Usman Institute of Technology

End-Term Examination Fall 2020 Semester

Course Code: MS121 **Course Title:** Multivariate Calculus **Date: 24-02-2021**

Maximum Marks: 60 **Max Time Allowed:** 3 hours

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Seat No. ST-18045 **Roll No. 18B-129-SE**

Batch: **2018** **Enrollment No.** UIT/147/2018-19

Important

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Declaration

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☒ I **have read and understood the Students Ethics Policy for Online Assessments.**

(paper will not be graded if the above is not checked)

Note: Submission of this paper certifies that you are agreed to the Students Ethics Policy for Online Assessments and are liable to be judged according to it.

AWARD

		Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9
Examiner										
ERC										
	Total Marks in Figures				**Total Marks in Words**					

Note: Attempt all questions, each question carries equal marks.

Q1)

a) Find the equation of line of intersection of the following planes

$$3x - 4y + 4z - 29 = 0 \quad , \quad 6x - 8y - 3z = 20$$

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Q2 a

$3x - 4y + 4z - 29 = 0$, $6x - 8y - 3z = 20$

$3x - 4y + 4z - 29 = 0$
$6x - 8y - 3z = 20$

$a(3, -4, 4)$
$b(6, -8, -3)$

$$v = |a \times b| = \begin{vmatrix} i & j & k \\ 3 & -4 & 4 \\ 6 & -8 & -3 \end{vmatrix}$$

$v = i(12+32) - j(-9-24) + k(-24+24)$
$= 44i + 33j + 0k$

$v_{12} = n_1 \times n_2 = \langle 44i + 33j + 0k \rangle$

$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\Rightarrow 44(x-3) + 33(y+4) + 0(z-3) = 0$

$\Rightarrow 44x - \cancel{132} + 33y + \cancel{132} = 0$

$\Rightarrow \boxed{44x + 33y = 0}$

b) Find the point of intersection and acute angle between Line and Plane

$$x + 1 = 4t,\ y - 3 = t,\ z - 1 = 0\ ,\ 2x - y + 5z = 21$$

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Question 1b:

$x+1=4t$
$y-3=t$
$z-1=0$

$2x-y+5z=21 \rightarrow (a)$

Consider (a)

$2x-y+5z=21$

Put x, y, z in (a)

$\Rightarrow 2(4t-1)-(t+3)+5(1)=21$
$\Rightarrow 7t=21$
$\Rightarrow \boxed{t=3}$

put "t" in x, y, z:

$\Rightarrow x+1=4(3)$
$\Rightarrow \boxed{x=11}$

$\Rightarrow y=3+3$
$\Rightarrow \boxed{y=6}$

$\Rightarrow \boxed{z=1}$

$\Rightarrow$ Point of Intersection $\langle 11, 6, 1\rangle$

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$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|}$$

$n_1 = \langle 4, 1, 0 \rangle$

$n_2 = \langle 2, -1, +5 \rangle$

$n_1 \cdot n_2 = \langle 8, -1, 0 \rangle$

$|n_1| = \sqrt{(4)^2 + (1)^2 + (0)^2}$

$|n_1| = \sqrt{17}$

$|n_2| = \sqrt{(2)^2 + (-1)^2 + (5)^2}$

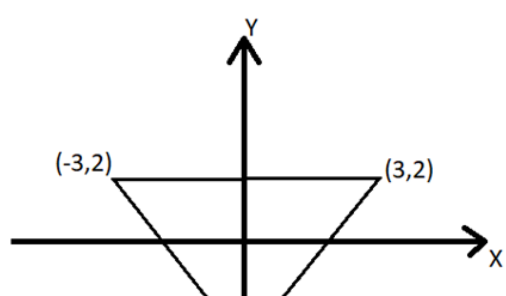
$|n_2| = \sqrt{30}$

$$\cos\theta = \frac{7}{\sqrt{17}\sqrt{30}}$$

$\boxed{\theta = 82.36}$

Q2)

a) Find $\frac{\partial^2 z}{\partial x\, \partial y}$ and $\frac{dz}{dt}$ Usin chain rule. If $z = e^{xy} - Sin\ xy$, $x = \frac{1}{t}$, $y = t^2$



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Question 2a

$\frac{\partial^2 y}{\partial x \partial y} = ??$ $\frac{dz}{\partial t} = ?$

for $\frac{dz}{dt}$:

$\Rightarrow \frac{dz}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} e^{xy} - \sin xy$

