SYDE/BME 411 – Assignment #2

November 12, 2020

Questions from the course textbook by Rao 4th Edition (freely accessible online at UW Library Website):

Problem 2.23
Problem 2.48- parts (a), (c)
Problem 2.50
Problem 6.23
Problem 6.47

Questions from the course textbook by Belegundu & Chandrupatla 2nd Edition (freely accessible online at UW Library Website):

Problem 3.2-part (iii)

A question from the course textbook by Arora 4th Edition (freely accessible online at UW Library Website):

Problem 4.140
Problem 8.52

Important Notes:

1- The written answers, calculations, etc. must be your own work, prepared by each student individually.

2- The use of Matlab, Python, or similar packages (by permission) is allowed, but related codes/files should be submitted.

3- The submission deadline is December 7, 2020 and it will be through a Dropbox on LEARN.

2.20 Determine whether the following matrix is positive definite:

$$[A] = \begin{bmatrix} -14 & 3 & 0 \\ 3 & -1 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

2.21 The potential energy of the two-bar truss shown in Fig. 2.11 is given by

$$f(x_1, x_2) = \frac{EA}{s}\left(\frac{1}{2s}\right)^2 x_1^2 + \frac{EA}{s}\left(\frac{h}{s}\right)^2 x_2^2 - Px_1 \cos\theta - Px_2 \sin\theta$$

where E is Young's modulus, A the cross-sectional area of each member, l the span of the truss, s the length of each member, h the height of the truss, P the applied load, θ the angle at which the load is applied, and x_1 and x_2 are, respectively, the horizontal and vertical displacements of the free node. Find the values of x_1 and x_2 that minimize the potential energy when $E = 207 \times 10^9$ Pa, $A = 10^{-5}$ m^2, $l = 1.5$ m, $h = 4.0$ m, $P = 10^4$ N, and $\theta = 30°$.

2.22 The profit per acre of a farm is given by

$$20x_1 + 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2$$

where x_1 and x_2 denote, respectively, the labor cost and the fertilizer cost. Find the values of x_1 and x_2 to maximize the profit.

2.23 The temperatures measured at various points inside a heated wall are as follows:

Distance from the heated surface as a percentage of wall thickness, d	0	25	50	75	100
Temperature, t (°C)	380	200	100	20	0

It is decided to approximate this table by a linear equation (graph) of the form $t = a + bd$, where a and b are constants. Find the values of the constants a and b that minimize the sum of the squares of all differences between the graph values and the tabulated values.

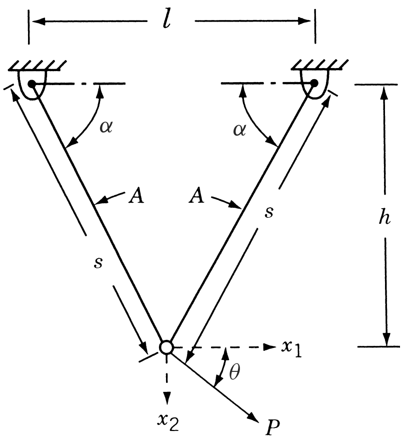

Figure 2.11 Two-bar truss.

2.42 Find the dimensions of an open rectangular box of volume V for which the amount of material required for manufacture (surface area) is a minimum.

2.43 A rectangular sheet of metal with sides a and b has four equal square portions (of side d) removed at the corners, and the sides are then turned up so as to form an open rectangular box. Find the depth of the box that maximizes the volume.

2.44 Show that the cone of the greatest volume that can be inscribed in a given sphere has an altitude equal to two-thirds of the diameter of the sphere. Also prove that the curved surface of the cone is a maximum for the same value of the altitude.

2.45 Prove Theorem 2.6.

2.46 A log of length l is in the form of a frustum of a cone whose ends have radii a and $b(a > b)$. It is required to cut from it a beam of uniform square section. Prove that the beam of greatest volume that can be cut has a length of $al/[3(a - b)]$.

2.47 It has been decided to leave a margin of 30 mm at the top and 20 mm each at the left side, right side, and the bottom on the printed page of a book. If the area of the page is specified as 5×10^4 mm^2, determine the dimensions of a page that provide the largest printed area.

2.48

$$\text{Minimize } f = 9 - 8x_1 - 6x_2 - 4x_3 + 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3$$

subject to

$$x_1 + x_2 + 2x_3 = 3$$

by (**a**) direct substitution, (**b**) constrained variation, and (**c**) Lagrange multiplier method.

2.49

$$\text{Minimize } f(\mathbf{X}) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

subject to

$$g_1(\mathbf{X}) = x_1 - x_2 = 0$$

$$g_2(\mathbf{X}) = x_1 + x_2 + x_3 - 1 = 0$$

by (**a**) direct substitution, (**b**) constrained variation, and (**c**) Lagrange multiplier method.

