Introduction to the Hamiltonian Circuit Problem and the Traveling Salesman Problem

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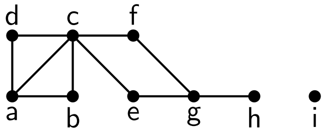
Overview

Overview

Basics

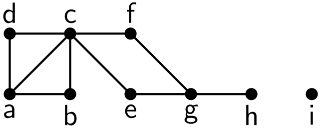
- Graph $G = (V, E)$ consists of
  - vertex set V
  - edge set $E \subseteq \binom{V}{2}$
- $n := |V|$, $m := |E|$.
- Let G be finite, i.e., $|V| < +\infty$.



Example graph with $n = 9$ and $m = 10$

Neighbor and Degree

- $u, v \in V$ neighboring, if $\{u, v\} \in E$.
- Degree of $v \in V$: Number of neighbors of v.
  Notation: $\deg(v)$.



$$\deg(c) = 5$$
$$\deg(a) = \deg(g) = 3$$
$$\deg(b) = \deg(d) = \deg(e) = \deg(f) = 2$$
$$\deg(h) = 1$$
$$\deg(i) = 0$$

- Minimum degree of G: minimum degree over all vertices of G.
  Notation: $\delta(G)$.

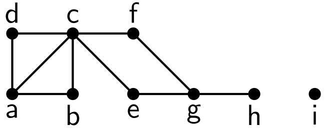
Path, Circuit and Cycle

- $v_1, v_2, \ldots, v_k \in V$, $k \leq n$,
  v_i pairwise distinct.
  $P := (v_1, v_2, \ldots, v_k)$: Path.
  (Notation: Successive vertices are connected by an edge.)
- $v_1, v_2, \ldots, v_k \in V$, $k \leq n$,
  v_i pairwise distinct.
  $C := (v_1, v_2, \ldots, v_k, v_1)$: Circuit.
- $v_1, v_2, \ldots, v_k \in V$,
  $D := (v_1, v_2, \ldots, v_k, v_1)$: Cycle.

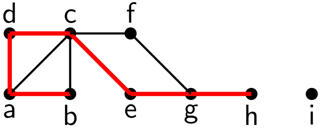
Exercise 1

- Task: Find a
  a) path
  b) circuit
  c) cycle

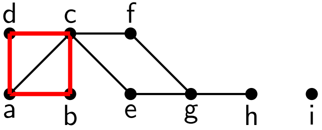
  with maximum number of vertices in the graph



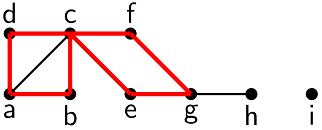
Solution:



Path (b,a,d,c,e,g,h)



Circuit (a,b,c,d,a)



Cycle (a,b,c,e,g,f,c,d,a)

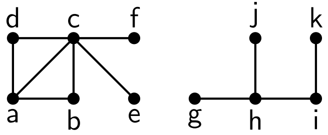
Connected Components

- G is called connected, if a path from v to w exists for each two different vertices $v, w \in V$.

- Remark:
  Each graph decomposes in k disjoint connected components $V_1, V_2, \ldots, V_k \subseteq V$ with $V = V_1 \cup V_2 \cup \cdots \cup V_k$.

  For two vertices a connecting path exists, if and only if they are contained in the same connected component.

  In the case $k = 1$ the graph is connected.



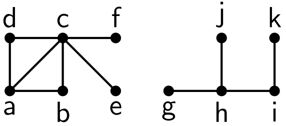
2 connected components

Bridge

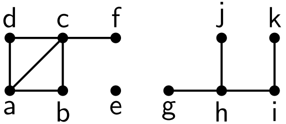
- $e \in E$ is called bridge, if omitting e increases the number of connected components by one.

Exercise 2

Task: How many bridges are contained in the graph



- Solution:
  6 bridges: (c, e), (c, f), (g, h), (h, j), (h, i), (i, k).
  E.g., without bridge (c, e): 3 connected components.

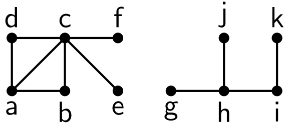


Articulation Point

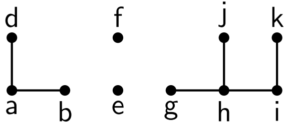
- $v \in V$ is called articulation point, if omitting v with all edges to neighboring vertices increases the number of connected components.

Exercise 3

Task: How many articulation points are contained in the graph



- Solution:
  3 articulation points: c, h, i.
  E.g., without articulation point c: 4 connected components.



