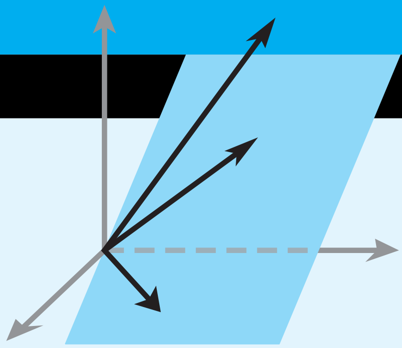


CHAPTER 4

# General Vector Spaces

CHAPTER CONTENTS



INTRODUCTION

Recall that we began our study of vectors by viewing them as directed line segments (arrows). We then extended this idea by introducing rectangular coordinate systems, which enabled us to view vectors as ordered pairs and ordered triples of real numbers. As we developed properties of these vectors we noticed patterns in various formulas that enabled us to extend the notion of a vector to an $n$-tuple of real numbers. Although $n$-tuples took us outside the realm of our "visual experience," it gave us a valuable tool for understanding and studying systems of linear equations. In this chapter we will extend the concept of a vector yet again by using the most important algebraic properties of vectors in $R^n$ as axioms. These axioms, if satisfied by a set of objects, will enable us to think of those objects as vectors.

## 4.1 Real Vector Spaces

In this section we will extend the concept of a vector by using the basic properties of vectors in $R^n$ as axioms, which if satisfied by a set of objects, guarantee that those objects behave like familiar vectors.

### *Vector Space Axioms*

The following definition consists of ten axioms, eight of which are properties of vectors in $R^n$ that were stated in Theorem 3.1.1. It is important to keep in mind that one does not *prove* axioms; rather, they are assumptions that serve as the starting point for proving theorems.

In this text scalars will be either real numbers or complex numbers. Vector spaces with real scalars will be called ***real vector spaces*** and those with complex scalars will be called ***complex vector spaces***. There is a more general notion of a vector space in which scalars can come from a mathematical structure known as a "field," but we will not be concerned with that level of generality. For now, we will focus exclusively on real vector spaces, which we will refer to simply as "vector spaces." We will consider complex vector spaces later.

**DEFINITION 1** Let $V$ be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by numbers called ***scalars***. By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in $V$ an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar $k$ and each object $\mathbf{u}$ in $V$ an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by $k$. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $V$ and all scalars $k$ and $m$, then we call $V$ a ***vector space*** and we call the objects in $V$ ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in $V$, then $\mathbf{u}+\mathbf{v}$ is in $V$.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in $V$, called a ***zero vector*** for $V$, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in $V$.
5. For each $\mathbf{u}$ in $V$, there is an object $-\mathbf{u}$ in $V$, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If $k$ is any scalar and $\mathbf{u}$ is any object in $V$, then $k\mathbf{u}$ is in $V$.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

Observe that the definition of a vector space does not specify the nature of the vectors or the operations. Any kind of object can be a vector, and the operations of addition and scalar multiplication need not have any relationship to those on $R^n$. The only requirement is that the ten vector space axioms be satisfied. In the examples that follow we will use four basic steps to show that a set with two operations is a vector space.

**To Show That a Set with Two Operations Is a Vector Space**

***Step 1.*** Identify the set $V$ of objects that will become vectors.

***Step 2.*** Identify the addition and scalar multiplication operations on $V$.

***Step 3.*** Verify Axioms 1 and 6; that is, adding two vectors in $V$ produces a vector in $V$, and multiplying a vector in $V$ by a scalar also produces a vector in $V$. Axiom 1 is called ***closure under addition***, and Axiom 6 is called ***closure under scalar multiplication***.

***Step 4.*** Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

**Hermann Günther Grassmann (1809–1877)**

**Historical Note** The notion of an "abstract vector space" evolved over many years and had many contributors. The idea crystallized with the work of the German mathematician H. G. Grassmann, who published a paper in 1862 in which he considered abstract systems of unspecified elements on which he defined formal operations of addition and scalar multiplication. Grassmann's work was controversial, and others, including Augustin Cauchy (p. 121), laid reasonable claim to the idea.

