# DSC210 HW1

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**Question 1: Property of triangular matrices**

*Given:* $L_1$ and $L_2$ are two lower triangular matrices of size $n \times n$.

**Solution:**

1. **Proof for $L_1L_2$ being a lower triangular matrix:**

A matrix is said to be a lower triangular matrix if all its entries above the main diagonal are zero. Let's denote the entry in the $i$-th row and $j$-th column of a matrix $M$ as $M_{ij}$.

Given that both $L_1$ and $L_2$ are lower triangular matrices, we have:

For $i < j$:

$$L_{1_{ij}} = 0 \quad \text{and} \quad L_{2_{ij}} = 0$$

Now, for the product matrix $L_1L_2$, the entry at $i$-th row and $j$-th column is given by:

$$(L_1L_2)_{ij} = \sum_{k=1}^{n} L_{1_{ik}} \times L_{2_{kj}}$$

For $i < j$, since $L_{1_{ik}}$ is zero for all $k \geq i$ and $L_{2_{kj}}$ is zero for all $k \leq j$, the sum becomes zero. Thus, all entries above the main diagonal in $L_1L_2$ are zero, which makes $L_1L_2$ a lower triangular matrix.

2. **Proof by induction for the multiplication of $m$ lower triangular matrices:**

**Base Step:** We have already shown that the product of two lower triangular matrices $L_1$ and $L_2$ is a lower triangular matrix.

**Inductive Step:** Assume the statement is true for $k$ matrices, i.e., the product of $k$ lower triangular matrices is also a lower triangular matrix. We need to prove it for $k + 1$ matrices.

Let the product of the first $k$ matrices be denoted by $L'$. From the inductive hypothesis, $L'$ is a lower triangular matrix. Now, the product of $k + 1$ matrices is $L' \times L_{k+1}$. Using the result from the first part, the product of two lower triangular matrices is also a lower triangular matrix. Thus, $L' \times L_{k+1}$ is also a lower triangular matrix.

This completes the induction, and we have shown that the multiplication of any $m$ ($m > 2$) lower triangular matrices results in a lower triangular matrix.

**Question 2: Matrix operations**

*Given:* Let $B$ be a $4 \times 4$ matrix undergoing the specified operations to yield matrix $D$.

**Solution:**

(a) **Express each operation as a matrix and the final matrix $D$ as a product of 8 matrices.**

(i) **Double column 1:** The elementary matrix for this operation is:

$$E_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) **Halve row 3:** The elementary matrix for this operation is:

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) **Add row 3 to row 1:** The elementary matrix for this operation is:

$$E_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iv) **Interchange columns 1 and 4:** The elementary matrix for this operation is:

$$E_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(v) **Subtract row 2 from each of the other rows:** The elementary matrix for this operation is:

$$E_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

(vi) **Replace column 4 by column 3:** The elementary matrix for this operation is:

$$E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(vii) **Delete column 1:** To represent this, we use an auxiliary $4 \times 3$ matrix $E_{7a}$:

$$E_{7a} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Thus, the matrix $D$ is represented as:

$$D = E_{7a}E_6E_5E_4E_3E_2E_1B$$

(b) **Write the final result again as a product of** $ABC$
We group the matrices as:

$$A = E_{7a}$$

$$B = E_6 \times E_5 \times E_4$$

$$C = E_3 \times E_2 \times E_1$$

Thus, matrix $D$ is represented as:

$$D = A \times B \times C$$

Where:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Question 3: Matrix properties**
*Given:* A matrix $A$ is both triangular and unitary.
**Solution:**

A matrix is said to be unitary if $A^*A = I$, where $A^*$ is the conjugate transpose (or adjoint) of $A$ and $I$ is the identity matrix.

Given that $A$ is both lower triangular and unitary, the matrix $A$ can be represented as:

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The conjugate transpose $A^*$ is:

$$A^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ 0 & \bar{a}_{22} & \cdots & \bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{a}_{nn} \end{bmatrix}$$

Given $A^*A = I$, the (1,1)-entry of $A^*A$ is:

$$\bar{a}_{11}a_{11} + 0 + \cdots + 0 = |\bar{a}_{11}|^2$$

For the (i,j)-entry of $A^*A$ where $i \neq j$, the product contains terms from non-diagonal elements of $A$ multiplied by diagonal elements of $A^*$ and vice-versa. As $A$ is lower triangular, these terms are zero for $i < j$.

For $A^*A$ to be the identity matrix $I$, all diagonal elements must be 1 and off-diagonal elements must be 0. This implies:

1. All diagonal elements of $A$ have a magnitude of 1, i.e., $|a_{ii}| = 1$ for all $i$.
2. All off-diagonal elements of $A$ are 0.

Thus, if $A$ is both lower triangular and unitary, it must be a diagonal matrix with diagonal entries of magnitude 1.

**Question 4: $p$-norm inequalities**

Given $x$ is a real $m$-vector, we have to verify the following inequalities:

(a) $\|x\|_\infty \leq \|x\|_2$ (b) $\|x\|_2 \leq \sqrt{m} \cdot \|x\|_\infty$

**Solution:**

(a) **Proof of** $\|x\|_\infty \leq \|x\|_2$

By definition, the infinity-norm is the maximum absolute value of the components of the vector:

$$\|x\|_\infty = \max_i |x_i|$$

The 2-norm of $x$ is:

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_m^2}$$

Since each term $x_i^2$ is non-negative and $|x_i| \leq \|x\|_\infty$ for all $i$, we have:

$$x_i^2 \leq \|x\|_\infty^2$$

Summing over all $i$:

$$x_1^2 + x_2^2 + \ldots + x_m^2 \leq m \cdot \|x\|_\infty^2$$

Taking the square root of both sides:

$$\sqrt{x_1^2 + x_2^2 + \ldots + x_m^2} \leq \sqrt{m} \cdot \|x\|_\infty$$

However, since $\sqrt{m} \geq 1$ (because $m$ is positive),

$$\|x\|_2 \leq \|x\|_\infty$$

For equality to hold, all components of $x$ must be equal. For example, consider $x = [1, 1, \ldots, 1]$.

(b) **Proof of** $\|x\|_2 \leq \sqrt{m} \cdot \|x\|_\infty$

From the above proof, we already derived that:

$$\|x\|_2 \leq \sqrt{m} \cdot \|x\|_\infty$$

For equality to hold, one component of $x$ must be equal to $\|x\|_\infty$ and all other components must be zero. For example, consider $x = [0, 0, \ldots, 0, \|x\|_\infty, 0, \ldots, 0]$.