of manufacturing products, \$3950 worth of agricultural products, and \$1975 worth of utilities.

(a) Can the economy meet this demand?

(b) If so, find a production vector $\mathbf{x}$ that will meet it exactly.

Solution The consumption matrix, production vector, and outside demand vector are

$$C = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 7900 \\ 3950 \\ 1975 \end{bmatrix} \tag{3}$$

To meet the outside demand, the vector $\mathbf{x}$ must satisfy the Leontief equation (2), so the problem reduces to solving the linear system

$$\underbrace{\begin{bmatrix} 0.5 & -0.1 & -0.1 \\ -0.2 & 0.5 & -0.3 \\ -0.1 & -0.3 & 0.6 \end{bmatrix}}_{I - C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 7900 \\ 3950 \\ 1975 \end{bmatrix}}_{\mathbf{d}} \tag{4}$$

(if consistent). We leave it for you to show that the reduced row echelon form of the augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 27{,}500 \\ 0 & 1 & 0 & 33{,}750 \\ 0 & 0 & 1 & 24{,}750 \end{array}\right]$$

This tells us that (4) is consistent, and the economy can satisfy the demand of the open sector exactly by producing \$27,500 worth of manufacturing output, \$33,750 worth of agricultural output, and \$24,750 worth of utilities output. ◀

Productive Open Economies

In the preceding discussion we considered an open economy with three product-producing sectors; the same ideas apply to an open economy with n product-producing sectors. In this case, the consumption matrix, production vector, and outside demand vector have the form

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

where all entries are nonnegative and

c_{ij} = the monetary value of the output of the ith sector that is needed by the jth sector to produce one unit of output

x_i = the monetary value of the output of the ith sector

d_i = the monetary value of the output of the ith sector that is required to meet the demand of the open sector

Remark Note that the jth column vector of C contains the monetary values that the jth sector requires of the other sectors to produce one monetary unit of output, and the ith row vector of C contains the monetary values required of the ith sector by the other sectors for each of them to produce one monetary unit of output.

As discussed in our example above, a production vector $\mathbf{x}$ that meets the demand $\mathbf{d}$ of the outside sector must satisfy the Leontief equation

$$(I - C)\mathbf{x} = \mathbf{d}$$

If the matrix $I - C$ is invertible, then this equation has the unique solution

$$\mathbf{x} = (I - C)^{-1}\mathbf{d} \tag{5}$$

for every demand vector $\mathbf{d}$. However, for $\mathbf{x}$ to be a valid production vector it must have nonnegative entries, so the problem of importance in economics is to determine conditions under which the Leontief equation has a solution with nonnegative entries.

It is evident from the form of (5) that if $I - C$ is invertible, and if $(I - C)^{-1}$ has nonnegative entries, then for every demand vector $\mathbf{d}$ the corresponding $\mathbf{x}$ will also have nonnegative entries, and hence will be a valid production vector for the economy. Economies for which $(I - C)^{-1}$ has nonnegative entries are said to be ***productive***. Such economies are desirable because demand can always be met by some level of production. The following theorem, whose proof can be found in many books on economics, gives conditions under which open economies are productive.

THEOREM 1.10.1 *If C is the consumption matrix for an open economy, and if all of the column sums are less than 1, then the matrix $I - C$ is invertible, the entries of $(I - C)^{-1}$ are nonnegative, and the economy is productive.*

Remark The jth column sum of C represents the total dollar value of input that the jth sector requires to produce \$1 of output, so if the jth column sum is less than 1, then the jth sector requires less than \$1 of input to produce \$1 of output; in this case we say that the jth sector is ***profitable***. Thus, Theorem 1.10.1 states that if all product-producing sectors of an open economy are profitable, then the economy is productive. In the exercises we will ask you to show that an open economy is productive if all of the row sums of C are less than 1 (Exercise 11). Thus, an open economy is productive if *either* all of the column sums or all of the row sums of C are less than 1.