2.50 Find the values of x, y, and z that maximize the function

$$f(x, y, z) = \frac{6xyz}{x + 2y + 2z}$$

when x, y, and z are restricted by the relation $xyz = 16$.

2.51 A tent on a square base of side $2a$ consists of four vertical sides of height b surmounted by a regular pyramid of height h. If the volume enclosed by the tent is V, show that the area of canvas in the tent can be expressed as

$$\frac{2V}{a} - \frac{8ah}{3} + 4a\sqrt{h^2 + a^2}$$

Also show that the least area of the canvas corresponding to a given volume V, if a and h can both vary, is given by

$$a = \frac{\sqrt{5}h}{2} \quad \text{and} \quad h = 2b$$

6.15 Find a suitable transformation or scaling of variables to reduce the condition number of the Hessian matrix of the following function to one:

$$f = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 + 10$$

6.16 Determine whether the following vectors serve as conjugate directions for minimizing the function $f = 2x_1^2 + 16x_2^2 - 2x_1x_2 - x_1 - 6x_2 - 5$.

(a) $\mathbf{S}_1 = \begin{Bmatrix} 15 \\ -1 \end{Bmatrix}, \quad \mathbf{S}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

(b) $\mathbf{S}_1 = \begin{Bmatrix} -1 \\ 15 \end{Bmatrix}, \quad \mathbf{S}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

6.17 Consider the problem:

$$\text{Minimize } f = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$

Find the solution of this problem in the range $-10 \le x_i \le 10$, $i = 1, 2$, using the random jumping method. Use a maximum of 10,000 function evaluations.

6.18 Consider the problem:

$$\text{Minimize } f = 6x_1^2 - 6x_1x_2 + 2x_2^2 - x_1 - 2x_2$$

Find the minimum of this function in the range $-5 \le x_i \le 5$, $i = 1, 2$, using the random walk method with direction exploitation.

6.19 Find the condition number of each matrix.

(a) $[A] = \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}$

(b) $[B] = \begin{bmatrix} 3.9 & 1.6 \\ 6.8 & 2.9 \end{bmatrix}$

6.20 Perform two iterations of the Newton's method to minimize the function

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

from the starting point $\begin{Bmatrix} -1.2 \\ 1.0 \end{Bmatrix}$.

6.21 Perform two iterations of univariate method to minimize the function given in Problem 6.20 from the stated starting vector.

6.22 Perform four iterations of Powell's method to minimize the function given in Problem 6.20 from the stated starting point.

6.23 Perform two iterations of the steepest descent method to minimize the function given in Problem 6.20 from the stated starting point.

6.24 Perform two iterations of the Fletcher–Reeves method to minimize the function given in Problem 6.20 from the stated starting point.

6.25 Perform two iterations of the DFP method to minimize the function given in Problem 6.20 from the stated starting vector.

6.26 Perform two iterations of the BFGS method to minimize the function given in Problem 6.20 from the indicated starting point.

6.45 Minimize $f = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$ starting from point (0, 0) using Powell's method. Perform four iterations.

6.46 Minimize $f(x_1, x_2) = x_1^4 - 2x_1^2x_2 + x_1^2 + x_2^2 + 2x_1 + 1$ by the simplex method. Perform two steps of reflection, expansion, and/or contraction.

6.47 Solve the following system of equations using Newton's method of unconstrained minimization with the starting point

$$\mathbf{X}_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$2x_1 - x_2 + x_3 = -1, \quad x_1 + 2x_2 = 0, \quad 3x_1 + x_2 + 2x_3 = 3$$

6.48 It is desired to solve the following set of equations using an unconstrained optimization method:

$$x^2 + y^2 = 2, \quad 10x^2 - 10y - 5x + 1 = 0$$

Formulate the corresponding problem and complete two iterations of optimization using the DFP method starting from $\mathbf{X}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$.

6.49 Solve Problem 6.48 using the BFGS method (two iterations only).

6.50 The following nonlinear equations are to be solved using an unconstrained optimization method:

$$2xy = 3, \quad x^2 - y = 2$$

Complete two one-dimensional minimization steps using the univariate method starting from the origin.

6.51 Consider the two equations

$$7x^3 - 10x - y = 1, \quad 8y^3 - 11y + x = 1$$

Formulate the problem as an unconstrained optimization problem and complete two steps of the Fletcher–Reeves method starting from the origin.

