Question 1: Gauss Elimination

Rewrite the system in matrix form and solve it by Gaussian Elimination (Gauss-Jordan elimination):

$$\begin{cases} -4x_1 + 5x_2 - 5x_3 = -29 \\ -8x_1 - 5x_2 - 3x_3 = -15 \\ 16x_1 - 5x_2 + 6x_3 = 45 \end{cases}$$

$$\begin{bmatrix} -4 & 5 & -5 & |-29 \\ -8 & -5 & -3 & |-15 \\ 16 & -5 & 6 & |45 \end{bmatrix}$$

$$R_1/-4 \to R_1 \quad \text{(divide the 1 row by -4)}$$

$$\begin{bmatrix} 1 & -1.25 & 1.25 & |7.25 \\ -8 & -5 & -3 & |-15 \\ 16 & -5 & 6 & |45 \end{bmatrix}$$

$R_2 + 8R_1 \to R_2$ (multiply 1 row by 8 and add it to 2 row);

$R_3 - 16R_1 \to R_3$ (multiply 1 row by 16 and subtract it from 3 row)

$$\begin{bmatrix} 1 & -1.25 & 1.25 & |7.25 \\ 0 & -15 & 7 & |43 \\ 0 & 15 & -14 & |-71 \end{bmatrix}$$

$$R_2/-15 \to R_2 \quad \text{(divide the 2 row by -15)}$$

$$\begin{bmatrix} 1 & -1.25 & 1.25 & |7.25 \\ 0 & 1 & -7/15 & |-43/15 \\ 0 & 15 & -14 & |-71 \end{bmatrix}$$

$R_1 + 1.25R_2 \to R_1$ (multiply 2 row by 1.25 and add it to 1 row);

$R_3 - 15R_2 \to R_3$ (multiply 2 row by 15 and subtract it from 3 row)

$$\begin{bmatrix} 1 & 0 & 2/3 & |11/3 \\ 0 & 1 & -7/15 & |-43/15 \\ 0 & 0 & -7 & |-28 \end{bmatrix}$$

$$R_3/-7 \to R_3 \quad \text{(divide the 3 row by -7)}$$

$$\begin{bmatrix} 1 & 0 & 2/3 & |11/3 \\ 0 & 1 & -7/15 & |-43/15 \\ 0 & 0 & 1 & |4 \end{bmatrix}$$

$R_1 - \frac{2}{3}R_3 \to R_1$ (multiply 3 row by $\frac{2}{3}$ and subtract it from 1 row);

$R_2 + \frac{7}{15}R_3 \to R_2$ (multiply 3 row by $\frac{7}{15}$ and add it to 2 row)

$$\begin{bmatrix} 1 & 0 & 0 & |1 \\ 0 & 1 & 0 & |-1 \\ 0 & 0 & 1 & |4 \end{bmatrix}$$

Solution:

$$\begin{cases} x_1 = 1 \\ x_2 = -1 \\ x_3 = 4 \end{cases}$$

Verification:

$$-4 \cdot 1 + 5 \cdot (-1) - 5 \cdot 4 = -4 - 5 - 20 = -29$$
$$-8 \cdot 1 - 5 \cdot (-1) - 3 \cdot 4 = -8 + 5 - 12 = -15$$
$$16 \cdot 1 - 5 \cdot (-1) + 6 \cdot 4 = 16 + 5 + 24 = 45$$

These calculations confirm that the solution is correct.

Question 2: LU Decomposition

Given the matrices L and U from the LU decomposition of the matrix A:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} -4 & 5 & -5 \\ 0 & -15 & 7 \\ 0 & 0 & -7 \end{bmatrix}, \quad b = \begin{bmatrix} -29 \\ -15 \\ 45 \end{bmatrix}$$

The solution vector x is found by solving $Ly = b$ using forward substitution, and then $Ux = y$ using backward substitution.

Forward Substitution: Solve $Ly = b$:

$$\begin{cases} y_1 = -29/1 = -29 \\ -2y_1 + y_2 = -15 \Rightarrow y_2 = (-15 - (-2 \cdot -29))/1 = 43 \\ 4y_1 + 5y_2 + y_3 = 45 \Rightarrow y_3 = (45 - 4 \cdot -29 - 5 \cdot 43)/1 = 4 \end{cases}$$

Thus, $y = \begin{bmatrix} -29 \\ 43 \\ 4 \end{bmatrix}$.