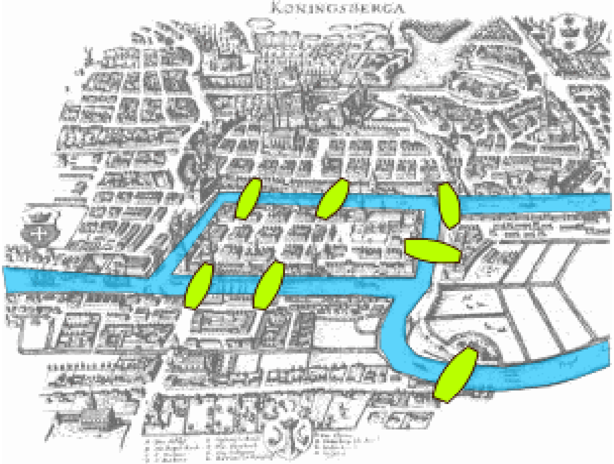
Eulerian Cycle Problem (ECP)

- Eulerian Cycle: Cycle traversing all edges.
- Eulerian Cycle Problem:

  Input.: Connected graph $G = (V, E)$.
  Question: Does in G a Eulerian cycle exist?

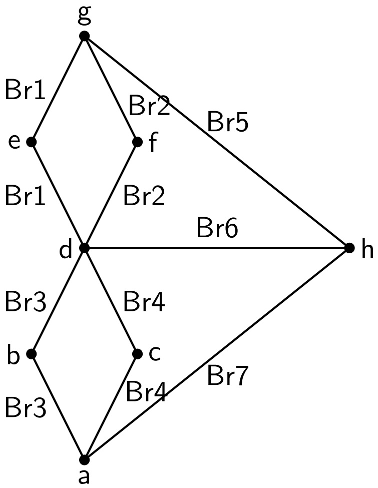
Theorem 1: Let G be a graph.
G contains a Eulerian cycle.
$\Leftrightarrow$ G ist connected and
each vertex has even degree.

Theorem 2: Let $G = (V, E)$ be a graph.
Then in time $\mathcal{O}(m)$ a Eulerian cycle can be found in G,
if one exists.



Leonhard Euler (1736): 7 bridges of Königsberg in 18-th century.
Attention: In graph theoretic sense this is not a bridge!

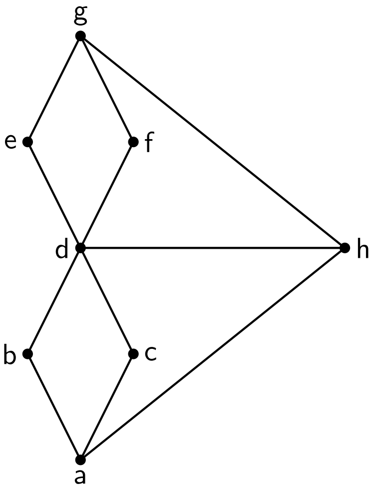
Question: Does a circular route exist in Königsberg traversing all bridges exactly once?



Model graph of the 7-bridges problem of Königsberg

Exercise 4

- Task: Find a Eulerian cycle in this graph or disprove its existence.



- Solution:

  No Eulerian cycle.

  Reason: Degree of vertices a, d, g and h is odd.

Hamiltonian Circuit Problem (HCP)

- Hamiltonian Circuit: Circuit containing all n vertices.
- Hamiltonian Circuit Problem:

  Input.: Graph $G = (V, E)$.

  Question: Does in G a Hamiltonian circuit exist?
- Comparison:

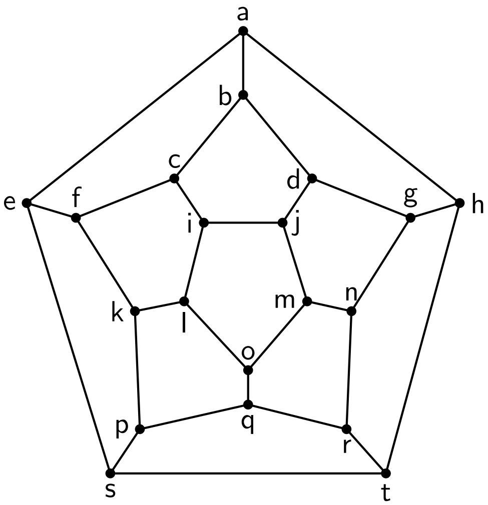
  ECP: Cycle traversing all edges.

Theorem 3: HCP is $\mathcal{NP}$-hart.

Proof: Polynomial reduction from 3-SAT to HCP.

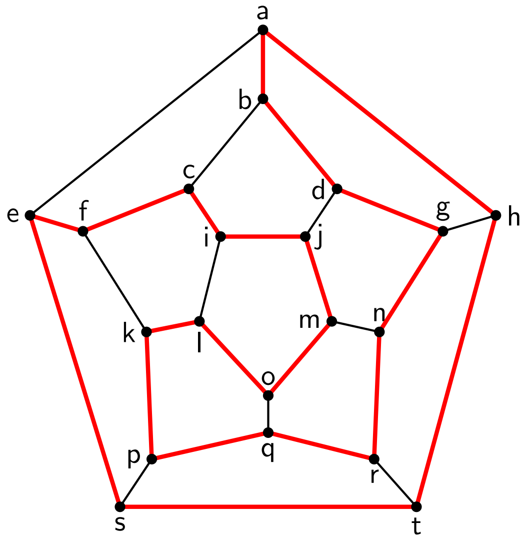
Exercise 5

- Task: Find a Hamiltonian circuit or disprove its existence in the graph



Sir William Rowan Hamilton (1857): Dodecahedron

- Solution:

