PROJECT REPORT

MONTE CARLO SIMULATION

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PART-B : Monte Carlo

Abstract: This report develops a Monte Carlo framework to price floating and fixed strike lookback options using Bitcoin. It applies path simulation, variance reduction, and sensitivity analysis to highlight how exotic options respond to volatility, time, and rate changes—offering a practical approach to advanced option pricing.

INTRODUCTION

This project extends by introducing a simulation-based approach for pricing exotic derivatives. Using Bitcoin as the underlying, a Monte Carlo framework was developed to price floating and fixed strike lookback call options—options whose payoffs depend on the minimum or maximum asset price observed over the life of the contract. The simulation incorporates key financial modeling techniques such as geometric Brownian motion (GBM) for path generation, antithetic variates for variance reduction, and numerical methods for calculating option Greeks. Control variate adjustments further improve result stability by leveraging the known price of a related geometric average option. The framework is built to assess how core market parameters—such as interest rate, volatility, time to maturity, and spot price—impact both the option's value and its sensitivities. The model provides a foundation for pricing path-dependent options where analytical solutions are either unavailable or unreliable.

Core Components

The Monte Carlo simulation framework developed for pricing two exotic options—Floating Strike Lookback Call and Fixed Strike Lookback Call—based on the asset Bitcoin (BTC-USD). These paths represent different potential evolutions of the asset’s price over the option's life, under realistic market conditions.

Simulation framework

The underlying asset (Bitcoin) follows a GBM process

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t$$

where:

- St: asset price at time t
- r: risk-free interest rate (set at 5%)
- q: dividend yield (set to 0%)
- σ\sigma: annual volatility (informed by earlier correlation analysis)
- dW_t: standard Brownian motion increment

Simulation uses the following discretization:

- T = 1.0 year (time to maturity)
- M = 252 steps (one step per trading day)
- 10,000 paths (for stability), and
- dt = T / M = 1/252
- For each time step:

$$S_{t+\Delta t} = S_t \cdot \exp\left((r - q - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} \cdot Z_t\right) \quad \text{where } Z_t \sim \mathcal{N}(0, 1)$$

Variance Reduction: Antithetic Variates

To improve the stability of the results and reduce the number of simulations needed, antithetic variates was used:

- Generates half the paths using standard normal random numbers Z, and the other half using their negatives −Z.
- This ensures that for every high-path, there's a symmetric low-path, which reduces variance without biasing the mean.

Exotic option Payoffs: Definitions

The two types of lookback options used in this project were Floating strike and Fixed strike. These are both path-dependent exotic options, meaning their payoffs depend not only on the final price but also on the entire trajectory of the asset over time, Hence the name “lookback” , meaning we can lookback over the options life to determine price and payoffs, making them more expensive than standard vanilla options.
The lookback options can be of both call and put type but for simplicity in this project call options has been implemented for both the options.
The payoff of both lookback options is compared to zero to ensure the holder receives a non-negative payout, aligning with the fundamental principle that options are not exercised at a loss.

- **Floating Strike Lookback Call Option**

  Payoff Definition:

$$\text{Payoff} = \max(S_T - \min(S_t), 0)$$

  Where,
  S_T : Final asset price
  min(St): Lowest price reached during the option's life

  - This option gives the right to buy the asset at its lowest observed price. It benefits from price surges—particularly when the asset rebounds strongly after a period of decline.

- **Fixed Strike Lookback Call Option**

  Payoff Definition:

$$\text{Payoff} = \max(\max(S_t) - K, 0)$$

  Where,
  max(S_t) : Maximum asset price during the option's life
  K: Fixed strike price, usually set to current spot price

  - This option allows the holder to capture the highest price reached by the asset during the life of the option and buy at a fixed strike. It's particularly valuable in volatile or trending markets.

Variance Reduction: Control Variate Method

Monte Carlo simulations are powerful but often noisy—the results can have high variance unless you use a very large number of simulations. To improve accuracy without increasing computational cost, The implementation smartly applies the control variate method.
A control variate is like a smart shortcut to reduce this noise.
We do this by:
Picking a related variable that:
  - behaves similarly to your target output (highly correlated),
  - but whose true value is already known.

In this case, were pricing a floating-strike lookback call option – something that depends on the minimum value of the path. So the control variate is geometric average of the asset path, which also depends on the whole path not just the endpoint. So:
- It's closely related to the behavior of a lookback option.
- It tends to rise and fall with the same movements.