▶ EXAMPLE 2 An Open Economy Whose Sectors Are All Profitable

The column sums of the consumption matrix C in (1) are less than 1, so $(I - C)^{-1}$ exists and has nonnegative entries. Use a calculating utility to confirm this, and use this inverse to solve Equation (4) in Example 1.

Solution We leave it for you to show that

$$(I - C)^{-1} \approx \begin{bmatrix} 2.65823 & 1.13924 & 1.01266 \\ 1.89873 & 3.67089 & 2.15190 \\ 1.39241 & 2.02532 & 2.91139 \end{bmatrix}$$

This matrix has nonnegative entries, and

$$\mathbf{x} = (I - C)^{-1}\mathbf{d} \approx \begin{bmatrix} 2.65823 & 1.13924 & 1.01266 \\ 1.89873 & 3.67089 & 2.15190 \\ 1.39241 & 2.02532 & 2.91139 \end{bmatrix} \begin{bmatrix} 7900 \\ 3950 \\ 1975 \end{bmatrix} \approx \begin{bmatrix} 27{,}500 \\ 33{,}750 \\ 24{,}750 \end{bmatrix}$$

which is consistent with the solution in Example 1. ◀

Exercise Set 1.10

1. An automobile mechanic (M) and a body shop (B) use each other's services. For each \$1.00 of business that M does, it uses \$0.50 of its own services and \$0.25 of B's services, and for each \$1.00 of business that B does it uses \$0.10 of its own services and \$0.25 of M's services.

(a) Construct a consumption matrix for this economy.

(b) How much must M and B each produce to provide customers with \$7000 worth of mechanical work and \$14,000 worth of body work?

2. A simple economy produces food (F) and housing (H). The production of \$1.00 worth of food requires \$0.30 worth of food and \$0.10 worth of housing, and the production of \$1.00 worth of housing requires \$0.20 worth of food and \$0.60 worth of housing.

(a) Construct a consumption matrix for this economy.

(b) What dollar value of food and housing must be produced for the economy to provide consumers \$130,000 worth of food and \$130,000 worth of housing?

3. Consider the open economy described by the accompanying table, where the input is in dollars needed for \$1.00 of output.

(a) Find the consumption matrix for the economy.

(b) Suppose that the open sector has a demand for \$1930 worth of housing, \$3860 worth of food, and \$5790 worth of utilities. Use row reduction to find a production vector that will meet this demand exactly.

Table Ex-3

Input Required per Dollar Output

Provider		Housing	Food	Utilities
	Housing	\$ 0.10	\$ 0.60	\$ 0.40
	Food	\$ 0.30	\$ 0.20	\$ 0.30
	Utilities	\$ 0.40	\$ 0.10	\$ 0.20

4. A company produces Web design, software, and networking services. View the company as an open economy described by the accompanying table, where input is in dollars needed for \$1.00 of output.

(a) Find the consumption matrix for the company.

(b) Suppose that the customers (the open sector) have a demand for \$5400 worth of Web design, \$2700 worth of software, and \$900 worth of networking. Use row reduction to find a production vector that will meet this demand exactly.

Table Ex-4

Input Required per Dollar Output

Provider		Web Design	Software	Networking
	Web Design	\$ 0.40	\$ 0.20	\$ 0.45
	Software	\$ 0.30	\$ 0.35	\$ 0.30
	Networking	\$ 0.15	\$ 0.10	\$ 0.20

▶ In Exercises 5–6, use matrix inversion to find the production vector $\mathbf{x}$ that meets the demand $\mathbf{d}$ for the consumption matrix C. ◀

5. $C = \begin{bmatrix} 0.1 & 0.3 \\ 0.5 & 0.4 \end{bmatrix}$; $\mathbf{d} = \begin{bmatrix} 50 \\ 60 \end{bmatrix}$

6. $C = \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}$; $\mathbf{d} = \begin{bmatrix} 22 \\ 14 \end{bmatrix}$

7. Consider an open economy with consumption matrix

$$C = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Show that the economy can meet a demand of $d_1 = 2$ units from the first sector and $d_2 = 0$ units from the second sector, but it cannot meet a demand of $d_1 = 2$ units from the first sector and $d_2 = 1$ unit from the second sector.

