Minor 1

Graded

Student

Manshi Sagar

Total Points

10 / 30 pts

Question 1

Q1

5 / 5 pts

+ **5 pts** Correct

- ✔ **+ 1.5 pts** Correct calculation of the condition number

+ **0.5 pts** Argued that the condition number is finite but no mention of a constant upper bound

- ✔ **+ 1 pt** Correct argument for well-conditioning that the condition number is always small and bounded.

+ **0.5 pts** Mentioned that unstable at x = 0, but unsatisfactory reasoning.

- ✔ **+ 1.5 pts** Mentioned that unstable at x = 0 and correct explanation.

- ✔ **+ 1 pt** Correct expression for alternative way to compute

+ **0 pts** Incorrect/Unattempted

Question 2

Q2(i)

2 / 2 pts

- ✔ **+ 2 pts** Correct

+ **1 pt** Correct for the first iteration but not for the second one.

+ **0 pts** Incorrect.

- **+ 0.5 pts** Point adjustment

1 how?

Question 3

Q2(ii)

Resolved **1** / 3 pts

✔ **+ 1 pt** Correct Invariant

+ 2 pts Correct proof of the induction hypothesis.

+ 0 pts Incorrect

Regrade Request **Submitted on: Mar 31**

please go through the solution again

the invariant is correct because
"no row interchange happened at some jth iteration" means the same as that "the lower (n-j)x(n-j) matrix has diagonal entry larger than sum of absolute values of off-diagonal entries"

(follows from part i and the further proof of part ii)

regraded

Reviewed on: May 05

Question 4

Q3 (i)

Resolved **1** / 3 pts

+ 3 pts Correct

+ 2 pts Correct computation of vector v and correct formula for final vector, but calculation mistake in computing it

✔ **+ 1 pt** Correct computation of the vector v

+ 0 pts Incorrect

Regrade Request **Submitted on: Mar 31**

i missed a factor of 2 in the formula of reflector
it is a small mistake (I wrote it wrong in my 1-page exam notes)
all other calculation and concept is correct
please give some partial marking for the reflection part

We cannot give marks for applying the wrong formula.

Reviewed on: May 06

Question 5

Q3(ii)

0 / 3 pts

✔ **+ 0 pts** Incorrect

+ 3 pts Correct

+ 0.5 pts Calculation error in computing k_f, but right formula

+ 1 pt Correctly computed σ_1

+ 0.5 pts Calculation error in computing σ_1, but right idea

+ 1 pt Correctly computed σ_2

+ 0.5 pts Calculation error in computing σ_2, but right idea

Question 6

Q4

0 / 5 pts

+ 5 pts Correct

+ 2 pts Correct algorithm but explanation incomplete

+ 4 pts Correct but running time not justified.

✔ **+ 0 pts** Incorrect/Unattempted.

+ 1 pt Correct Direction

+ 5 pts Solution using Matrix Calculus

Question 7

Q5

0 / 5 pts

+ 5 pts Correct

+ 2 pts Correct explanation for infinite number of solutions

✔ **+ 0 pts** Incorrect/Unattempted

+ 1.5 pts Part b correct direction

+ 3 pts Part b

Question 8

Q6

1 / 4 pts

- **+ 4 pts** Correct
- **+ 0 pts** Incorrect
- **+ 3 pts** correct algorithm but no running time analysis
- **+ 2 pts** Idea ok but no clear description. No running time analysis.
- **+ 0.5 pts** Mentioned zeroing out using Householder but details are incorrect.
- **+ 3 pts** Idea ok, no clear description, running time analysis done.

+ 1 pt Point adjustment

2 Not at all clear what is going on here. No clear descritpion.

COL 726 Minor Exam

Name: MANSHI SAGAR Entry Number: 2020CS50429

Give justifications for your answers. Without a valid reasoning, you may not get any marks. There are 6 pages in this question paper.

1. **(5 marks)** Consider $f(x) = 1 - \sqrt{1+x}$ for $x \geq 0$. (a) Show that this function is well-conditioned for all non-negative values of x. (b) If the function is computed according to the formula above, for which values of x will the calculation be unstable? Why? (c) How will you compute this function such that the relative error is always $O(\varepsilon_{mach})$, where ε_{mach} denotes the machine precision (just mention the method, need not give any justification).