And most importantly:
- We know how to calculate the exact (closed-form) price of a geometric average option under the Black-Scholes model – no simulation needed.

The idea is to "anchor" the simulation using this related variable to reduce fluctuation in the result. By adjusting the raw Monte Carlo payoff using a related known-value variable, we are essentially subtracting noise that is shared between them.

- **Application and its results:**

Metric	Standard MC	Control Variate
Floating Price	17,292.12	17,292.12
Error Estimate (±)	**160.0750**	**119.7116**
95% Confidence Interval	Wider **(16978.3762, 17605.8701)**	Tighter **(17057.4884, 17526.7579)**

(Example for Floating strike call price)

- Significant improvements that we can see is error reduction and tightening of the 95% confidence interval improving accuracy of the final value.

- **Mathematical calculations:**

  - Adjusted drift term for pricing the geometric average option

$$\mu_{\text{hat}} = \frac{1}{2}(r - q - 0.5\sigma^2) + \frac{1}{2} \cdot \sigma^2_{\text{hat}}$$

  - Calculates the standard Black-Scholes d1 and d2 terms for the geometric average option.

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

  - Closed-form price of a geometric average call option**(control variate)**

$$\text{Price}_{geo} = e^{-rT}\left[S_0 e^{\mu_{\text{hat}} T} N(d_1) - K N(d_2)\right]$$

Option pricing

- Option pricing is done by Risk-Neutral Monte Carlo Pricing Formula (Standard discounting)

$$\text{Option Price} = e^{-rT} \cdot \mathbb{E}\left[\max\left(S(T) - \min(S), 0\right)\right]$$

Error estimation and Confidence intervals

Monte Carlo simulations provide estimates that are inherently random. To evaluate the reliability of your pricing results, it’s critical to compute the standard error and construct confidence intervals.

- **Standard error calculation** (Point estimation)

$$\hat{V} = e^{-rT} \cdot \frac{1}{N} \sum_{i=1}^{N} \text{payoff}_i$$

To evaluate how accurate this estimate is, we calculate the standard error (SE):

$$\mathrm{SE} = \frac{\text{standard deviation of payoffs}}{\sqrt{N}}$$

Error falls with more simulations (as 1/sqrt[N]), but slowly. This is why variance reduction is powerful—it gives you the same accuracy for lower N.

- **95% Confidence interval**

  To make results interpretable, we construct a 95% confidence interval using the standard normal Z-value z = 1.96

$$\mathrm{CI} = \left[\hat{V} - z \cdot \mathrm{SE},\ \hat{V} + z \cdot \mathrm{SE}\right]$$

  As the name suggests it gives us the interval in which the final value lies with 95% probability. Therefore smaller the interval greater then accuracy

Convergence analysis

Monte Carlo simulations depend heavily on the number of iterations (N). The higher the number of simulations, the closer the estimate gets to the true theoretical price. However, increasing N comes with computational cost, so it's crucial to evaluate how fast the simulation converges.

The goal is to:
- Track how price estimates stabilize as N increases
- Observe how standard error reduces
- Decide the minimum number of simulations needed for reliable pricing

The simulation range used in the implementation is-

[1000, 2000, 5000, 10000, 15000, 20000]

The results derived from this analysis help us determine around how many simulations start bringing us closer to the result, explained in detail in the results section.

