Major Exam

Graded

Student

Manshi Sagar

Total Points

10.5 / 30 pts

Question 1

Q1

2 / 2 pts

✔ **+ 2 pts** Correct

+ 1 pt Correct calculation of condition number

+ 1 pt Correct answer for when the condition number is large.

+ 0.5 pts Instead of saying condition number is large near 1, the answer says just for $x = 1$.

+ 0 pts Incorrect

Question 2

Q2

Resolved **2.5** / 7 pts

+ 2 pts Part (a) Correct

✔ **+ 1 pt** Part (a) partially correct.

+ 2 pts Part (b) correct

✔ **+ 1 pt** In part (b), Wrote $||(I-A)^{-1}|| = 1/||I-A||$, but assuming inequality, rest of the argument is correct.

+ 0.5 pts wrote $||I-A|| \leq 1 + ||A||$ in part (b)

+ 3 pts Part (c) correct

✔ **+ 0.5 pts** wrote $||I-A|| \geq 1 - ||A||$ in part (c)

+ 0 pts Incorrect

3 how does this imply invertible?

4 norm is not same as determinant. $||-A||$ is same as $||A||$.

5 ??

Regrade Request **Submitted on: May 06**

In part b, i have written that | |I-A| | <= 1 + | |A| | but i have not got marks for that rubric

You have written for only even n and that means you are interpreting it incorrectly.

Reviewed on: May 07

Question 3

Q3

2 / 3 pts

✔ **+ 3 pts** Correct

+ 0 pts Incorrect

+ 2 pts partially correct

✔ **− 1 pt** rate of convergence not mentioned or is wrong.

Question 4

Q4

0 / 4 pts

+ 4 pts Correct

+ 3 pts Showed that $e_{k+1} = (I - \alpha A)e_k$ but subsequent argument is missing/not formally correct.

✔ **+ 0 pts** Incorrect

+ 1.5 pts Correctly wrote the iterative step including expression for gradient.

Question 5

Q5

Resolved **1.5** / 3 pts

+ 3 pts Correct

+ 0 pts Incorrect

+ 0.5 pts Major mistake in y'' calculation.

+ 1.5 pts Only showed y'', y''' is incorrect.

+ 2 pts y'' correct but serious errors in y'''

✔ **+ 1.5 pts** did not substitute y' and y''

− 0.5 pts calculated y'' and y''' correctly but ODE iteration not shown.

Regrade Request **Submitted on: May 06**

-0.5 marks have been given with remark that = ODE iteration not showed.
But I have written the ODE:
yk+1 = yk +....
i did not substitute y' and y'' but marks for that have been deducted.
Please verify my answer.

You did not calculate the expressions for y' and y'' explicitly.

Reviewed on: May 07

Question 6

Q6

1 / 4 pts

- ✔ **+ 1 pt** part (i) Correct
- **+ 0.5 pts** minor error in part (i)
- **+ 0 pts** incorrect part (a)
- **+ 3 pts** Part (ii) correct
- ✔ **+ 0 pts** incorrect part (ii)
- **+ 1.5 pts** Part (ii), correctly figured out A^k times the vector.
- **– 0.5 pts** Concludes linear convergence or does not mention whether it is linear or not.

Question 7

Q7

1.5 / 7 pts

- **+ 3 pts** part (a) Correct
- **+ 1 pt** part(a) serious flaws, but some argument is ok.
- **+ 2 pts** Minor flaws in part(a)
- ✔ **+ 1 pt** part (b) correct
- **+ 0.5 pts** part (b) partially correct.
- **+ 3 pts** part (c) correct.
- ✔ **+ 0.5 pts** in part (c), mentions $AQ_k = Q_k H_k$, but rest of the argument is wrong.
- **+ 0 pts** Incorrect
- **– 0.5 pts** In part (c), missing case of $Q_k v = 0$
- **+ 1.5 pts** partly correct part (c)

1 Not a valid argument

2 incorrect argument.

COL 726 Major Exam

Name: MANSHI SAGAR   Entry Number: 2020CS50429

Unless otherwise specified, give justifications for your answers. Without a valid reasoning, you may not get any marks. All norms are 2-norms. There are 5 pages in this question paper.

1. **(2 marks)** When is the the function $f(x) = x - \sqrt{x^2 - 1}$ ill-conditioned? Assume $x \geq 1$.

Ill conditioned at x close to 1

~~$f(x)$ is well conditioned $\forall x \geq 1$~~

$$f'(x) = 1 - \frac{1}{2\sqrt{x^2-1}} \times 2x = 1 - \frac{x}{\sqrt{x^2-1}}$$

$$\text{Condition number} = \frac{x f'(x)}{f(x)} = \frac{x \times \left(1 - \frac{x}{\sqrt{x^2-1}}\right)}{x - \sqrt{x^2-1}} = \frac{\cancel{x}\left(1 - \frac{x}{\sqrt{x^2-1}}\right)}{\cancel{x}\left(1 - \frac{\sqrt{x^2-1}}{x}\right)}$$

$$= \frac{-x}{\sqrt{x^2-1}}$$ for x very close to 1, $\sqrt{x^2-1} \approx 0$

$\Rightarrow$ condition number is very large

2. Let A be an $n \times n$ matrix and I denote the $n \times n$ identity matrix. Assume that $||A|| < 1$.

(a) **(2 marks)** Show that $I - A$ is invertible.