$= ye^{xy} - \cos xy \cdot y$

$\Rightarrow \frac{\partial z}{\partial x} = y(e^{xy} - \cos xy) \longrightarrow (i)$

$\Rightarrow \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} e^{xy} - \sin xy$

$= xe^{xy} - \cos xy \cdot x$

$\Rightarrow \frac{\partial z}{\partial y} = x(e^{xy} - \cos xy)$

$\Rightarrow \frac{\partial x}{\partial t} = \frac{\partial}{\partial t}\frac{1}{t}$

$$= \frac{\partial}{\partial t} t^{-1} = -t^{-1-1} = \boxed{-\frac{1}{t^2}}$$

$$\frac{\partial y}{\partial t} = \frac{d}{dt} t^2$$

$$\boxed{\frac{\partial y}{\partial t} = 2t}$$

$$\Rightarrow \frac{\partial z}{\partial t} = y(e^{xy} - \cos xy) \cdot -\frac{1}{t^2} \cdot + x(e^{xy} - \cos xy) \cdot 2t$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial t} = (e^{xy} - \cos xy)\left(-\frac{y}{t^2} + 2xt\right)}$$

$$\because \frac{\partial z}{\partial x} = y(e^{xy} - \cos xy)$$

$$\frac{\partial^2 z}{\partial x} = y(e^{xy} \cdot y + \sin xy \cdot y)$$

$$= y(y(e^{xy} + \sin xy))$$

$$\frac{\partial^2 z}{\partial x} = y^2(e^{xy} + \sin xy)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x} = y^2(e^{xy} + \sin^{xy})$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = y^2 \frac{\partial}{\partial y}\left(e^{xy} + \sin xy\right) + \left(e^{xy} + \sin xy\right)\frac{\partial}{\partial y} y^2$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial x \partial y} = y^2\left(e^{xy} + \cos xy\right) + 2y\left(e^{xy} + \sin xy\right)}$$

b) Locate all relative extrema and saddle points of $\mathrm{f(x, y)} = \frac{9}{4}x^4 - y^3 - \mathrm{xy} + 6$

$y = a\left(-\frac{1}{3}\right)^3$

$\boxed{y = -1/3}$

$\boxed{y = 0}$

$244x^5 + 1 = 0$

$244x^5 = -1$

$x^5 = \frac{-1}{244}$

$\boxed{x = -1/3}$

$\Rightarrow (x, y) = (-1/3, -1/3)$

At P1 $(-1/3, -1/3)$

$\Rightarrow \boxed{D = 5}$

At P(0,0)

$\Rightarrow 0 - (-1)^2 = -1$

Saddle Point $(0, 0)$
Max at $(-1/3, -1/3)$

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Question 2b:

$f(x,y) = \frac{9}{4}x^4 - y^3 - xy + 6$

$f_x = \frac{9}{4} \cdot 4x^3 - 0 - y + 0$

$\boxed{f_x = 9x^3 - y}$; $\boxed{f_{xx} = 27x^2}$

$f_y = 0 - 3y^2 - x + 0$; $f_{yy} = -3 \cdot 2y - 0$

$\boxed{f_y = -3y^2 - x}$ $\boxed{f_{yy} = -6y}$

$f_x = 9x^3 - y$

$f_{xy} = 0 - 1$

$\boxed{f_{xy} = -1}$

$\therefore D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$= (27x^2)(-6y) - (-1)^2$

$\boxed{D = -162x^2y - 1}$

Let:

$f_x = 0$ $\qquad$ $f_y = 0$

$9x^2 - y = 0$ $\qquad$ $-3y^2 - x = 0$

$\boxed{y = 9x^3}$ $\qquad$ $\boxed{x = 0}$

Q3)

$\boldsymbol{a}$) $Evaluate \iint_R (2x - y^2)\, dA$,

$where\ R\ is\ Triangular\ region\ given\ in\ figure$ 1.