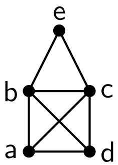


Hamiltonian circuit: (a,b,d,g,n,r,q,p,k,l,o,m,j,i,c,f,e,s,t,h,a)

Exercise 6

- Task: Find a
  a) Eulerian cycle
  b) Hamiltonian circuit

  or disprove its existence in the graph

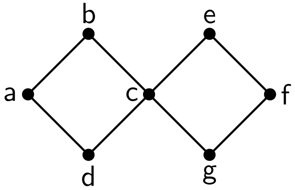


- Solution:
  a) Hamiltonian circuit (a, b, e, c, d, a).
  b) No Eulerian cycle.
  Reason: Degree of vertices a and d is odd.

Exercise 7

- Task: Find a
  a) Eulerian cycle
  b) Hamiltonian circuit

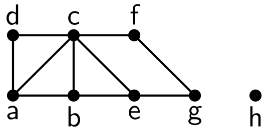
  or disprove its existence in the graph



- Solution:
  a) No Hamiltonian circuit.
  Reason: Vertex c is an articulation point of the graph.
  $\Rightarrow$ Vertex c has to be traversed twice.
  b) Eulerian cycle $(a, b, c, e, f, g, c, d, a)$.
- Proposition 1: If a graph G contains an articulation point, then G contains no Hamiltonian circuit.

Exercise 8

- Task: Find a Hamiltonian circuit or disprove its existence in the graph



- Solution:

  No Hamiltonian circuit.

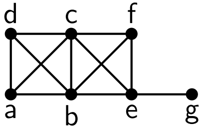
  Reason I: Vertex h has degree 0.

  Reason II: Graph is not connected.

- Proposition 2: If a graph G contains a vertex of degree 0, then G contains no Hamiltonian circuit.
- Proposition 3: If a graph G is not connected, then G contains no Hamiltonian circuit.

Exercise 9

- Task: Find a Hamiltonian circuit or disprove its existence in the graph



- Solution:

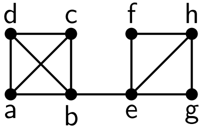
  No Hamiltonian circuit.

  Reason: Vertex g has degree 1.

- Proposition 4: If a graph G contains a vertex of degree 1, then G contains no Hamiltonian circuit.

Exercise 10

- Task: Find a Hamiltonian circuit or disprove its existence in the graph



- Solution:

  No Hamiltonian circuit.

  Reason: Edge (b, e) is a bridge of the graph.

- Proposition 5: If a graph G contains a bridge, then G contains no Hamiltonian circuit.

Theorem 4 (Dirac, 1952): Let $G = (V, E)$ be a graph with minimum degree $\delta(G)$.
If $\delta(G) \geq n/2$, then G contains a Hamiltonian circuit.

Proof: a) Show: G is connected.

b) Choose a path P_1 in G.

c) Construct from P_1 a circuit C.

d) Show: C is a Hamiltonian circuit.

a) Assume G is not connected.
Then a connected component F exists with $|F| \leq n/2$.
$\Rightarrow \forall v \in F$: $\deg(v) < n/2$. $\lightning$ to $\delta(G) \geq n/2$.
$\Rightarrow$ G is connected.

b) Choose

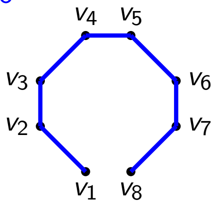
$$P_1 \quad := \quad (v_1, v_2, \dots, v_k)$$

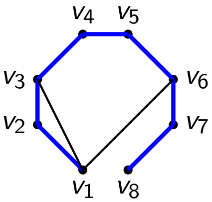
so that the number of vertices k of the path is maximal.

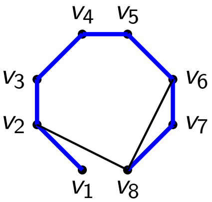
c) Let

$$\begin{aligned} M &:= \{v_1, v_2, \ldots, v_{k-1}\} \\ M_1 &:= \{v \in M \mid v \text{ is predecessor (on } P_1\text{) of a neighbor of } v_1\} \\ M_2 &:= \{v \in M \mid v \text{ is neighbor of } v_k\} \end{aligned}$$

Example: Path P_1 for $k = 8$



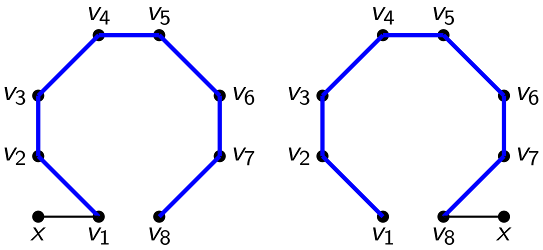




It holds $v_2 \in M_1; v_5 \in M_1$ It holds $v_2 \in M_2; v_6 \in M_2$

It holds:

(i) Both, v_1 and v_k have at least $n/2$ neighbors on P_1.