$B = I - A$ is invertible if $|B| \neq 0$

$\|B\| = \|I - A\| \leq \|I\| + \|-A\| = 1 - (-1)^n \|A\|$ → $1 + \|A\|$ → positive; → $1 - \|A\|$ → positive

we know $\|A\| < 1 \Rightarrow 0 < 1 - \|A\|$

we want to prove $\|B\| > 0$ $\|I - A\| > 0$

(b) **(2 marks)** Show that $||(I-A)^{-1}|| \geq \frac{1}{1+||A||}$.

$$\boxed{\|I - A\|^{-1} = \frac{1}{\|I - A\|}}$$

using norm properties) we know $\|I - A\| \leq \|I\| + (-1)^n \|A\| = 1 + (-1)^n \|A\|$

$n =$ even $\Rightarrow \|I - A\| \leq 1 + \|A\|$

$$\Rightarrow \boxed{\frac{1}{1 + \|A\|} \leq \frac{1}{\|I - A\|}} = \|(I - A)^{-1}\|$$

~~$n =$ odd $1 + \|A\| \geq 1 - \|A\| \Rightarrow \frac{1}{1 - \|A\|} \geq \frac{1}{1 + \|A\|}$~~

[redacted]

(c) **(3 marks)** Show that $||(I-A)^{-1}|| \leq \frac{1}{1-||A||}$. $\Rightarrow$ $\frac{1}{||I-A||} \leq \frac{1}{1-||A||}$

$\Rightarrow$ $\boxed{1-||A|| \leq ||I-A||}$

we know that

$||I-A|| \leq ||I|| + ||-A||$

$= 1 + (-1)^n ||A||$

$n = \text{odd}$ $\quad ||I-A|| \leq 1-||A||$

3. **(3 marks)** What is the rate of convergence of Newton's method for minimizing the function $f(x) = x^{10}$?

$f(x) = x_{k+1} = x_k^{10}$

$s_k = -\frac{f'(x_k)}{f''(x_k)} = -\frac{10 \times x_k^9}{10 \times 9 \times x_k^8} = -\frac{x_k}{9}$

$x_{k+1} = x_k + s_k = x_k - \frac{x_k}{9} = \frac{8 x_k}{9}$

convergence rate $= \frac{x_{k+1}}{x_k} = \frac{8}{9}$

4. **(4 marks)** Suppose we run gradient descent iteration $x_{k+1} = x_k - \alpha \nabla f(x_k)$ to minimize $f(x) = \frac{1}{2}x^T Ax + b^T x$, where A is a symmetric positive definite $n \times n$ matrix and b is a vector of length n. Let the eigenvalues of A be $0 < \lambda_1 \leq \ldots \leq \lambda_n$. Show that if we choose $\alpha = \frac{1}{\lambda_n}$, then we obtain linear convergence, where the constant in the linear convergence may depend on the eigenvalues of A.

Convergence rate $= \frac{x_{k+1}}{x_k} = \frac{x_k - \alpha \nabla f(x_k)}{x_k}$

$\nabla f(x) = \frac{1}{2}Ax + b$

$= 1 - \frac{\alpha}{x_k}\left(\frac{1}{2}Ax + b\right)$

linear convergence $\Rightarrow$

5. **(3 marks)** Consider the following ODE: $y' = -2t^2y^2$. Write the method for solving this ODE using Taylor series such that it is third order accurate.

$$\boxed{y' = -2t^2y^2} = f$$

$$y'' = (-2t^2)(2yy') + (-4t)y^2 = \cancel{f_t + f_y \cdot f} = -4(t^2yy' + ty^2)$$

$$= \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial t} + \frac{\partial f}{\partial t} = f_y \cdot f + f_t$$

$$\cancel{y''' = (-4t)(2yy')}$$

$$y''' = -4(2tyy' + t^2(yy'' + y' \cdot y') + 2tyy' + y^2)$$

$$\boxed{y''' = -4(4tyy' + t^2yy'' + t^2y'^2 + y^2)}$$

let $y' = f$

$y'' = f_2$

$y''' = f_3$

then

$$\boxed{y_{k+1} = y_k + h \cdot f + \frac{h^2}{2} \cdot f_2 + \frac{h^3}{3!} \cdot f_3} \quad (1)$$

Use this expression in Taylor Series method

Third order accuracy is ensured by adding the term $\frac{h^3}{3!} \times f'''$ in the taylor series expansion

start with arbitrary y_0 and execute (1) till convergence

6. Let A be the matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where λ is a non-zero real number.

(i) **(1 mark)** Write closed form expression for $A^k \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for some positive integer k.
[No explanation necessary]

$$A^k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k\lambda^{k-1} \\ \lambda^k \end{pmatrix}$$

(ii) **(3 marks)** Suppose we run power iteration on A starting with a randomly chosen initial vector. Will it converge to an eigenvector of A? If so, will it converge linearly?