[*Image: © Sueddeutsche Zeitung Photo/The Image Works*]

Our first example is the simplest of all vector spaces in that it contains only one object. Since Axiom 4 requires that every vector space contain a zero vector, the object will have to be that vector.

### EXAMPLE 1 The Zero Vector Space

Let $V$ consist of a single object, which we denote by $\mathbf{0}$, and define

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0}$$

for all scalars $k$. It is easy to check that all the vector space axioms are satisfied. We call this the ***zero vector space***. ◀

Our second example is one of the most important of all vector spaces—the familiar space $R^n$. It should not be surprising that the operations on $R^n$ satisfy the vector space axioms because those axioms were based on known properties of operations on $R^n$.

### EXAMPLE 2 $R^n$ Is a Vector Space

Let $V = R^n$, and define the vector space operations on $V$ to be the usual operations of addition and scalar multiplication of $n$-tuples; that is,

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$
$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

The set $V = R^n$ is closed under addition and scalar multiplication because the foregoing operations produce $n$-tuples as their end result, and these operations satisfy Axioms 2, 3, 4, 5, 7, 8, 9, and 10 by virtue of Theorem 3.1.1. ◀

Our next example is a generalization of $R^n$ in which we allow vectors to have infinitely many components.

### EXAMPLE 3 The Vector Space of Infinite Sequences of Real Numbers

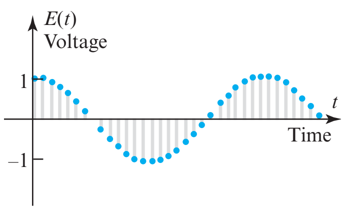
Let $V$ consist of objects of the form

$$\mathbf{u} = (u_1, u_2, \ldots, u_n, \ldots)$$

in which $u_1, u_2, \ldots, u_n, \ldots$ is an infinite sequence of real numbers. We define two infinite sequences to be *equal* if their corresponding components are equal, and we define addition and scalar multiplication componentwise by

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, \ldots, u_n, \ldots) + (v_1, v_2, \ldots, v_n, \ldots)\\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots)\\ k\mathbf{u} &= (ku_1, ku_2, \ldots, ku_n, \ldots)\end{aligned}$$

In the exercises we ask you to confirm that $V$ with these operations is a vector space. We will denote this vector space by the symbol $R^\infty$. ◀




▲ Figure 4.1.1

Vector spaces of the type in Example 3 arise when a transmitted signal of indefinite duration is digitized by sampling its values at discrete time intervals (Figure 4.1.1).

In the next example our vectors will be matrices. This may be a little confusing at first because matrices are composed of rows and columns, which are themselves vectors (row vectors and column vectors). However, from the vector space viewpoint we are not concerned with the individual rows and columns but rather with the properties of the matrix operations as they relate to the matrix as a whole.

### ▶ EXAMPLE 4 The Vector Space of 2 × 2 Matrices

Note that Equation (1) involves *three* different addition operations: the addition operation on vectors, the addition operation on matrices, and the addition operation on real numbers.

Let $V$ be the set of $2 \times 2$ matrices with real entries, and take the vector space operations on $V$ to be the usual operations of matrix addition and scalar multiplication; that is,

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix} \tag{1}$$

$$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

The set $V$ is closed under addition and scalar multiplication because the foregoing operations produce $2 \times 2$ matrices as the end result. Thus, it remains to confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold. Some of these are standard properties of matrix operations. For example, Axiom 2 follows from Theorem 1.4.1($a$) since

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v}+\mathbf{u}$$

Similarly, Axioms 3, 7, 8, and 9 follow from parts ($b$), ($h$), ($j$), and ($e$), respectively, of that theorem (verify). This leaves Axioms 4, 5, and 10 that remain to be verified.