Q # 3(a)

$$\iint_R (2x - y^2)\, dA \quad \text{where } R \text{ is } \Delta \text{ region}$$

$-y = -x-2$, $-y = x-2$, $y = 2$

$\Rightarrow x = y-2 \qquad \Rightarrow x = -y+2$

$\Rightarrow y = 2 \qquad y = -2$

$$\Rightarrow \int_{y=-2}^{2} \int_{y-2}^{-y+2} (2x - y^2)\, dx\, dy$$

$$\Rightarrow \int_{-2}^{2} \left| 2\frac{x^2}{2} - xy^2 \right|_{y-2}^{-y+2} dy$$

$$\Rightarrow \int_{-2}^{2} \left| x^2 - y^2 x \right|_{y-2}^{-y+2} dy$$

$$\Rightarrow \int_{-2}^{2} (4 - 4y + y^2 - 2y^2 + y^3) - (y^2 - 4y + 4 - y^3 + 2y^2)\, dy$$

$$\Rightarrow \int_{-2}^{2} (2y^3 - 4y^2)\, dy$$

$$\Rightarrow \left| \frac{2y^4}{4} - \frac{4y^3}{3} \right|_{-2}^{3}$$

$$\Rightarrow \left| \frac{1}{2} y^4 - \frac{4}{3} y^3 \right|_{-2}^{2}$$

$$= \left\{ \frac{1(2)^4}{2} - \frac{4(2)^3}{3} \right\} - \left\{ \frac{(-2)^4}{2} - \frac{4(-2)^3}{3} \right\}.$$

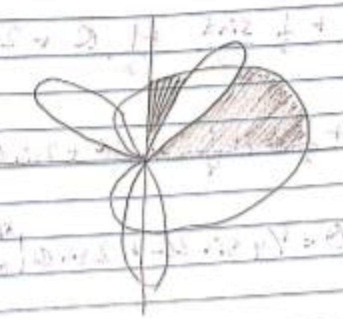
$$= -\left\{ \frac{64}{3} \right\} \quad \text{Ans}$$

b) $Evaluate \iint_R 2 \ dA$, $where\ R\ is\ region\ in\ the\ first\ \ quadrant\ that\ is$

$out\ side\ the\ Petal\ r = Sin(3\theta)\ and\ inside\ the\ cardoid\ r = (1 + cos\theta).$

$Also\ sktech\ its\ rough\ graph.$

Question 3b:



$\iint 2\,dA$

$r = \sin 3\theta \rightarrow$ rose with 3 petals

$r = 1 + \cos\theta \rightarrow$ cardioid

$$\iint_R 2\,dA$$

$$\int_0^{\pi/3} \int_0^{1+\cos\theta} 2r\,dr\,d\theta$$

$$= \int_0^{\pi/3} \left| \frac{2r^2}{2} \right|_0^{1+\cos\theta} d\theta$$

$$= \int_0^{\pi/3} (1+\cos\theta)^2 - 0^2 \, d\theta$$

$$= \int_0^{\pi/3} 1 + \frac{\cos 2\theta + 1}{2} + 2\cos\theta$$

$$= \left| \theta + \frac{1}{2}\,\frac{\sin\theta}{2} + \frac{1}{2}\theta + 2\sin\theta \right|_0^{\pi/3}$$

$$= \left| \theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + 2\sin\theta \right|_0^{\pi/3}$$

$$= \left| 3/2\,\theta + 1/4\,\sin 2\theta + 2\sin\theta \right|_0^{\pi/3}$$

$$= \left\{ 3/2 \cdot \pi/3 + 1/4\,\sin 2\pi/3 + 2\sin \pi/3 \right\} - \left\{ 0 - \frac{1}{4}\sin 2\theta) + 2\sin 0 \right\}$$

$$= \left\{ \pi/2 + \frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\cdot 2 \right\} - \left\{ 0 + 0 + 0 \right\}$$

$$= \frac{\pi}{2} + \frac{\sqrt{3}}{8} + \sqrt{3}$$

$$= \boxed{\frac{4\pi + 9\sqrt{3}}{8}}$$

Q4)

a) Evaluate $\iiint\limits_G x^2yz\, dV$, $\quad$ *where G is solid in the first octant that is bounded by parabolic cylinder* $z = 4 - x^2$, $y = x$ *and the coordinate planes*

