MAT-MEK4270 - Mandatory 1

Mansur Dudajev

October 2023

1.2.1

Given the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{1}$$

and the exact solution

$$u(x, y, t) = \sin(k_x x) \sin(k_y y) \cos(\omega t)$$

we can derive an equation for the dispersion coefficient ω as a function of c, k_x and k_y.
We start by differentiating the exact solution in time:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \sin(k_x x) \sin(k_y y)(-\omega \sin(\omega t)) \\
\Rightarrow \frac{\partial^2 u}{\partial t^2} &= \sin(k_x x) \sin(k_y y)(-\omega^2 \cos(\omega t)) \\
&= -\omega^2 \sin(k_x x) \sin(k_y y) \cos(\omega t) \\
\Rightarrow \frac{\partial^2 u}{\partial t^2} &= -\omega^2 u
\end{aligned}$$

Now we find the double derivative with respect to both of the spatial variables:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= (k_x \cos(k_x x)) \sin(k_y y) \cos(\omega t) \\
\Rightarrow \frac{\partial^2 u}{\partial x^2} &= (-k_x^2 \sin(k_x x)) \sin(k_y y) \cos(\omega t) \\
&= -k_x^2 \sin(k_x x) \sin(k_y y) \cos(\omega t) \\
\Rightarrow \frac{\partial^2 u}{\partial x^2} &= -k_x^2 u
\end{aligned}$$

The last derivation holds for y, giving us the double derivative:

$$\frac{\partial^2 u}{\partial y^2} = -k_y^2 u$$

We can now use these relations in the wave equation 1:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) u \\
\Rightarrow -\omega^2 u &= c^2(-k_x^2 u - k_y^2 u) \\
\Rightarrow -\omega^2 u &= -c^2(k_x^2 + k_y^2) u
\end{aligned}
$$

Assuming $u \neq \mathbf{0}$, we get the relation:

$$
\begin{aligned}
\omega^2 &= c^2(k_x^2 + k_y^2) \\
\omega &= \pm c\sqrt{k_x^2 + k_y^2}
\end{aligned}
$$

1.2.3

To show that the complex wave

$$
u(x, y, t) = e^{\imath(k_x x + k_y y - \omega t)} \tag{2}
$$

is a solution to the wave equation 1, we start by rewriting the complex representation into a product

$$
u(x, y, t) = e^{\imath k_x x} \cdot e^{\imath k_y y} \cdot e^{-\imath \omega t}
$$

Now we have the complex wave as a product of three functions that are each a function of only one of the variables. We can now differentiate u with respect to each of the variables.

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= e^{\imath k_x x} \cdot e^{\imath k_y y} \cdot (-\imath\omega)^2 e^{-\imath \omega t} \\
&= -\omega^2 \cdot e^{\imath k_x x} \cdot e^{\imath k_y y} \cdot e^{-\imath \omega t} \\
&= -\omega^2 \cdot u \\
\frac{\partial^2 u}{\partial x^2} &= (\imath k_x)^2 e^{\imath k_x x} \cdot e^{\imath k_y y} \cdot e^{-\imath \omega t} \\
&= -k_x^2 \cdot e^{\imath k_x x} \cdot e^{\imath k_y y} \cdot e^{-\imath \omega t} \\
&= -k_x^2 \cdot u \\
\frac{\partial^2 u}{\partial y^2} &= e^{\imath k_x x} \cdot (\imath k_y)^2 e^{\imath k_y y} \cdot e^{-\imath \omega t} \\
&= -k_y^2 \cdot e^{\imath k_x x} \cdot e^{\imath k_y y} \cdot e^{-\imath \omega t} \\
&= -k_y^2 \cdot u
\end{aligned}
$$

using these relations in the wave equation 1, we get

$$
\begin{aligned}
-\omega^2 u &= c^2(-k_x^2 u - k_y^2 u) \\
\Rightarrow \omega^2 u &= c^2(k_x^2 + k_y^2) u
\end{aligned}
$$

we see that this holds true if $c = \pm\omega/\sqrt{k_x^2 + k_y^2}$, which means that the complex wave 2 is a solution to the wave equation 1.

1.2.4

The discrete version of the complex wave 2 is given as

$$u_{i,j}^n = e^{\imath(kh(i+j)-\bar{\omega}n\Delta t)} \tag{3}$$

where $k = k_x = k_y$, and $\bar{\omega}$ is the numerical dispersion coefficient. We will prove that $\bar{\omega} = \omega$ if $k_x = k_y$ and $C = 1/\sqrt{2}$. We will start by deriving a formula for the dispersion relation ω, given the constraints. From task 1.2.1, we have

$$\begin{aligned} \omega &= c\sqrt{k_x^2 + k_y^2} \\ \Rightarrow \omega &= c\sqrt{k^2 + k^2} \\ \Rightarrow \omega &= \sqrt{2}ck \end{aligned}$$

using that $C = \frac{c\Delta t}{h}$ and $C = 1/\sqrt{2}$, we get

$$\begin{aligned} c &= \frac{Ch}{\Delta t} \\ \Rightarrow c &= \frac{1}{\sqrt{2}}\frac{h}{\Delta t} \end{aligned}$$

which gives us

$$\begin{aligned} \omega &= \sqrt{2}\frac{1}{\sqrt{2}}\frac{h}{\Delta t}k \\ \Rightarrow \omega &= \frac{kh}{\Delta t} \end{aligned}$$

The discrete version of the wave equation is:

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2}\right)$$

We can rewrite this as

$$u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1} = \frac{c^2\Delta t^2}{h^2}(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n + u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n)$$

And using $C = c\Delta t/h$, we get

$$u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1} = C^2(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n + u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n)$$