$f(x) = 1 - \sqrt{1+x}$

(a)

$$K_f(x) = \left|\frac{x \times f'(x)}{f(x)}\right| = \left|\left(\frac{x}{1-\sqrt{1+x}}\right) \times \left(\frac{-1}{2\sqrt{1+x}}\right)\right| \times \left(\frac{1+\sqrt{1+x}}{1+\sqrt{1+x}}\right)$$

$$= \left|\frac{\cancel{-x}(1+\sqrt{1+x})}{(2\sqrt{1+x})(\cancel{-x})}\right| = \left|\frac{1+\sqrt{1+x}}{2\sqrt{1+x}}\right|$$

for $x \geq 1$ $K_f(x)$ is finite and small (close to $\frac{1}{2}$)

~~for x~~ for $0 \leq x < 1$, specially for x close to 0, $K_f(x)$ is close to 1

$\Rightarrow$ well conditioned for all $x \geq 0$

(b) y = value computed by $f = \left(1 - \sqrt{(1+x)(1+\varepsilon_1)}\,(1+\varepsilon_2)\right)(1+\varepsilon_3)$

ε_1 due to error in addition $(1+x)$ $\quad |\varepsilon_i| \leq \varepsilon_{mach}$

ε_2 due to error in square root

ε_3 due to error in subtraction

$$\frac{y - f(x)}{f(x)} = \frac{(\cancel{1}+\varepsilon_3) - \sqrt{1+x}\left(1+\frac{\varepsilon_1}{2}\right)(1+\varepsilon_2)(1+\varepsilon_3) - \cancel{1} + \sqrt{1+x}}{1-\sqrt{1+x}}$$

$$= \frac{\varepsilon_3 - \sqrt{1+x}\,(O(\varepsilon_{mach}))}{1-\sqrt{1+x}} = \left(\frac{\varepsilon_3}{1-\sqrt{1+x}}\right) - \frac{\sqrt{1+x}\,O(\varepsilon_{mach})}{1-\sqrt{1+x}}$$

For x close to 0, $1-\sqrt{1+x}$ is close to 0

$\Rightarrow x \lll 1$ then $1-\sqrt{1+x} \lll 0$ so the term $\frac{\varepsilon_3}{1-\sqrt{1+x}}$ can be very large

$\frac{y - f(x)}{f(x)}$ can be large for $0 \le x << 1$, So algorithm is unstable

also, from the expression of y, we can't find $\hat{x}$ st

$$\left|\frac{\hat{x} - x}{x}\right| = O(\epsilon_{mach})$$

(c) To make this stable

$$f(x) = (1 - \sqrt{1+x}) \times \left(\frac{1 + \sqrt{1+x}}{1 + \sqrt{1+x}}\right) = \frac{-x}{1 + \sqrt{1+x}}$$

If we compute $f(x)$ using this algorithm, it will be stable because the cancellation error is not seen here,

as in $(1 - \sqrt{1+x})$ when $x \simeq 0$

2. Suppose you are given an $n \times n$ matrix A such that the absolute value of every diagonal entry is larger than the sum of the absolute values of the non-diagonal entries in that column. In other words, for every column j, $|A_{jj}| > \sum_{i,\, i \neq j} |A_{ij}|$. We now perform Gaussian elimination with partial pivoting on A. The method has n iterations where in each iteration i, a suitable multiple of row i is subtracted from rows $i+1, \ldots, n$.

(i) **(2 marks)** Show that no row interchange will be done in the first and the second iterations of the method.

$$A = \begin{bmatrix} a_{11} & \cdots \\ a_{21} & \cdots \\ a_{31} \\ \vdots \end{bmatrix} \qquad |a_{11}| > |a_{21}| + |a_{31}| - - + |a_{n1}|$$

$$\Rightarrow |a_{11}| > |a_{i1}| \quad \forall i \neq 1$$

so, there is no need to row interchange at first iterations as the max abs value is already at a_{11}

$$L_1 = \begin{bmatrix} 1 \\ -a_{21}/a_{11} & 1 \\ -a_{31}/a_{11} & & 1 \\ \vdots & & & 1 \end{bmatrix} \qquad L_1 = \begin{bmatrix} 1 \\ \$ & 1 \\ \$ & & 1 \\ & & & 1 \end{bmatrix}$$

now for second iteration

$$A' = \begin{bmatrix} * & a_{12} \\ 0 & a'_{22} \\ 0 & a'_{32} \\ 0 & \vdots \end{bmatrix} \qquad a_{22}' = -\frac{a_{21}}{a_{11}} \cdot a_{12} + a_{22}$$

$$a_{32}' = -\frac{a_{31}}{a_{11}} a_{12} + a_{32}$$

$$a_{i2}' = -\frac{a_{i1}}{a_{11}} \cdot a_{12} + a_{i2}$$

here also, $a_{22}' > a_{i2}'$ $\forall i > 2$

so, no need to ~~swap~~ do row inter change

(iii) **(3 marks)** Show that no row interchange happens during any iteration of the algorithm (the proof should clearly state the induction hypothesis or the invariant condition).

