Quiz 2

● Graded

Student

Manshi Sagar

Total Points

9 / 15 pts

Question 1

Q1

5 / 5 pts

✔ **+ 5 pts** Correct

+ 2 pts first part correct.

+ 0 pts Incorrect.

+ 1.5 pts only one out of 2ns and 3rd parts correct.

+ 3 pts Last two parts correct.

Question 2

Q2

1 / 5 pts

+ 5 pts Correct

+ 2 pts first part correct

+ 0 pts Incorrect.

✔ **+ 1 pt** First part partly correct.

+ 1 pt In second part, mentions that $H_{i,i} = q_i^* A q_i$.

+ 3 pts Second part correct.

1. But we can have diagonal entries with imaginary parts.
2. No. This is not correct.

Question 3

Q3

3 / 5 pts

+ **5 pts** Correct

+ **0 pts** Incorrect.

+ **1 pt** Correct calculation of eigenvalues

✔ + **2 pts** Correct description of Krylov Space

✔ + **1 pt** Correct description of H_k matrix

+ **1 pt** Correct description of eigenvalues of H_k

+ **0.5 pts** Computed eignevalues of the 5×5 matrix only.

+ **1 pt** Correct description of Krylov space but no reasoning

Quiz 2 (COL 726)

Name Manshi Sagar

Entry No. 2020CS50429

$A: \lambda_1, \lambda_2, \lambda_3, \lambda_4$

1. (**5 marks**) Let A be a real symmetric 4×4 matrix with eigenvalues $5, 3, 1, -5$. We run inverse iteration with shifts (i.e., estimates) μ as follows: (i) $\mu = 0$, (ii) $\mu = 2$, (iii) $\mu = 4$. In each of the cases, mention whether the algorithm converges to an eigenvalue, and if so, to which eigenvalue (assume that we start with a random initial vector). In which case will the convergence be fastest ? [No reasoning required for any part, just mention the answers].

$\frac{1}{|\lambda_i - \mu|}$ = eigenvalues of Inverse iteration

~~(i) $\mu = 0$~~
~~|eigen values| = $\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4$: 5, 5, 3, 1~~ ~~does not converge to eigen value (due to repetition) may converge to linear combination of v_1, v_2 (depend on initial vector)~~

(ii) $\mu = 2$
|eigen values| = $\frac{1}{7}, \frac{1}{3}, \frac{1}{1}, \frac{1}{1}$ ~~converge to 1 con time $\propto \frac{3}{7}$~~
~~does~~ does not converge (due to repetition)

(iii) $\mu = 4$
|eigen values| = $\frac{1}{9}, \frac{1}{3}, 1, 1$ does not converge

(i) $\mu = 0$
|eigen values| = $\frac{1}{5}, \frac{1}{5}, \frac{1}{3}, \frac{1}{1}$ converges to 1 (largest)
convergence time $\propto \left(\frac{1}{3}\right)$

convergence fastest when $\mu = 0$

$\lambda = 1$ in A

2. (**5 marks**) Let A be a real skew-symmetric matrix (i.e., $A = -A^T$). Show that all eigenvalues of A are purely imaginary (this includes 0 as well). Suppose we run Arnoldi iteration on this matrix. Recall that if $q_1, q_2, \ldots$ are the orthonormal vectors generated by this process (started with a real vector), then we express: $Aq_i = \sum_{j=1}^{i+1} H_{j,i} q_j$. Show that $H_{i,i} = 0$ for all i.

A is real $A^T = A^*$ $\quad A = -A^T = -A^*$ $\quad AA^* = AA^T$ $\quad A^*A = A^TA$ $\Rightarrow$ A is normal
$\Rightarrow A = VDV^*$
D is diagonal

$A = VDV^*$

& $A^T = A^* = \cancel{V^T D V^{*T}}$ $(VDV^*)^* = V^{**} D V^* = VDV^*$

~~$VDV^* = -V^T D V^*$~~

Now $A = -A^T \Rightarrow VDV^* = -(VDV^*) \Rightarrow \boxed{D = -D} \Rightarrow$ all diagonal entries are zero

~~Eа~~ Eigenvalues of A = diagonal entries of D = zero

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we know that eigen values of A = eigen values of H

we know that all ~~diagonal~~ eigen values of A = 0

$\Rightarrow$ we need to show that H is diagonal and H_{ii} = eigenvalues of H = 0

$A = -A^T$

$Aq_i = (A^T) q_i = h_{1i} q_1 + h_{2i} q_2 \cdots + h_{i+1\,i}\, q_{i+1}$

3. (**5 marks**) What are the eigenvalues of the $m \times m$ matrix A where the only non-zero entries are $A_{i+1,i}, i = 1, \ldots, m-1$ and $A_{1,m}$, and all these entries are 1 [No reasoning needed]. The matrix A for $m = 5$ is shown below:

eigen values of A : $m \times m$ $= 1$ (all eigen values $= 1$)

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$A \cdot P$ permute, eigen values don't change

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Suppose we run the Arnoldi iteration on this matrix starting with vector $b = e_1$, i.e., $(1, 0, 0, \ldots, 0)$, for $k < m$ iterations. What will be the Krylov space? What will be the estimated eigenvalues generated by this algorithm? Give reasons.

Krylov space $= \langle b, Ab, \ldots \; A^{n-1} b \rangle$

$q_1 = \dfrac{b}{\|b\|}$

$b = e_1$

$$Ab = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} = e_2$$

$$A(Ab) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}}_{e_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = e_3$$

$$\Rightarrow \boxed{\begin{array}{l} A^n b = e_{n+1} \\ \text{Krylov space} = \langle e_1, e_2, \ldots e_n \rangle \end{array}} \text{ after } n \text{ iterations}$$

$$\tilde{H}_n = \begin{bmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & \ddots & \\ \vdots & \vdots & 0 & \ddots & 1 \\ 0 & 0 & 0 & 0 & 0 \; 0 \end{bmatrix}$$

$(n+1) \times n$

H_n ~~this~~ is a diagonal matrix

$\Rightarrow$ ~~esti~~ ~~of~~ Ritz values = estimations of eigenvalues of A

= ~~eigen~~ eigen values of H_n

$= (1, 1, \ldots)$

$\Rightarrow$ The estimation is correct

$$H_{ij} = \cancel{(Aq_i)^* q_j} \; (A^i q_1) q_j^* = \cancel{(e_i)^* e_j} = e_i e_j^* = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$