$\lightning$ to maximality of P_1.

(ii) $|M| = k - 1 \leq n - 1$.

(iii) $|M_1| \geq n/2$.
Follows from (i), as v_k can be no predecessor on P_1.

(iv) $|M_2| \geq n/2$.
Follows from (i), as v_k is no neighbor of itself.

Assume $M_1 \cap M_2 = \emptyset$

It follows:

$$n \;=\; n/2 + n/2 \;\le\; |M_1| + |M_2| \;=\; |M_1 \cup M_2| \;\le\; |M| \;\le n - 1$$

We have: $n \le n - 1 \quad ↯$

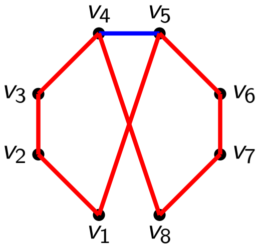
$\Rightarrow\; M_1 \cap M_2 \neq \emptyset.$

$\Rightarrow\; \exists v_i \in M_1 \cap M_2,\; 1 \le i < k,$

i.e., $\{v_1, v_{i+1}\} \in E$ and $\{v_k, v_i\} \in E$.

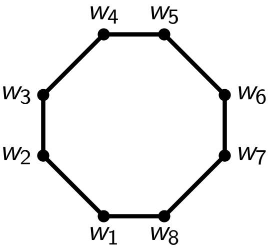
Then $C := (v_1, v_{i+1}, v_{i+2}, \ldots, v_{k-1}, v_k, v_i, v_{i-1}, \ldots, v_2, v_1)$ is a circuit with k vertices.

Example: Path P_1 for $k = 8$, $i = 4$



For convenience, renumber the circuit: $C = (w_1, w_2, \ldots, w_k, w_1)$.

Example: Cycle C after renumbering for $k = 8$



d) Assume C is not Hamiltonian.

Then $k < n$.

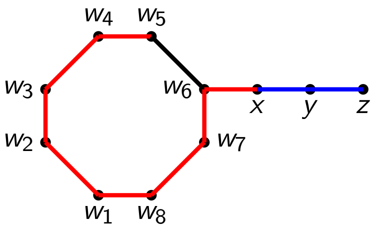
As by a), the graph is connected, $j \in \mathbb{N}$ and $x \in V \setminus C$ exist with $\{x, w_j\} \in E$. Then

$$P_2 \quad := \quad (u, w_j, w_{j+1}, \ldots, w_k, w_1, w_2, \ldots, w_{j-1})$$

is a path with $k+1$ vertices.

$\lightning$ to maximality of P_1

Example: Circuit C for $k = 8$



Path P_2 with $k + 1 = 9$ vertices

$\Rightarrow$ C is Hamiltonian. $\square$

Exercise 11

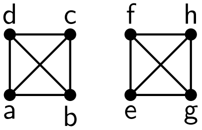
- Task: Find out whether Theorem of Dirac is sharp,

  i.e., construct a graph

  where the minimum degree is maximized,

  with the condition that the graph contains no Hamiltonian circuit.
  Consider $n = 8$.

- Solution:

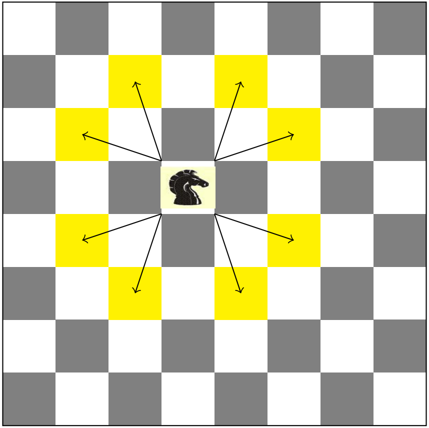
  

  Minimum degree $\delta(G_1) = 3 = n/2 - 1$.

  Can be generalized to arbitrary n.

- Proposition 6: The degree bound of Theorem of Dirac is sharp.

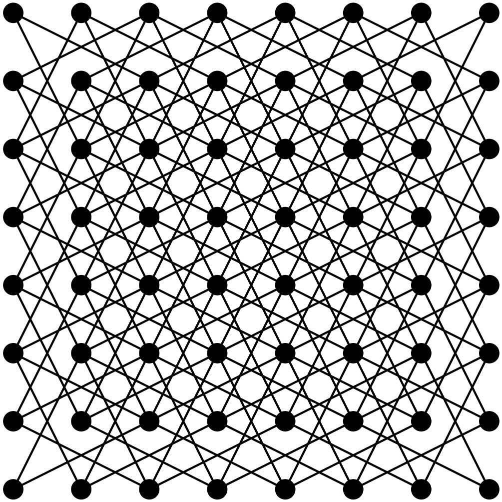
Application I: Problem of Knight's Tour



Possible moves of a knight on a chessboard

Question: Does a knight's tour exist so that the knight

- starts at a cell of the chessboard,
- traverses each cell of the chessboard exactly once
- and finally returns to his starting cell.



