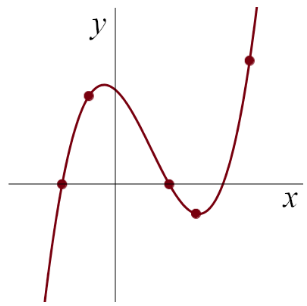
MATH 1030 – Application Assignment 1
Polynomial Interpolation

Due: Thursday, May 19, 2022 at 11:59pm Atlantic time (submit through Brightspace)
You must show your work for full marks.

Students are welcome to collaborate with their peers for assignments, but each student **must** submit their own work. This means you can discuss questions with your peers, but to ensure the work is your own, you should later sit down alone and write out your individual solution. The work you submit must be your own, and not a collaborative effort.

The goal of this assignment is to find functions passing through a number of given points.



Suppose we want to find an equation of the form $y = ax + b$ for the line that goes through the points $(x_1, y_1) = (2, 5)$ and $(x_2, y_2) = (4, -1)$. We could use previous knowledge we might have that the slope of the line will satisfy $a = \frac{y_2 - y_1}{x_2 - x_1}$ (which is a perfectly acceptable way to solve this problem), but since this is a linear algebra class, let's try to set up this problem as a system of linear equations. Substituting our points into our equation, we get

$$\begin{aligned} y_1 &= ax_1 + b & \Longrightarrow && 5 &= 2a + b \\ y_2 &= ax_2 + b & \Longrightarrow && -1 &= 4a + b \end{aligned}$$

This gives us a system of linear equations in the variables a and b, which we can solve using whichever method we'd like (you can try this on your own). Solving this system, we get $a = -3$ and $b = 11$, so the equation of our line is $y = -3x + 11$.

1. Consider the three points $(x_1, y_1) = (0, 3)$, $(x_2, y_2) = (1, -1)$ and $(x_3, y_3) = (2, 4)$. If you plot these points, you will see that they do not fit on a straight line. We will try to find the equation of a quadratic polynomial (i.e., degree 2 polynomial) going through the points. We want to find the coefficients in the polynomial

$$y = ax^2 + bx + c.$$

   (a) Substitute the three known points into this equation to find the corresponding system of linear equations.

   (b) Solve this system using the method of your choice, and write out the resulting polynomial.

2. In the introductory example we saw that two points were used to determine the coefficients of a line, and in Question 1 we used three points to determine the coefficients of a quadratic. As you may have guessed, there is a relationship between the number of points and the degree of the polynomial that passes through those points. Consider a general polynomial

$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

of degree n, where $a_n \neq 0$. You have a list of points that this polynomial passes through, but you do not know the coefficients of the polynomial. What is the minimum number of points you need to use in order to find the equation for $p(x)$? Justify your answer by explaining your reasoning (your reasoning should include linear algebra!).

3. (a) Consider the three points $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (2, 2)$ and $(x_3, y_3) = (3, -6)$. Use an augmented matrix to find the quadratic polynomial $p(x)$ that goes through these three points.

   (b) Keep the first two points the same, but now instead consider the third point to be $(x_3', y_3') = (-3, 6)$ (so our three points are $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (2, 2)$ and $(x_3', y_3') = (-3, 6)$). Use an augmented matrix to find the quadratic polynomial $p(x)$ that goes through these three points.

   (c) Use graphing software (such as Desmos) to graph the two polynomials you found. Sketch or include an image of your resulting graph in your submission, labelling the particular points used to find these two polynomials. Notice that though our polynomial interpolations used two out of three of the same points, our final polynomials look quite different!

   (d) Write out, but do not solve, the augmented matrix for the system of equations that will result in the coefficients of the polynomial of that goes through all four points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, and (x_3', y_3').