To confirm that Axiom 4 is satisfied, we must find a $2 \times 2$ matrix $\mathbf{0}$ in $V$ for which $\mathbf{u}+\mathbf{0} = \mathbf{0}+\mathbf{u}$ for all $2 \times 2$ matrices in $V$. We can do this by taking

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With this definition,

$$\mathbf{0}+\mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

and similarly $\mathbf{u}+\mathbf{0} = \mathbf{u}$. To verify that Axiom 5 holds we must show that each object $\mathbf{u}$ in $V$ has a negative $-\mathbf{u}$ in $V$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$ and $(-\mathbf{u})+\mathbf{u} = \mathbf{0}$. This can be done by defining the negative of $\mathbf{u}$ to be

$$-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$$

With this definition,

$$\mathbf{u}+(-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

and similarly $(-\mathbf{u})+\mathbf{u} = \mathbf{0}$. Finally, Axiom 10 holds because

$$1\mathbf{u} = 1\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

### ▶ EXAMPLE 5 The Vector Space of $m \times n$ Matrices

Example 4 is a special case of a more general class of vector spaces. You should have no trouble adapting the argument used in that example to show that the set $V$ of all $m \times n$ matrices with the usual matrix operations of addition and scalar multiplication is a vector space. We will denote this vector space by the symbol $M_{mn}$. Thus, for example, the vector space in Example 4 is denoted as $M_{22}$.

**EXAMPLE 6 The Vector Space of Real-Valued Functions**

Let $V$ be the set of real-valued functions that are defined at each $x$ in the interval $(-\infty, \infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two functions in $V$ and if $k$ is any scalar, then define the operations of addition and scalar multiplication by

$$(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x) \tag{2}$$

$$(k\mathbf{f})(x) = kf(x) \tag{3}$$

One way to think about these operations is to view the numbers $f(x)$ and $g(x)$ as "components" of $\mathbf{f}$ and $\mathbf{g}$ at the point $x$, in which case Equations (2) and (3) state that two functions are added by adding corresponding components, and a function is multiplied by a scalar by multiplying each component by that scalar—exactly as in $R^n$ and $R^\infty$. This idea is illustrated in parts (*a*) and (*b*) of Figure 4.1.2. The set $V$ with these operations is denoted by the symbol $F(-\infty, \infty)$. We can prove that this is a vector space as follows:

**Axioms 1 and 6:** These closure axioms require that if we add two functions that are defined at each $x$ in the interval $(-\infty, \infty)$, then sums and scalar multiples of those functions must also be defined at each $x$ in the interval $(-\infty, \infty)$. This follows from Formulas (2) and (3).

**Axiom 4:** This axiom requires that there exists a function $\mathbf{0}$ in $F(-\infty, \infty)$, which when added to any other function $\mathbf{f}$ in $F(-\infty, \infty)$ produces $\mathbf{f}$ back again as the result. The function whose value at every point $x$ in the interval $(-\infty, \infty)$ is zero has this property. Geometrically, the graph of the function $\mathbf{0}$ is the line that coincides with the $x$-axis.

In Example 6 the functions were defined on the entire interval $(-\infty, \infty)$. However, the arguments used in that example apply as well on all subintervals of $(-\infty, \infty)$, such as a closed interval $[a, b]$ or an open interval $(a, b)$. We will denote the vector spaces of functions on these intervals by $F[a, b]$ and $F(a, b)$, respectively.

**Axiom 5:** This axiom requires that for each function $\mathbf{f}$ in $F(-\infty, \infty)$ there exists a function $-\mathbf{f}$ in $F(-\infty, \infty)$, which when added to $\mathbf{f}$ produces the function $\mathbf{0}$. The function defined by $-\mathbf{f}(x) = -f(x)$ has this property. The graph of $-\mathbf{f}$ can be obtained by reflecting the graph of $\mathbf{f}$ about the $x$-axis (Figure 4.1.2*c*).