Model graph of the problem of knight's tour

- Formal modeling as HCP:

  Define $G_1 = (V, E_1)$ with $|V| = 64$

$$V = \{v_{i,j} \mid 1 \le i, j \le 8\}$$

and

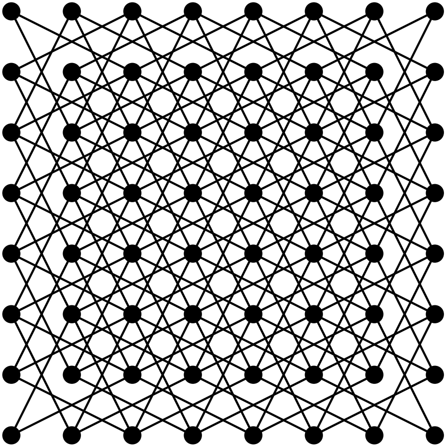
$$\begin{aligned} E_1 = \{\{v_{i,j}, v_{i',j'}\} \mid 1 \le i, j, i', j' \le 8 \\ \wedge \big(|i - i'| = 2,\ |j - j'| = 1 \\ \vee |i - i'| = 1,\ |j - j'| = 2 \big)\} \end{aligned}$$

- It holds:

  A knight's tour exists.

  $\Leftrightarrow$ G_1 contains a Hamiltonian circuit.

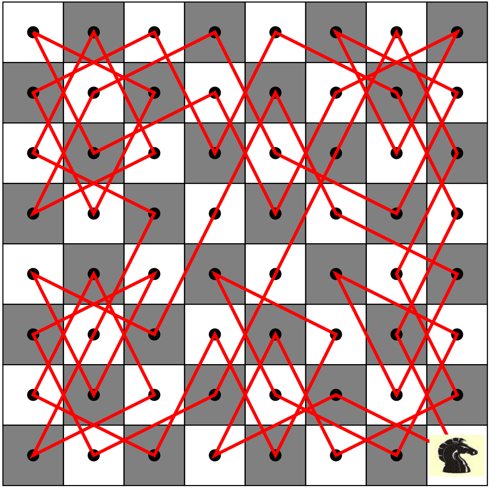
- Existence of a Hamiltonian circuit?



- No vertices of degree 0 or 1.
- No bridges.
- No articulation points.
- Application of Theorem of Dirac not possible: $\delta(G_1) = 2 < 32$.

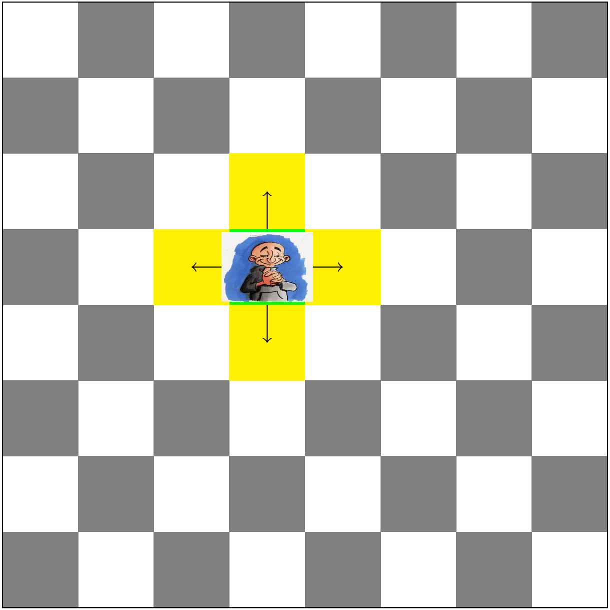
- Solution:

  Knight's tour exists.

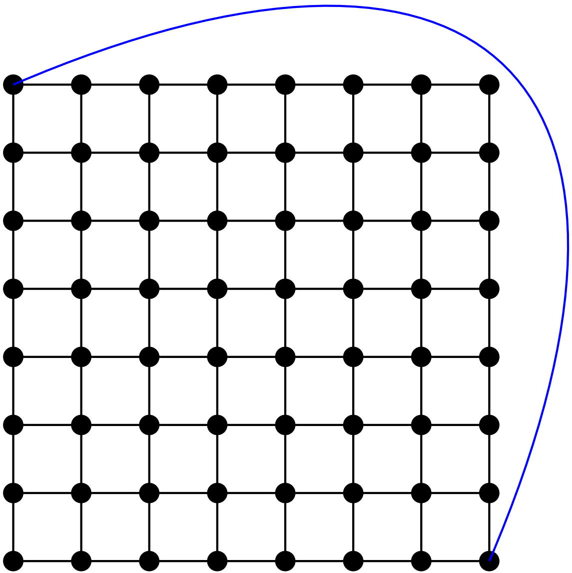


Application II: Problem of Mr. No

- Mr. No and Mr. Go are two mystic Japanese detectives.
- Mr. No lives in the right lower cell of a chessboard and Mr. Go in the left upper cell.
- Mr. No wants to visit Mr. Go, but before also all different cells of the chessboard.
- In each move he may move only one step vertical or one step horizontal.
- Question: Does such a path exist for Mr. No?