(b) Give both a mathematical and an economic explanation of the result in part (a).

8. Consider an open economy with consumption matrix

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}$$

If the open sector demands the same dollar value from each product-producing sector, which such sector must produce the greatest dollar value to meet the demand? Is the economy productive?

9. Consider an open economy with consumption matrix

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & 0 \end{bmatrix}$$

Show that the Leontief equation $\mathbf{x} - C\mathbf{x} = \mathbf{d}$ has a unique solution for every demand vector $\mathbf{d}$ if $c_{21}c_{12} < 1 - c_{11}$.

Working with Proofs

10. (a) Consider an open economy with a consumption matrix C whose column sums are less than 1, and let $\mathbf{x}$ be the production vector that satisfies an outside demand $\mathbf{d}$; that is, $(I - C)^{-1}\mathbf{d} = \mathbf{x}$. Let $\mathbf{d}_j$ be the demand vector that is obtained by increasing the jth entry of $\mathbf{d}$ by 1 and leaving the other entries fixed. Prove that the production vector $\mathbf{x}_j$ that meets this demand is

$$\mathbf{x}_j = \mathbf{x} + j\text{th column vector of } (I - C)^{-1}$$

(b) In words, what is the economic significance of the jth column vector of $(I - C)^{-1}$? [*Hint:* Look at $\mathbf{x}_j - \mathbf{x}$.]

11. Prove: If C is an $n \times n$ matrix whose entries are nonnegative and whose row sums are less than 1, then $I - C$ is invertible and has nonnegative entries. [*Hint:* $(A^T)^{-1} = (A^{-1})^T$ for any invertible matrix A.]

True-False Exercises

TF. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) Sectors of an economy that produce outputs are called open sectors.

(b) A closed economy is an economy that has no open sectors.

(c) The rows of a consumption matrix represent the outputs in a sector of an economy.

(d) If the column sums of the consumption matrix are all less than 1, then the Leontief matrix is invertible.

(e) The Leontief equation relates the production vector for an economy to the outside demand vector.

Working with Technology

T1. The following table describes an open economy with three sectors in which the table entries are the dollar inputs required to produce one dollar of output. The outside demand during a 1-week period if \$50,000 of coal, \$75,000 of electricity, and \$1,250,000 of manufacturing. Determine whether the economy can meet the demand.

Input Required per Dollar Output

Provider		Electricity	Coal	Manufacturing
Provider	**Electricity**	\$ 0.1	\$ 0.25	\$ 0.2
	Coal	\$ 0.3	\$ 0.4	\$ 0.5
	Manufacturing	\$ 0.1	\$ 0.15	\$ 0.1

Chapter 1 Supplementary Exercises

▶ In Exercises 1–4 the given matrix represents an augmented matrix for a linear system. Write the corresponding set of linear equations for the system, and use Gaussian elimination to solve the linear system. Introduce free parameters as necessary. ◀

1. $\begin{bmatrix} 3 & -1 & 0 & 4 & 1 \\ 2 & 0 & 3 & 3 & -1 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 4 & -1 \\ -2 & -8 & 2 \\ 3 & 12 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -4 & 1 & 6 \\ -4 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix}$ **4.** $\begin{bmatrix} 3 & 1 & -2 \\ -9 & -3 & 6 \\ 6 & 2 & 1 \end{bmatrix}$

5. Use Gauss–Jordan elimination to solve for x' and y' in terms of x and y.

$$x = \tfrac{3}{5}x' - \tfrac{4}{5}y'$$
$$y = \tfrac{4}{5}x' + \tfrac{3}{5}y'$$

6. Use Gauss–Jordan elimination to solve for x' and y' in terms of x and y.

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

7. Find positive integers that satisfy

$$\begin{aligned} x + y + z &= 9 \\ x + 5y + 10z &= 44 \end{aligned}$$