**Axioms 2, 3, 7, 8, 9, 10:** The validity of each of these axioms follows from properties of real numbers. For example, if $\mathbf{f}$ and $\mathbf{g}$ are functions in $F(-\infty, \infty)$, then Axiom 2 requires that $\mathbf{f} + \mathbf{g} = \mathbf{g} + \mathbf{f}$. This follows from the computation

$$(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x) = \mathbf{g}(x) + \mathbf{f}(x) = (\mathbf{g} + \mathbf{f})(x)$$

in which the first and last equalities follow from (2), and the middle equality is a property of real numbers. We will leave the proofs of the remaining parts as exercises. ◀

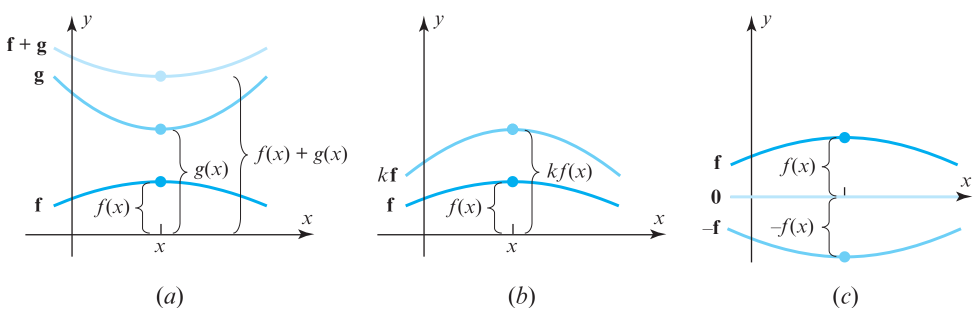



Figure 4.1.2

It is important to recognize that you cannot impose any two operations on any set $V$ and expect the vector space axioms to hold. For example, if $V$ is the set of $n$-tuples with *positive* components, and if the standard operations from $R^n$ are used, then $V$ is not closed under scalar multiplication, because if $\mathbf{u}$ is a nonzero $n$-tuple in $V$, then $(-1)\mathbf{u}$ has at least one negative component and hence is not in $V$. The following is a less obvious example in which only one of the ten vector space axioms fails to hold.

### EXAMPLE 7 A Set That Is Not a Vector Space

Let $V = R^2$ and define addition and scalar multiplication operations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if $k$ is any real number, then define

$$k\mathbf{u} = (ku_1, 0)$$

For example, if $\mathbf{u} = (2, 4)$, $\mathbf{v} = (-3, 5)$, and $k = 7$, then

$$\mathbf{u} + \mathbf{v} = (2 + (-3), 4 + 5) = (-1, 9)$$
$$k\mathbf{u} = 7\mathbf{u} = (7 \cdot 2, 0) = (14, 0)$$

The addition operation is the standard one from $R^2$, but the scalar multiplication is not. In the exercises we will ask you to show that the first nine vector space axioms are satisfied. However, Axiom 10 fails to hold for certain vectors. For example, if $\mathbf{u} = (u_1, u_2)$ is such that $u_2 \neq 0$, then

$$1\mathbf{u} = 1(u_1, u_2) = (1 \cdot u_1, 0) = (u_1, 0) \neq \mathbf{u}$$

Thus, $V$ is not a vector space with the stated operations. ◀

Our final example will be an unusual vector space that we have included to illustrate how varied vector spaces can be. Since the vectors in this space will be real numbers, it will be important for you to keep track of which operations are intended as vector operations and which ones as ordinary operations on real numbers.

### EXAMPLE 8 An Unusual Vector Space

Let $V$ be the set of positive real numbers, let $\mathbf{u} = u$ and $\mathbf{v} = v$ be any vectors (i.e., positive real numbers) in $V$, and let $k$ be any scalar. Define the operations on $V$ to be

$$u + v = uv \quad \text{[Vector addition is numerical multiplication.]}$$
$$ku = u^k \quad \text{[Scalar multiplication is numerical exponentiation.]}$$

Thus, for example, $1 + 1 = 1$ and $(2)(1) = 1^2 = 1$—strange indeed, but nevertheless the set $V$ with these operations satisfies the ten vector space axioms and hence is a vector space. We will confirm Axioms 4, 5, and 7, and leave the others as exercises.