Possible steps of Mr. No



Model graph of the problem of Mr. No

- Formal modeling as HCP:

  Define $G_2 = (V, E_2)$ with $|V| = 64$

$$V = \{v_{i,j} \mid 1 \leq i, j \leq 8\}$$

and

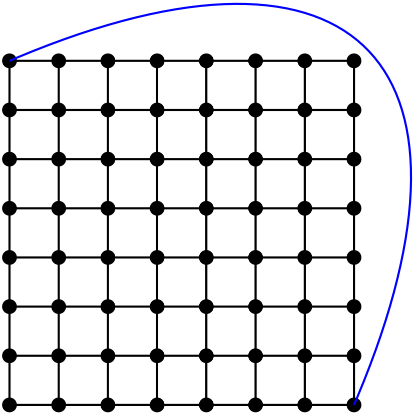
$$\begin{aligned} E_2 = \{\{v_{i,j}, v_{i',j'}\} \mid 1 \leq i, j, i', j' \leq 8 \\ \wedge \ (\ |i - i'| = 1, \ |j - j'| = 0 \\ \vee |i - i'| = 0, \ |j - j'| = 1 \\ \vee \ i = 1, j = 8, \ i' = 8, j' = 1 \ )\} \end{aligned}$$

- It holds:

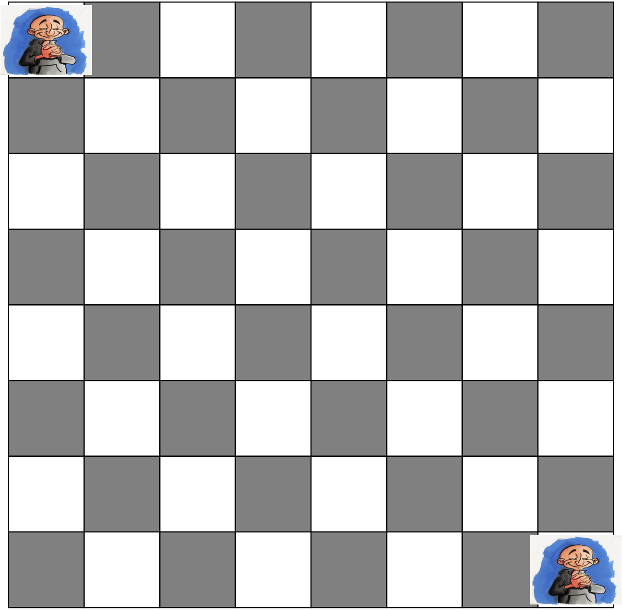
  A path exists for Mr. No.

  $\Leftrightarrow$ G_2 contains a Hamiltonian circuit.

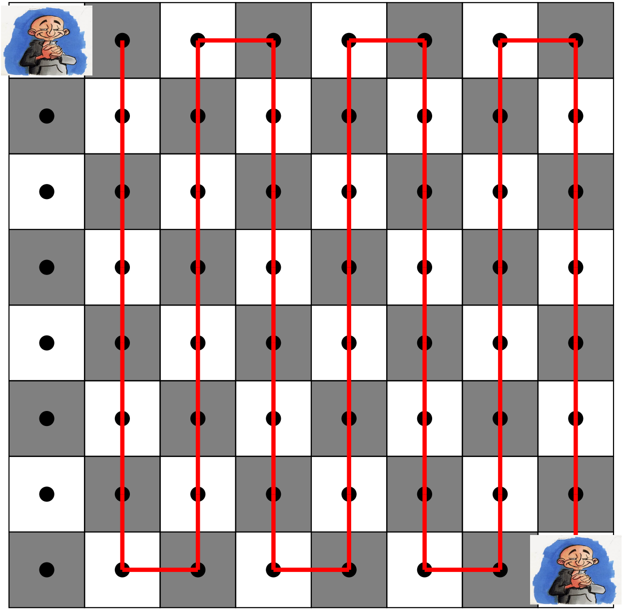
- Existence of a Hamiltonian circuit?