We will now use equation 3 to derive expressions for each of the terms in the discreet PDE. We will shorten the derivation somewhat using the notation $n \pm 1$ or $n \mp 1$, to derive formulas for $n+1$ and $n-1$ in parallel. For $n+1$ and

$n-1$, we get

$$
\begin{aligned}
u_{i,j}^{n\pm1} &= e^{\imath(kh(i+j)-\bar{\omega}(n\pm1)\Delta t)} \\
&= e^{\imath(kh(i+j)-\bar{\omega}n\Delta t\mp\bar{\omega}\Delta t)} \\
&= e^{\imath(kh(i+j)-\bar{\omega}n\Delta t)} \\
&= e^{\imath(kh(i+j)-\bar{\omega}n\Delta t)}\cdot e^{\mp\imath\bar{\omega}\Delta t} \\
\Rightarrow u_{i,j}^{n\pm1} &= u_{i,j}^{n}\cdot e^{\mp\imath\bar{\omega}\Delta t}
\end{aligned}
$$

For $i+1$ and $i-1$, we get

$$
\begin{aligned}
u_{i\pm1,j}^{n} &= e^{\imath(kh(i\pm1+j)-\bar{\omega}n\Delta t)} = e^{\imath(kh(i+j\pm1)-\bar{\omega}n\Delta t)} \\
&= e^{\imath(kh(i+j)\pm kh-\bar{\omega}n\Delta t)} \\
&= e^{\imath(kh(i+j)-\bar{\omega}n\Delta t)}\cdot e^{\pm\imath kh} \\
\Rightarrow u_{i\pm1,j}^{n} &= u_{i,j}^{n}\cdot e^{\pm\imath kh}
\end{aligned}
$$

For $j+1$ and $j-1$, we get

$$
\begin{aligned}
u_{i,j\pm1}^{n} &= e^{\imath(kh(i+j\pm1)-\bar{\omega}n\Delta t)} = u_{i\pm1,j}^{n} \\
\Rightarrow u_{i,j\pm1}^{n} &= u_{i,j}^{n}\cdot e^{\pm\imath kh}
\end{aligned}
$$

This gives us six terms we can use in the discrete PDE:

$$
\begin{aligned}
u_{i,j}^{n+1} &= u_{i,j}^{n}\cdot e^{-\imath\bar{\omega}\Delta t} \\
u_{i,j}^{n-1} &= u_{i,j}^{n}\cdot e^{+\imath\bar{\omega}\Delta t} \\
u_{i+1,j}^{n} &= u_{i,j}^{n}\cdot e^{+\imath kh} \\
u_{i-1,j}^{n} &= u_{i,j}^{n}\cdot e^{-\imath kh} \\
u_{i,j+1}^{n} &= u_{i,j}^{n}\cdot e^{+\imath kh} \\
u_{i,j-1}^{n} &= u_{i,j}^{n}\cdot e^{-\imath kh}
\end{aligned}
$$

Using these, we get, for the left hand side of the PDE:

$$
\begin{aligned}
&u_{i,j}^{n}\cdot e^{-\imath\bar{\omega}\Delta t} - 2u_{i,j}^{n} + u_{i,j}^{n}\cdot e^{+\imath\bar{\omega}\Delta t} \\
&\quad = u_{i,j}^{n}(e^{-\imath\bar{\omega}\Delta t} + e^{+\imath\bar{\omega}\Delta t} - 2)
\end{aligned}
$$

using the trigonometric identity $2\cos(x) = e^{\imath x}+e^{-\imath x}$, the left hand side becomes

$$
u_{i,j}^{n}(2\cos(\bar{\omega}\Delta t)-2) = 2u_{i,j}^{n}(\cos(\bar{\omega}\Delta t)-1)
$$

For the right hand side, we have

$$
\begin{aligned}
&C^2(u_{i,j}^{n}\cdot e^{+\imath kh} - 2u_{i,j}^{n} + u_{i,j}^{n}\cdot e^{-\imath kh} \\
&\quad + u_{i,j}^{n}\cdot e^{+\imath kh} - 2u_{i,j}^{n} + u_{i,j}^{n}\cdot e^{-\imath kh}) \\
&= C^2(2[u_{i,j}^{n}\cdot e^{+\imath kh} - 2u_{i,j}^{n} + u_{i,j}^{n}\cdot e^{-\imath kh}]) \\
&= 2u_{i,j}^{n}C^2(e^{+\imath kh} + e^{-\imath kh} - 2) \\
&= 2u_{i,j}^{n}C^2(2\cos(kh)-2) = 4u_{i,j}^{n}C^2(\cos(kh)-1)
\end{aligned}
$$

And the discrete PDE now becomes

$$\begin{aligned}
2u_{i,j}^n(\cos(\bar{\omega}\Delta t) - 1) &= 4u_{i,j}^n C^2(\cos(kh) - 1) \\
\Rightarrow (\cos(\bar{\omega}\Delta t) - 1) &= 2C^2(\cos(kh) - 1)
\end{aligned}$$

Using $C = 1/\sqrt{2} \Rightarrow C^2 = 1/2$, we get

$$\begin{aligned}
\cos(\bar{\omega}\Delta t) - 1 &= 2\frac{1}{2}(\cos(kh) - 1) \\
\Rightarrow \cos(\bar{\omega}\Delta t) - 1 &= \cos(kh) - 1 \\
\Rightarrow \cos(\bar{\omega}\Delta t) &= \cos(kh) \\
\Rightarrow \bar{\omega}\Delta t &= kh \\
\Rightarrow \bar{\omega} &= \frac{kh}{\Delta t} = \omega
\end{aligned}$$