- Axiom 4—The zero vector in this space is the number 1 (i.e., $\mathbf{0} = 1$) since

$$u + 1 = u \cdot 1 = u$$

- Axiom 5—The negative of a vector $u$ is its reciprocal (i.e., $-u = 1/u$) since

$$u + \frac{1}{u} = u\left(\frac{1}{u}\right) = 1\ (= \mathbf{0})$$

- Axiom 7—$k(u + v) = (uv)^k = u^k v^k = (ku) + (kv)$. ◀

## *Some Properties of Vectors*

The following is our first theorem about vector spaces. The proof is very formal with each step being justified by a vector space axiom or a known property of real numbers. There will not be many rigidly formal proofs of this type in the text, but we have included this one to reinforce the idea that the familiar properties of vectors can all be derived from the vector space axioms.

**THEOREM 4.1.1** *Let $V$ be a vector space, $\mathbf{u}$ a vector in $V$, and $k$ a scalar; then:*

(*a*) $0\mathbf{u} = \mathbf{0}$

(*b*) $k\mathbf{0} = \mathbf{0}$

(*c*) $(-1)\mathbf{u} = -\mathbf{u}$

(*d*) *If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.*

We will prove parts (*a*) and (*c*) and leave proofs of the remaining parts as exercises.

***Proof (a)*** We can write

$$\begin{aligned} 0\mathbf{u} + 0\mathbf{u} &= (0+0)\mathbf{u} && \textbf{[Axiom 8]} \\ &= 0\mathbf{u} && \textbf{[Property of the number 0]} \end{aligned}$$

By Axiom 5 the vector $0\mathbf{u}$ has a negative, $-0\mathbf{u}$. Adding this negative to both sides above yields

$$[0\mathbf{u} + 0\mathbf{u}] + (-0\mathbf{u}) = 0\mathbf{u} + (-0\mathbf{u})$$

or

$$\begin{aligned} 0\mathbf{u} + [0\mathbf{u} + (-0\mathbf{u})] &= 0\mathbf{u} + (-0\mathbf{u}) && \textbf{[Axiom 3]} \\ 0\mathbf{u} + \mathbf{0} &= \mathbf{0} && \textbf{[Axiom 5]} \\ 0\mathbf{u} &= \mathbf{0} && \textbf{[Axiom 4]} \end{aligned}$$

***Proof (c)*** To prove that $(-1)\mathbf{u} = -\mathbf{u}$, we must show that $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$. The proof is as follows:

$$\begin{aligned} \mathbf{u} + (-1)\mathbf{u} &= 1\mathbf{u} + (-1)\mathbf{u} && \textbf{[Axiom 10]} \\ &= (1 + (-1))\mathbf{u} && \textbf{[Axiom 8]} \\ &= 0\mathbf{u} && \textbf{[Property of numbers]} \\ &= \mathbf{0} && \textbf{[Part (a) of this theorem]} \end{aligned}$$

◀

## *A Closing Observation*

This section of the text is important to the overall plan of linear algebra in that it establishes a common thread among such diverse mathematical objects as geometric vectors, vectors in $R^n$, infinite sequences, matrices, and real-valued functions, to name a few. As a result, whenever we discover a new theorem about general vector spaces, we will at the same time be discovering a theorem about geometric vectors, vectors in $R^n$, sequences, matrices, real-valued functions, and about any new kinds of vectors that we might discover.