Parameters used

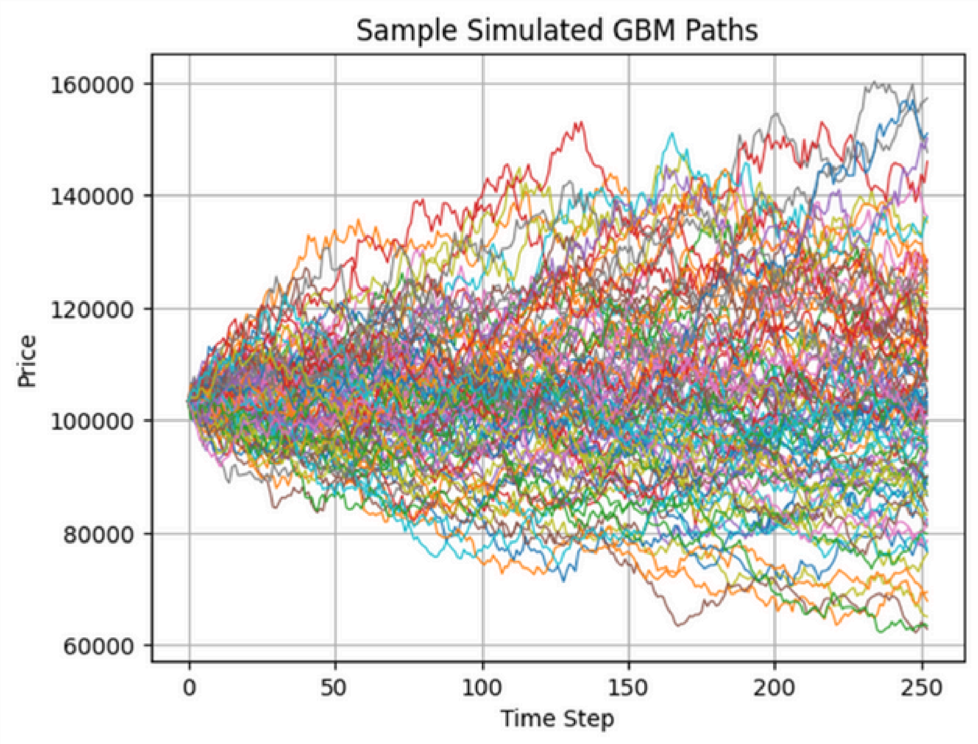
r = 0.05	#Standard risk free rate of 5%
T = 1.0	#Time to maturity = 1 yr
M = 252	#Time steps = 252
N = 10000	#Simulation steps
dividend_yield = 0.0	#Dividend yield 0 for bitcoin
z = 1.96	#Normal distribution Z score

The Spot price for bitcoin is determined by taking real latest values and its volatility is determined by the following formula for a period of 1 yr

$$\sigma_{\text{annual}} = \sqrt{\frac{252}{n-1}\sum_{t=1}^{n}\left(\ln\left(\frac{P_t}{P_{t-1}}\right) - \bar{r}\right)^2}$$

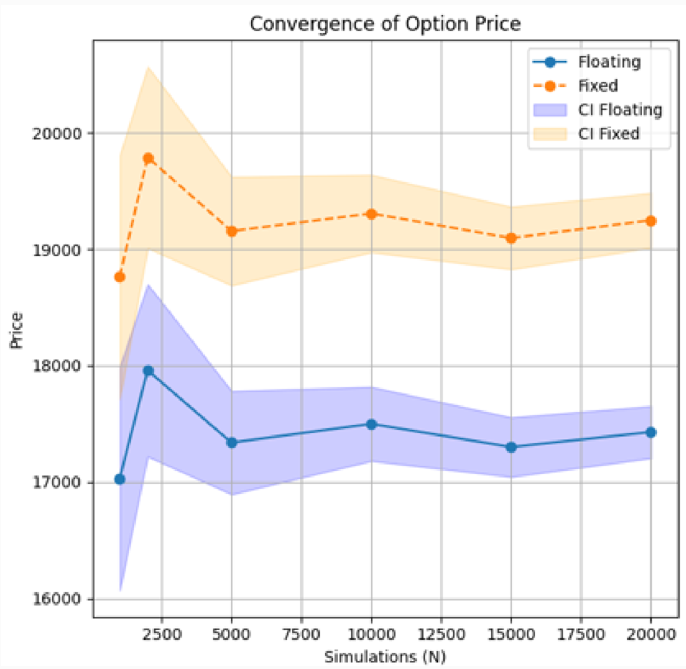
Results Summary

Sample simulated GBM paths



- 100 sample paths were plotted to visualise how these paths are constructed and how they diversify. The paths start at the same point (S_0= 102768.2) but quickly diverge. Some rise gradually, others fall — this reflects realistic volatility and drift i.e, variance of the asset price grows with time.
- Some paths display extreme divergence as much as 160,000 as well as 60,000 reflecting the high volatility that bitcoin has (0.41)
- the graph also demonstrates the 252 time steps for each future value prediction

Convergence of option price



Results Summary

Convergence of option price

- The **final value** of the options was calculated as

Floating Strike Call Price : 17292.1231 ± 160.0750
Fixed Strike Call Price : 19074.4286 ± 167.4487

Floating (Control Variate) Price : 17292.1231 ± 119.7116
Fixed (Control Variate) Price : 19074.1231 ± 126.8238