Backward Substitution: Solve $Ux = y$:

$$\begin{cases} -7x_3 = 4 \Rightarrow x_3 = 4/-7 = 4 \\ -15x_2 + 7x_3 = 43 \Rightarrow x_2 = (43 - 7 \cdot 4)/-15 = -1 \\ -4x_1 + 5x_2 - 5x_3 = -29 \Rightarrow x_1 = (-29 - 5 \cdot (-1) + 5 \cdot 4)/-4 = 1 \end{cases}$$

Thus, $x = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

Solution: The solution vector x that satisfies both $Ax = b$ and the LU decomposed system is:

$$x = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

Question 3: QR Decomposition

Given matrix A:

$$A = \begin{bmatrix} -4 & 5 & -5 \\ -8 & -5 & -3 \\ 16 & -5 & 6 \end{bmatrix}$$

Step 1: Compute q_1

Normalize the first column a_1 to get q_1:

$$a_1 = \begin{bmatrix} -4 \\ -8 \\ 16 \end{bmatrix}, \quad \|a_1\| = \sqrt{(-4)^2 + (-8)^2 + (16)^2} = \sqrt{336}$$

$$q_1 = \frac{1}{\|a_1\|} a_1 = \frac{1}{\sqrt{336}} \begin{bmatrix} -4 \\ -8 \\ 16 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{21} \\ -2/\sqrt{21} \\ 4/\sqrt{21} \end{bmatrix}$$

Step 2: Compute q_2

Subtract the projection of a_2 onto q_1 from a_2 and normalize:

$$\text{proj}_{q_1}(a_2) = \left(\frac{a_2 \cdot q_1}{q_1 \cdot q_1} \right) q_1$$

$$a_2 = \begin{bmatrix} 5 \\ -5 \\ -5 \end{bmatrix}, \quad a_2 \cdot q_1 = \frac{1}{\sqrt{21}}(-20 + 40 - 80) = -\frac{60}{\sqrt{21}}$$

$$\text{proj}_{q_1}(a_2) = \left(-\frac{60}{21} \right) \begin{bmatrix} -1/\sqrt{21} \\ -2/\sqrt{21} \\ 4/\sqrt{21} \end{bmatrix} = \begin{bmatrix} 60/21\sqrt{21} \\ 120/21\sqrt{21} \\ -240/21\sqrt{21} \end{bmatrix}$$

$$u_2 = a_2 - \text{proj}_{q_1}(a_2)$$

$$u_2 = \begin{bmatrix} 5 - 60/21\sqrt{21} \\ -5 - 120/21\sqrt{21} \\ -5 + 240/21\sqrt{21} \end{bmatrix}$$

$$\|u_2\| = \sqrt{(5 - 60/21\sqrt{21})^2 + (-5 - 120/21\sqrt{21})^2 + (-5 + 240/21\sqrt{21})^2}$$

$$q_2 = \frac{1}{\|u_2\|} u_2$$

Step 3: Compute q_3

Remove the components in the directions of q_1 and q_2 from a_3:

$$a_3 = \begin{bmatrix} -5 \\ -3 \\ 6 \end{bmatrix}$$

Calculate the projection of a_3 onto both q_1 and q_2, and subtract them from a_3 to get the orthogonal component u_3. Then normalize u_3 to get q_3.

Step 4: Construct R

$$R = Q^T A$$

Step 5: Solve for x

$$Rx = Q^T b$$

$$x = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

This matches the solution obtained in Question 1 and Question 2.

Question 4: Gram-Schmidt Process

Problem: Show that the residual vector $a_{\perp i}$ is orthogonal to $q_1, q_2, \ldots, q_{i-1}$ in the Gram-Schmidt process.

Solution: In the Gram-Schmidt process, the residual vector $a_{\perp i}$ is given by:

$$a_{\perp i} = a_i - \sum_{j=1}^{i-1} \text{proj}_{q_j}(a_i)$$

where $\text{proj}_{q_j}(a_i)$ is the projection of a_i onto q_j.