To illustrate this idea, consider what the rather innocent-looking result in part (*a*) of Theorem 4.1.1 says about the vector space in Example 8. Keeping in mind that the vectors in that space are positive real numbers, that scalar multiplication means numerical exponentiation, and that the zero vector is the number 1, the equation

$$0\mathbf{u} = \mathbf{0}$$

is really a statement of the familiar fact that if $u$ is a positive real number, then

$$u^0 = 1$$

## Exercise Set 4.1

**1.** Let $V$ be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2), \quad k\mathbf{u} = (0, ku_2)$$

(a) Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (3, 4)$, and $k = 3$.

(b) In words, explain why $V$ is closed under addition and scalar multiplication.

(c) Since addition on $V$ is the standard addition operation on $R^2$, certain vector space axioms hold for $V$ because they are known to hold for $R^2$. Which axioms are they?

(d) Show that Axioms 7, 8, and 9 hold.

(e) Show that Axiom 10 fails and hence that $V$ is not a vector space under the given operations.

**2.** Let $V$ be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1), \quad k\mathbf{u} = (ku_1, ku_2)$$

(a) Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (0, 4)$, $\mathbf{v} = (1, -3)$, and $k = 2$.

(b) Show that $(0, 0) \neq \mathbf{0}$.

(c) Show that $(-1, -1) = \mathbf{0}$.

(d) Show that Axiom 5 holds by producing an ordered pair $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for $\mathbf{u} = (u_1, u_2)$.

(e) Find two vector space axioms that fail to hold.

▶ In Exercises 3–12, determine whether each set equipped with the given operations is a vector space. For those that are not vector spaces identify the vector space axioms that fail. ◀

**3.** The set of all real numbers with the standard operations of addition and multiplication.

**4.** The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on $R^2$.

**5.** The set of all pairs of real numbers of the form $(x, y)$, where $x \geq 0$, with the standard operations on $R^2$.

**6.** The set of all $n$-tuples of real numbers that have the form $(x, x, \ldots, x)$ with the standard operations on $R^n$.

**7.** The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = (k^2x, k^2y, k^2z)$$

**8.** The set of all $2 \times 2$ invertible matrices with the standard matrix addition and scalar multiplication.

**9.** The set of all $2 \times 2$ matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with the standard matrix addition and scalar multiplication.

**10.** The set of all real-valued functions $f$ defined everywhere on the real line and such that $f(1) = 0$ with the operations used in Example 6.

**11.** The set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y') \quad \text{and} \quad k(1, y) = (1, ky)$$

**12.** The set of polynomials of the form $a_0 + a_1x$ with the operations

$$(a_0 + a_1x) + (b_0 + b_1x) = (a_0 + b_0) + (a_1 + b_1)x$$

and

$$k(a_0 + a_1x) = (ka_0) + (ka_1)x$$

**13.** Verify Axioms 3, 7, 8, and 9 for the vector space given in Example 4.

**14.** Verify Axioms 1, 2, 3, 7, 8, 9, and 10 for the vector space given in Example 6.

**15.** With the addition and scalar multiplication operations defined in Example 7, show that $V = R^2$ satisfies Axioms 1–9.

**16.** Verify Axioms 1, 2, 3, 6, 8, 9, and 10 for the vector space given in Example 8.

**17.** Show that the set of all points in $R^2$ lying on a line is a vector space with respect to the standard operations of vector addition and scalar multiplication if and only if the line passes through the origin.

**18.** Show that the set of all points in $R^3$ lying in a plane is a vector space with respect to the standard operations of vector addition and scalar multiplication if and only if the plane passes through the origin.

▶ In Exercises 19–20, let $V$ be the vector space of positive real numbers with the vector space operations given in Example 8. Let $\mathbf{u} = u$ be any vector in $V$, and rewrite the vector statement as a statement about real numbers. ◀

**19.** $-\mathbf{u} = (-1)\mathbf{u}$

**20.** $k\mathbf{u} = \mathbf{0}$ if and only if $k = 0$ or $\mathbf{u} = \mathbf{0}$.

### Working with Proofs

**21.** The argument that follows proves that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in a vector space $V$ such that $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$ (the ***cancellation law*** for vector addition). As illustrated, justify the steps by filling in the blanks.