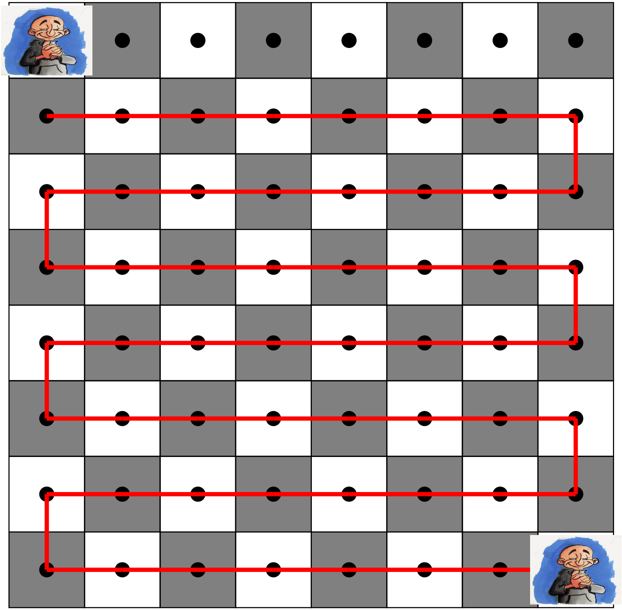
- No vertices of degree 0 or 1.
- No bridges.
- No articulation points.
- Application of Theorem of Dirac not possible: $\delta(G_1) = 2 < 32$.



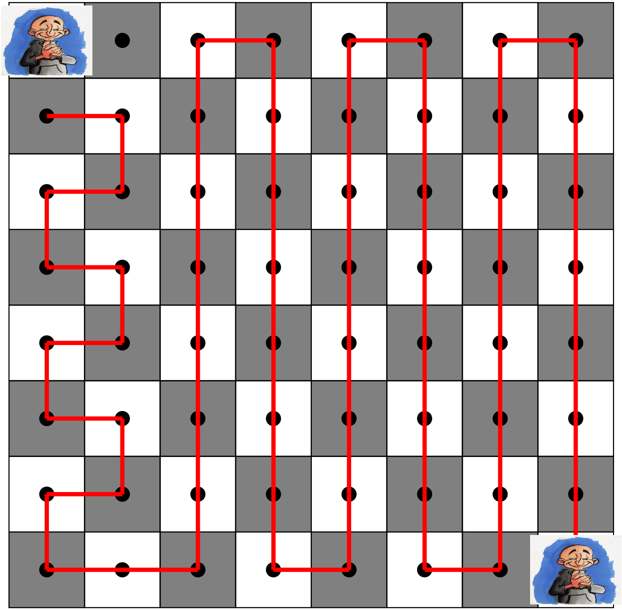
Mr. Go and Mr. No on a chessboard



Trial 1 for a path for Mr. No



Trial 2 for a path for Mr. No



Trial 3 for a path for Mr. No

Exercise 12

- Task: Does a path exist for Mr. No?
- Solution:

  Path does not exist.

  Reason:

  Each step of Mr. No changes the color of the traversed cell.

  When starting on a white cell, he would reach after 63 steps a black cell.

  But the upper left cell is white.

Traveling Salesman Problem (TSP):

Input: Graph $G = (V, E)$, cost function $c : E \to \mathbb{R}$.

Find: Hamiltonian circuit or tour $(v_1, v_2, \ldots, v_n, v_1)$

with minimum costs

$c(v_n, v_1) + \sum_{i=1}^{n-1} c(v_i, v_{i+1})$.

- Easy to understand.
- Hard to solve:

  Theorem 5: TSP is $\mathcal{NP}$-hard.

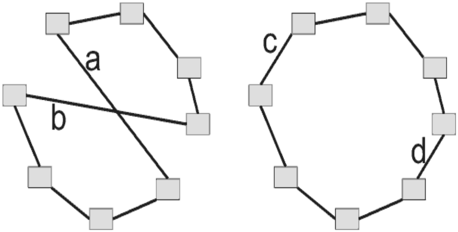
  Proof: Polynomial reduction from HCP to TSP.

- Many important applications:
  - public transport
  - tour planning
  - design of microchips
  - genome sequencing
- Gap between
  - few performance guarantees
  - phenomenal results

Shortest tour through 15.112 cities in Germany

1. Start with an arbitrary vertex.
2. In each step go to the nearest non-traversed vertex.
3. If all vertices are visited, return to the starting point.
4. Use the resulted tour as starting tour for the next steps.
5. For $k \leq n$ apply a k-OPT step, i.e.:
   Replace tour edges by non-tour edges so that
   - the edges are still a tour
   - the tour is better than the original one
6. Repeat step 5 as long as improving steps can be found.

Example of a 2-OPT step

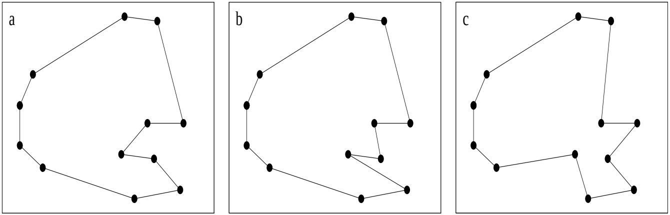


- Best TSP heuristic: [Helsgaun, 1998, improved: 2007]
- Main ideas: [Lin, Kernighan, 1971]
- Optimizations:
  1. Choose k small.
  2. For each vertex consider only the s best neighboring edges, the so-called candidate system.
     Helsgaun's main improvement:
     For each vertex do not consider the s shortest neighboring edges, but the s neighboring edges with a criterion based on so-called tolerances of the minimum spanning tree.
  3. Apply t (nearly) independent runs of the algorithm.
- The larger the algorithm parameters k, s and t are, the slower, but more effective is Helsgaun's Heuristic.