Base case : No row interchange needed at col = 1

Induction : after $(j-1)$ iterations, (col=1 to j)

$$a_{jj}' = a_{jj} - \sum_{i=1}^{j-1} \frac{a_{ji} \cdot a_{ij}}{a_{ii}}$$

no row interchange was needed for $(j-1)^{th}$ iter.

$A_{j-1} = (L_{j-1} \cdots L_2 L_1 A) \rightarrow$ in A_{j-1}, $|a_{jj}| > \sum_{i \neq j} |a_{ij}|$

then for $A_j = L_j \cdot A_{j-1}$

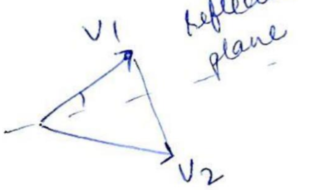
is $|a'_{jj}|$ max abs value in the column

if yes, then ~~by in~~ we don't need to do row interchange at j^{th} iteration

then by induction we don't need to do that for any iteration

3. Answer the following questions:

(i) **(3 marks)** Suppose there is a Housuholder transformation H (i.e., reflection operation) that maps the vector $\underset{V_1}{\begin{pmatrix}2\\1\\2\end{pmatrix}}$ to the vector $\underset{V_2}{\begin{pmatrix}3\\0\\0\end{pmatrix}}$. Where will the vector $\begin{pmatrix}2\\1\\-2\end{pmatrix}$ get mapped to by this operation H ?



normal of the reflection plane $= V_1 - V_2 = \begin{pmatrix}-1\\1\\2\end{pmatrix} = V_3$

$$Q_1 = I - \frac{V_3 V_3^T}{|V_3|^2} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} - \begin{bmatrix}1/6&-1/6&-2/6\\-1/6&1/6&2/6\\-2/6&2/6&4/6\end{bmatrix}$$

$$\text{Reflection of} \begin{pmatrix}2\\1\\-2\end{pmatrix} = \cancel{\begin{pmatrix}2\\1\\-2\end{pmatrix}}\left(I - \frac{V_3V_3^T}{6}\right)\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}2\\1\\-2\end{pmatrix} - \left(\frac{V_3V_3^T}{6}\right)\begin{pmatrix}2\\1\\-2\end{pmatrix}$$

$$= \begin{pmatrix}7/6\\-12/6\\-2/6\end{pmatrix}$$

(ii) **(3 marks)** Let A be a 3×2 matrix which maps the vector $\overset{x_1}{\begin{pmatrix}1\\2\end{pmatrix}}$ to $\overset{b_1}{\begin{pmatrix}4\\1\\2\end{pmatrix}}$ and the vector $\overset{x_2}{\begin{pmatrix}4\\-2\end{pmatrix}}$ to $\overset{b_2}{\begin{pmatrix}1\\-2\\-1\end{pmatrix}}$. What is the condition number of A ?

$Ax = b$

$$k(A) = \frac{|\Delta b|/|b|}{|\Delta x|/|x|}$$

$\Delta x = x_2 - x_1 = \begin{pmatrix}3\\-4\end{pmatrix}$

$\Delta b =$

$x_1 x_2 = 0$

$Ax_1 = b_1$ $\quad \cancel{(Ax_1)}\, x_2$

$Ax_2 = b_2$

$m\times n \quad n\times n$

$A = QR$

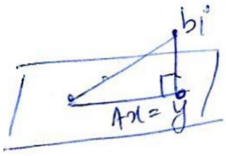
4. **(5 marks)** Suppose you are given the reduced QR factorization of an $m \times n$ full rank matrix A, where $m > n$. Given vectors $b_1, b_2, \ldots, b_r \in \Re^m$, show how to find a vector x that minimizes

$=$rank

$$|Ax - b_1|^2 + |Ax - b_2|^2 + \ldots + |Ax - b_r|^2 \to \text{min if all } |Ax-b_i|^2 \text{ are min}$$

Your algorithm should make $O(m(n + r))$ floating point operations.