$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ → upper triangular ⇒ eigenvalues $= \lambda, \lambda$
$\lambda_1 = \lambda$
$\lambda_2 = \lambda$

λ is a repeated eigen value
→ it will have 2 eigen vectors say v_1, v_2
Power iteration will converge to a linear combination of v_1 and v_2 $\quad v = av_1 + bv_2$

In power iteration
$$|w_k - v| = \left|\frac{\lambda_2}{\lambda_1}\right|^k = \left|\frac{\lambda}{\lambda}\right|^k$$

then $|\lambda_k - \lambda| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right) = O\left(\left|\frac{\lambda}{\lambda}\right|^k\right)$ ⇒ linear convergence

0/1

Q7(a) continued

$q_1 = \frac{b}{|b|}$ $\langle q_1 v_1 \rangle = 0$ q_1 orthogonal to ~~q~~ v_1

$q_2 = Aq_1 - \langle (Aq_1) q_1 \rangle q_1$

$\Rightarrow \langle q_2 v_1 \rangle = \cancel{\langle (A q_1) q_1 \rangle} \cancel{e}$

$= \langle (Aq_1 - *q_1) v_1 \rangle$

$= 0$ $\Rightarrow$ q_2 orthogonal to v_1

similarly for all q_i

~~q_{i+1} = thes~~ s

$q_{i+1} = *q_1 + *q_2 + \cdots *q_i$

where each $q_1, q_2, \cdots q_i$ is orthogonal to v_1

$\Rightarrow$ q_{i+1} is orthogonal to v_1

$\Rightarrow v_1 \notin K_i$ for any i

A = real & symm

& real ⟹ $A = VDV^*$ (diagonalizable)

7. Let A be a **symmetric** $m \times m$ matrix and λ_1 be an eigenvalue of A. Let v_1 be an eigenvector of A corresponding to eigenvalue λ_1. Suppose we run Arnoldi iteration starting with a real vector b which is orthogonal to v_1.

(a) **(3 marks)** Show that v_1 does not belong to any of the Krylov spaces K_k.

$q_1 = \frac{b}{|b|}$ $\quad \langle b\, v_1 \rangle = \langle q_1, v_1 \rangle = 0$

Using Arnoldi iterations, we will keep adding a vector $q_i \perp (q_1, \dots q_{i-1})$ to Basis / ~~krylo~~

~~ku~~ ~~Kr~~ krylov space$_i$ = span $(q_1, q_2, \dots q_i)$

~~So~~ It is sufficient to show that all q_i are orthogonal to v_1

⟹ v_1 cannot be expressed as linear combination of q_i (as it has a component perpendicular to all q_i)

⟹ $v_1 \notin$ ~~Kk~~ K_i ∀ any i

(b) **(1 mark)** Show that the Arnoldi iteration must terminate before m steps, i.e., $K_k = K_{k+1}$ for some $k < m$.

From part (a) ⟹ we can not get more than $(m-1)$ ortho. vectors in krylov space as there will be no q_i along v_1 ⟹ Arnoldi iteration will converge at $\le (m-1)$ iterations ~~tota~~

⟹ $K_k = K_{k+1}$ for some $k < m$

(c) **(3 marks)** Suppose Arnoldi iteration terminates after k iterations and let H_k be the corresponding $k \times k$ Hessenberg matrix. Show that if λ_1 is a simple eigenvalue of A, then it is not an eigenvalue of H_k. An eigenvalue of a matrix is said to be simple if the subspace of eigenvectors corresponding to this eigenvalue is 1-dimensional.

$AQ_k = Q_k H_n$

λ_1 is a simple eigenvalue ⟹ only 1 eigenvector corresponding to λ_1, say v_1 $\quad Aq_1 = \lambda_1 v_1$

we know that $v_1 \notin$ span $\langle q_1\, q_2 - q_k \rangle$

⟹ $A\underline{v_1}$ $\quad A\cancel{q_1 = \lambda_1}$ $\quad \cancel{Aq_1} = \lambda_1 v_1$ does not hold

the equation $A[q_1 | q_2 | \dots q_k] = [q_1 | q_2 \dots q_k] H_n$

holds only when eigenvectors ∈ krylov space

eigen values of A = eigenvalues of H when $AQ_n = Q_n H_n$ holds

But $v_1 \in$ krylov space ⟹ eigen value of $A (\lambda_1) \neq$ eigenvalue of H_n