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Q#4A

$$\iiint_G x^2 y z \, dv$$

$z = 4-x^2, \; y = x$

$\Rightarrow z = 4-x^2, \; y = x, \; z = 0$

$\Rightarrow 0 = 4-x^2$

$\Rightarrow \boxed{x = \pm 2}$

$$= \int_{x=0}^{2}\int_{y=0}^{x}\int_{z=0}^{4-x^2} x^2 y z \, dz \, dy \, dx$$

$$= \int_0^2\int_0^x \left[x^2 y \frac{z^2}{2} \right]_0^{4-x^2} dy \, dx$$

$$= \int_0^2\int_0^x x^2 y \left(\frac{4-x^2}{2} \right) dy \, dx$$

$$= \int_0^2 x^2 \frac{y^2}{2} \left(\frac{4-x^2}{2} \right) y \Big|_0^x dx$$

$$= \frac{1}{4} \int_0^2 x^4 (4-x^2)^2 \, dx$$

$$= \frac{1}{4} \left[\frac{x^5}{5} (-x^4 - 8x^2 + 16) \right]_0^2$$

$$= \frac{1024}{315}$$

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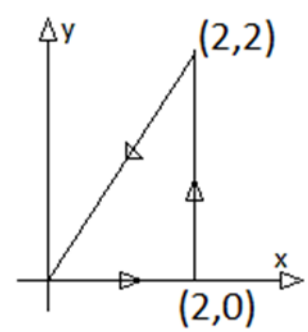
b) Evaluate $\oint_c x^2 y \, dx + x \, dy$, *where c is triangular* shown in adjacent figure using green's theorem.



Q4b)

$$\oint x^2y\,dx + x\,dy$$

Since $f(x,y) = x^2y$

$g(x,y) = x$

So

$$= \int_C x^2y\,dx + x\,dy = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x^2y)\right] dA$$

$$= \int_0^2\int_0^{2x} (1-x^2)\,dy\,dx$$

$$= \int_0^2 y - x^2y \Big|_0^{2x} dx$$

$$= \int_0^2 (2x - x^2(2x)) - [0 - x^2(0)]\,dx$$

$$= \int_0^2 2x - 2x^3\,dx$$

$$= \left[\frac{2x^2}{2} - \frac{2x^4}{4}\right]_0^2$$

$$= \boxed{-4}$$

Q5)

a) Use potential function to evaluate the following.

$$\int_{(\frac{\pi}{2},0)}^{\left(\frac{\pi}{2},\,\frac{\pi}{2}\right)} e^x \,\mathrm{Sin}(y)\,dx + e^x Cos(y) dy$$

Q 5(b)

$F(x,y,z) = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$

$\Phi = \iint_{\sigma} \vec{F}\cdot\vec{n}\,ds = \iiint_{G} \vec{\nabla}\cdot\vec{F}\,dV$

where

$G: \{(x,y,z) \mid -\sqrt{2} \le x \le \sqrt{2},\ -\sqrt{2-x^2} \le y \le \sqrt{2-x^2},$
$0 \le z \le 2\}$

USING CYLINDRICAL CORDINATES:

$G = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi,$
$0 \le r \le \sqrt{2},\ 0 \le z \le 2\}$

taking the value

$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$

$= 3(x^2+y^2+z^2)$

$= 3(r^2+z^2) \quad \because x^2+y^2=r^2$

The fluxs:

$$\oint = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_0^{2} 3(r^2+z^2)r \, dz\, dr\, d\theta$$

$$\oint = \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2}} \left[3r^3(2) + \frac{2r}{3}\right] - [0]\, dr\, d\theta$$

$$\oint = \int_{\theta=0}^{2\pi} \frac{4r^4}{4}\Big|_0^{\sqrt{2}} d\theta$$

$\oint = 8\pi$ flux

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b) Let σ be the surface of the solid enclosed by the circular cylinder $x^2+y^2=2$ and planes $z=0$ and $z=2$ oriented outward. Use the divergence theorem to find the flux of vector field F across σ

$$F(x,y,z)=x^3\,i+y^3\,j+z^3\,k$$

Question 5a

$$\int_{(\pi/2,0)}^{(\pi/2,\pi/2)} e^x \sin(y)\, dx + e^x \cos(y)$$

$\phi = ?$

$f = e^x \sin y$
$g = e^x \cos y$

$\frac{\partial f}{\partial y} = e^x \sin y \qquad , \qquad \frac{\partial g}{\partial x} = e^x \cos y$

$= \boxed{e^x \cos y} \qquad \boxed{e^x \cos y}$

$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = e^x \cos y$

Integral is independent as well as field is conservative.

$\frac{\partial \phi}{\partial x} = e^x \sin y$

$= e^x \sin y + K(y)$
$= e^x \cos y + K'(y)$

$\cancel{e^x \cos y} + K'(y) = \cancel{e^x \cos y}$

$\boxed{K(y) = K}$

$\phi = e^x \sin y + k(y)$

$\phi = e^x \sin y + k$

$$\int_{(\pi/2,0)}^{(\pi/2,\pi/2)} F\,dr = \phi(\pi/2,\pi/2) - \phi(\pi/2,0)$$

$= (e^{\pi/2} \sin \pi/2 + k) - (e^{\pi/2} \cdot \sin 0)$

$= e^{\pi/2}$