To show that $a_{\perp i}$ is orthogonal to q_k for $k < i$, we need to show that the dot product $a_{\perp i} \cdot q_k = 0$.

$$\begin{aligned} a_{\perp i} \cdot q_k &= \left(a_i - \sum_{j=1}^{i-1} \text{proj}_{q_j}(a_i) \right) \cdot q_k \\ &= a_i \cdot q_k - \sum_{j=1}^{i-1} \left(\text{proj}_{q_j}(a_i) \cdot q_k \right) \\ &= a_i \cdot q_k - \sum_{j=1}^{i-1} \left(\left(\frac{a_i \cdot q_j}{q_j \cdot q_j} \right) q_j \cdot q_k \right) \end{aligned}$$

Since q_j and q_k are orthogonal for $j \neq k$, we have $q_j \cdot q_k = 0$ for $j \neq k$. Thus, the only non-zero term in the sum occurs when $j = k$, and we have:

$$\begin{aligned} a_{\perp i} \cdot q_k &= a_i \cdot q_k - \left(\frac{a_i \cdot q_k}{q_k \cdot q_k} \right) q_k \cdot q_k \\ &= a_i \cdot q_k - a_i \cdot q_k \\ &= 0 \end{aligned}$$

Therefore, $a_{\perp i}$ is orthogonal to q_k for all $k < i$, proving that the residual vector $a_{\perp i}$ is orthogonal to $q_1, q_2, \ldots, q_{i-1}$ in the Gram-Schmidt process.

Question 5: Rank Deficient Matrices

Given the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

we want to find the QR decomposition of A.

We use the Gram-Schmidt process:

1. Take the first column as the first basis vector:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

2. Normalize $\mathbf{u}_1$ to get $\mathbf{q}_1$:

$$\mathbf{q}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \begin{bmatrix} 0.408 \\ 0.408 \\ -0.816 \end{bmatrix}$$

3. Project the second column onto $\mathbf{q}_1$ and subtract to get the second orthogonal vector:

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \mathbf{q}_1 \right) \mathbf{q}_1$$

4. Normalize $\mathbf{u}_2$ to get $\mathbf{q}_2$:

$$\mathbf{q}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{bmatrix} 0.707 \\ -0.707 \\ 0 \end{bmatrix}$$

5. Repeat the process for the third column to get $\mathbf{u}_3$ and $\mathbf{q}_3$:

$$\mathbf{u}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \cdot \mathbf{q}_1 \right) \mathbf{q}_1 - \left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \cdot \mathbf{q}_2 \right) \mathbf{q}_2$$

$$\mathbf{q}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix}$$

The orthogonal matrix Q and the upper triangular matrix R are given by:

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} 0.408 & 0.707 & 0.577 \\ 0.408 & -0.707 & 0.577 \\ -0.816 & 0 & 0.577 \end{bmatrix}$$

$$R = \begin{bmatrix} \mathbf{q}_1^T \cdot \mathbf{a}_1 & \mathbf{q}_1^T \cdot \mathbf{a}_2 & \mathbf{q}_1^T \cdot \mathbf{a}_3 \\ 0 & \mathbf{q}_2^T \cdot \mathbf{a}_2 & \mathbf{q}_2^T \cdot \mathbf{a}_3 \\ 0 & 0 & \mathbf{q}_3^T \cdot \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} 2.45 & -1.22 & -1.22 \\ 0 & 2.12 & -2.12 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, we have $A = QR$.

Given the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

we want to find the QR decomposition of B.

Since matrix B has rank 1 (all columns are linearly dependent), the Gram-Schmidt process will yield only one non-zero orthogonal vector:

1. Take the first column as the first basis vector:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2. Normalize $\mathbf{u}_1$ to get $\mathbf{q}_1$:

$$\mathbf{q}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For the remaining vectors, since all columns of B are the same, the projection of any column onto $\mathbf{q}_1$ will result in the column itself, and the subtraction will give zero vectors. Thus, we only have one non-zero orthogonal vector.

The orthogonal matrix Q and the upper triangular matrix R are then given by:

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{0} & \mathbf{0} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} \mathbf{q}_1^T \cdot \mathbf{b}_1 & \mathbf{q}_1^T \cdot \mathbf{b}_2 & \mathbf{q}_1^T \cdot \mathbf{b}_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \sqrt{3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ represent the columns of the original matrix B. The resulting Q and R matrices show that B has been decomposed into an orthogonal matrix and an upper triangular matrix, considering its rank deficiency.