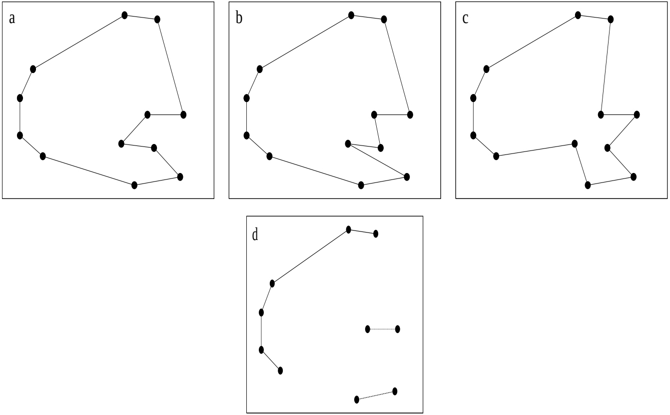
Joint work with

Changxing Dong, Boris Goldengorin, Paul Molitor, Dirk Richter

1. Using known heuristics, e.g., Helsgaun's Heuristic, find good starting tours.

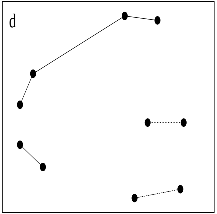


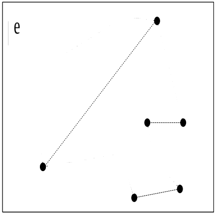
2. Find all common edges in these starting tours.



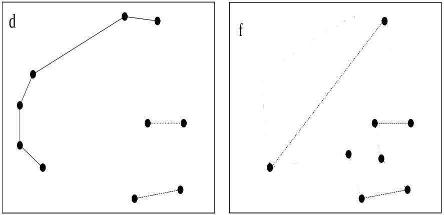
Such edges are called pseudo backbone edges.

3. Contract all edges of paths of pseudo backbone edges to one edge.



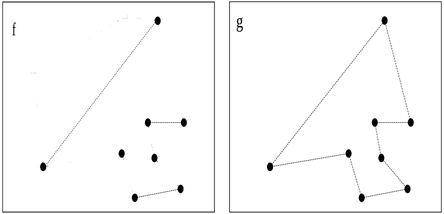


4. Create a new (reduced) instance by omitting the vertices, which lie on a path of pseudo backbone edges:

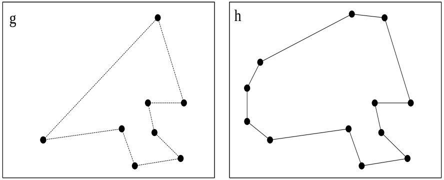


Fix the contracted edges, i.e., force them to be in the final tour.

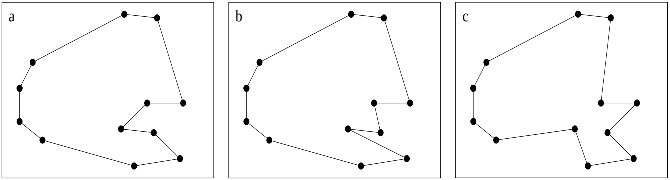
5. Apply Helsgaun's Heuristic to the new instance.



6. Re-contract the tour of the new instance to a tour of the original instance.



The last tour is the optimum one.



- Two advantages:
  1. Reduction of the set of vertices.
  2. Fixing of a part of the edges.

  $\Rightarrow$ Helsgaun's Heuristic can be applied with larger algorithm parameters k, s and t than for the original instance.
- The algorithm works rather good, if the starting tours are
  1. good ones
  2. not too similar
     (as otherwise the search space is restricted too strongly)

Competition: TSP homepage

(http://www.tsp.gatech.edu/)

- Large TSP datasets from practice:
  for comparison of exact algorithms and heuristics.
- 74 unsolved example instances:
  VLSI and national instances
- For 18 of 74 instances we have set a new record.
- 10 of 18 records are still up to date.

Our new records

Date	# Vertices	Date	# Vertices
24.05.2006	6,880	14.07.2008	28,534
22.10.2006	19,289	14.07.2008	39,603
20.03.2008	17,845	14.07.2008	56,769
28.03.2008	13,584	19.07.2008	104,815
01.04.2008	19,402	14.08.2008	52,057
27.05.2008	38,478	14.08.2008	238,025
23.06.2008	21,215	10.11.2008	29,514
24.06.2008	28,924	12.12.2008	47,608
30.06.2008	34,656	26.08.2009	32,892

Thanks
for your
attention!