for b_i $|Ax-b_i|^2$ is min if $(Ax-b_i) \perp \text{Range}(A)$



least squares problem

$\cancel{A^+ = (A^TA)^{-1}A^T}$

we have Q, R,

$\boxed{Rx = Q^T b_i}$ $\quad x = R^{-1} Q^T b_i$

① solve for y $\quad Q^T b_i = y$ $\quad Q$ is unitary

$\cancel{m\times n}$ $m\times m$ $\quad m\times 1$

$O(m)$

this will take $\cancel{(m\times n)}$ operations

② solve for $\cancel{y}$ x

$Rx = y$

$[\nabla][\,]$

$n\times n \quad n\times 1$

this will take $O(n^2)$ operations

$\cancel{m>n}$

so

$m>n \Rightarrow m.n > n.\cancel{n}$

so, ① and ② will take $O(m.n)$ operations

after getting x, $\|Ax-b\|_2^2$

$\nearrow m$ $\qquad\qquad$ $m < n$

5. **(5 marks)** Suppose you are given a full-rank $m \times n$ matrix A where m is **smaller** than n, i.e., $m < n$. Given a vector $b \in \Re^m$, we would like to find a solution x to $Ax = b$ such that $|x|$ is minimized. Show that there are infinitely many solutions to this system of equations. Suppose you are given the reduced SVD of A. Show how you can find the desired solution x in $O(mn)$ time.

$$A = U \Sigma V^T = \hat{U} \hat{\Sigma} V^T \qquad \underset{m\times n}{A} = \underset{m \times n}{\left[\hat{U}\right]} \underset{n\times n}{\left[\hat{\Sigma}\right]} \underset{n\times n}{\left[V^T\right]}$$

$$\underset{m\times n}{[\quad]}\underset{\Sigma}{[\quad]} \qquad \underset{m\times n}{\left[\; A \;\right]} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

we have m equations and n variables $\Rightarrow$ there will be infinity many solutions for $Ax = b$

$m < n$

(we need at least n equations to find $x_1, x_2, \ldots x_n$)

let us consider 1 norm

$$|x|_1 = |x_1| + |x_2| + \ldots + |x_n|$$

$\cancel{|Ax| = |b|}$

$\cancel{|A||x| = |b|}$

$|x|$ will be minimum when

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ \vdots \\ x_n \end{bmatrix}$$

$|x_1|, \ldots |x_m|$ are the m smallest values of $|x_i|$

~~x_m~~

x_{m+1} to x_n will be multiplied by 0

(there are m singular values $\{\sigma_i\}_{i=1}^m$)

To ~~sto~~ To solve

① $\hat{\Sigma} w = \hat{U}^T b$ solve for w

② $x = Vw$ solve for x

step ① will take $O(m \cdot n)$ time

step ② will take $O(n)$ time

$\rightarrow$ total time $= O(m \cdot n)$

6. **(4 marks)** Let A be an $m \times n$ matrix which is of the form $\begin{pmatrix} U \\ B \end{pmatrix}$, where U is an $n \times n$ **upper triangular** matrix and B is an arbitrary $k \times n$ matrix where $k = m - n$. Assume that $m \geq n$ but $k = m - n$ is much smaller than n. Give an $O(kn^2)$ time algorithm to compute the reduced QR factorization of A.

$$\underset{m \times n}{A} = \underset{m \times n}{\begin{bmatrix} \boxed{U} \\ \boxed{B} \end{bmatrix}} \quad \begin{matrix} n \times n \\ \\ (m-n) \times (n) \end{matrix}$$

$m > n$

$m - n < n$

$m < 2n$

$n > \frac{m}{2}$

extend A to A' $(m \times m)$

$$\underset{A'}{\begin{bmatrix} \boxed{U} & \\ \boxed{B} & \boxed{} \end{bmatrix}}$$

$m-n$, $m-n$

for QR factorisation, we want to premultiply A' by Q_i st

$\prod Q_i A' = [\nabla]$ is upper Δ

we only need $(a_1$ to $a_{m-n})$ to

convert $B \to [0]$ zero matrix

~~and~~ w/o disturbing U

For $j = 1$ to $m - n$

convert j^{th} column

$$\begin{bmatrix} * \\ * \\ * \\ * \\ * \end{bmatrix} \Big\} n \quad \text{to} \quad \begin{bmatrix} * \\ * \\ * \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \} n \\ \} n-m \end{matrix}$$

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \\ a_{n+1,j} \\ \vdots \\ a_{m,i} \end{bmatrix}$$

$a_{n+1,i} \to$

~~$a_{n+1,i} - \frac{a_{n+1,i}}{a_{ii}} \times$~~

~~for~~ $i =$

$$a_{n+1,j} \leftarrow a_{n+1,i} - \frac{a_{ii}\, a_{n+1,i}}{a_{ii}}$$