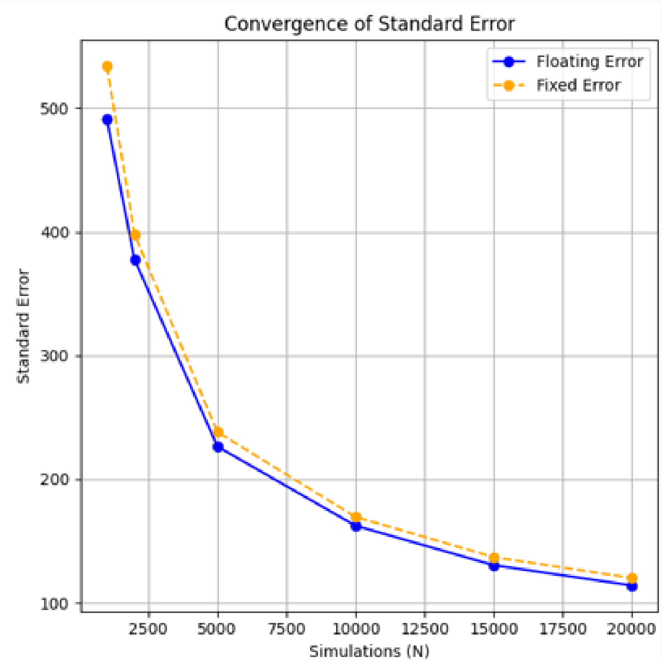
- **From the graph**

Blue line: Floating strike lookback call
Orange line: Fixed strike lookback call
Shaded regions:
CI Floating: 95% confidence band for floating
CI Fixed: 95% confidence band for fixed

For the price :
- As the number of simulations is increased, the estimated option price starts to stabilize.
- Floating strike price (blue) is lower than fixed strike (orange) — this is typical, because the strike price for fixed option was a constant value which was set **At the Money,** but since our asset is so volatile the final prices diverged a lot from the original.
- Prices are usually expected to converge around 10,000 simulations but our simulation takes a bit more to converge. This is expected because of volatile nature of bitcoin

For the Confidence interval:
- At low simulation counts (e.g., 1,000), the price estimate is highly unstable. At 10,000+, prices start to converge, and CI narrows — which confirms the control variate method is working.
- Fixed strike has a tighter band overall — likely because its payoff (max(S) - K) is slightly less volatile than the floating payoff (S_T - min(S)).

Error convergence



Results Summary

Error convergence

- Standard error decreases as the number of simulations increases — this is expected due to the law of large numbers.
- Both curves follow a 1/√N pattern, indicating that variance shrinks with more simulations.
- Floating strike has a slightly higher error than fixed at each N — because its payoff is more path-sensitive (depends on min(St))

Error reduced :

160.0750 to 119.7116 in case of Floating strike option
167.4487 to 126.8238 in case of Fixed strike option

Greeks calculated

Delta : -63.7113
Gamma : 69.2393
Vega : 79141.1798
Rho : 43953.0346
Theta : -3631.0268

These Greek values represent how the price of a floating-strike lookback call option responds to changes in key market parameters, based on Monte Carlo simulation. A Delta of −63.71 means the option's value drops significantly as the spot price rises, reflecting the payoff's dependence on the minimum price over time. The high Gamma (69.24) indicates extreme sensitivity of Delta to spot price changes. A very large Vega (79,141) shows the option is highly sensitive to volatility changes, which is expected for path-dependent options. Rho (43,953) suggests the price increases strongly with rising interest rates. The large negative Theta (−3,631) reflects rapid time decay — the option loses significant value as it nears maturity. These metrics help assess and hedge the risk of the exotic option.

Conclusion

This project implemented a Monte Carlo simulation framework to price exotic lookback options and evaluate their behavior under varying market conditions. By combining geometric Brownian motion with antithetic variates and control variate adjustments, the framework provided reliable price estimates and significantly reduced variance in simulation output. The use of finite difference methods to compute Greeks offered detailed insights into the sensitivity of floating strike lookback options to changes in market parameters. The model also included convergence analysis, ensuring that pricing accuracy improved consistently with larger simulation sizes. Overall, this approach demonstrated how simulation-based techniques can effectively handle complex, path-dependent instruments where analytical solutions may not exist.

Key Outcomes :

- Floating and fixed strike prices estimated with 95% confidence intervals.
- Control variate reduced error from ±160.08 to ±119.71.
- Greeks showed non-standard values (e.g., Delta −17.58, Vega 58.5K).
- High sensitivity to volatility and time (strong Vega, positive Theta).
- Convergence confirmed accuracy with higher simulations.