6.52 Solve the equations $5x_1 + 3x_2 = 1$ and $4x_1 - 7x_2 = 76$ using the BFGS method with the starting point (0, 0).

6.53 Indicate the number of one-dimensional steps required for the minimization of the function $f = x_1^2 + x_2^2 - 2x_1 - 4x_2 + 5$ according to each scheme:

(a) Steepest descent method
(b) Fletcher–Reeves method
(c) DFP method
(d) Newton's method
(e) Powell's method
(f) Random search method
(g) BFGS method
(h) Univariate method

For instance, we can supply the gradient in the user subroutine and avoid the possibly expensive automatic divided difference scheme (the default) by switching on the corresponding feature as

```
options=optimset('GradObj', 'on')
```

fminunc is then executed using the command

```
[Xopt,fopt,iflag,output] = fminunc('testfun', X, options)
```

with a subroutine getfun that provides the analytical gradient in the vector DF as

```
function [f, Df] = getfun(X)
 f = ...
 Df(1) = ...;  Df(2) = ...; Df(N) = ...;
```

COMPUTER PROGRAMS

STEEPEST, FLREEV, DFP

PROBLEMS

P3.1. Plot contours of the function $f = x_1^2 x_2 - x_1 x_2 + 8$, in the range $0 < x_1 < 3$, $0 < x_2 < 10$. You may use Matlab or equivalent program.

P3.2. For the functions given in the following, determine (a) all stationary points and (b) check whether the stationary points that you have obtained are strict local minima, using the sufficiency conditions:

(i) $f = 3x_1 + \dfrac{100}{x_1 x_2} + 5x_2$

(ii) $f = (x_1 - 1)^2 + x_1 x_2 + (x_2 - 1)^2$

(iii) $f = \dfrac{x_1 + x_2}{3 + x_1^2 + x_2^2 + x_1 x_2}$

P3.3. (a) What is meant by a "descent direction"? (Answer this using an inequality.)

(b) If $\mathbf{d}$ is a solution of $\mathbf{W}\,\mathbf{d} = -\nabla f$, then state a sufficient condition on $\mathbf{W}$ that guarantees that $\mathbf{d}$ is a descent direction. Justify/prove your statement.

4.125 Exercise 4.72
4.126 Exercise 4.73
4.127 Exercise 4.74
4.128 Exercise 4.75
4.129 Exercise 4.76
4.130 Exercise 4.77
4.131 Exercise 4.78

Section 4.8 Global Optimality

4.132 *Answer true or false.*

1. A linear inequality constraint always defines a convex feasible region.
2. A linear equality constraint always defines a convex feasible region.
3. A nonlinear equality constraint cannot give a convex feasible region.
4. A function is convex if and only if its Hessian is positive definite everywhere.
5. An optimum design problem is convex if all constraints are linear and the cost function is convex.
6. A convex programming problem always has an optimum solution.
7. An optimum solution for a convex programming problem is always unique.
8. A nonconvex programming problem cannot have global optimum solution.
9. For a convex design problem, the Hessian of the cost function must be positive semidefinite everywhere.
10. Checking for the convexity of a function can actually identify a domain over which the function may be convex.

4.133 Using the definition of a line segment given in Eq. (4.71), show that the following set is convex $S = \{\mathbf{x} \mid x_1^2 + x_2^2 - 1.0 \le 0\}$

4.134 Find the domain for which the following functions are convex: (1) sin x, (2) cos x.

Check for convexity of the following functions. If the function is not convex everywhere, then determine the domain (feasible set S) over which the function is convex.

4.135 $f(x_1, x_2) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + 7$

4.136 $f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + 3$

4.137 $f(x_1, x_2) = x_1^3 + 12x_1x_2^2 + 2x_2^2 + 5x_1^2 + 3x_2$

4.138 $f(x_1, x_2) = 5x_1 - \dfrac{1}{16}x_1^2x_2^2 + \dfrac{1}{4x_1}x_2^2$

4.139 $f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$

4.140 $U = \dfrac{(21.9 \times 10^7)}{V^2C} + (3.9 \times 10^6)C + 1000\ V$

4.141 Consider the problem of designing the "can" formulated in Section 2.2. Check convexity of the problem. Solve the problem graphically and check the KKT conditions at the solution point.

Formulate and check convexity of the following problems; solve the problems graphically and verify the KKT conditions at the solution point.

4.142 Exercise 2.1

$2x_1 + x_2 \le 6$
$x_1, x_2 \ge 0$

8.45 Minimize $f = -x_1 - 4x_2$
subject to $x_1 + x_2 \le 16$
$x_1 + 2x_2 \le 28$
$24 \ge 2x_1 + x_2$
$x_1, x_2 \ge 0$

8.46 Minimize $f = x_1 - x_2$
subject to $4x_1 + 3x_2 \le 12$
$x_1 + 2x_2 \le 4$
$4 \ge 2x_1 + x_2$
$x_1, x_2 \ge 0$

8.47 Maximize $z = 2x_1 + 3x_2$
subject to $x_1 + x_2 \le 16$
$-x_1 - 2x_2 \ge -28$
$24 \ge 2x_1 + x_2$
$x_1, x_2 \ge 0$

8.48 Maximize $z = x_1 + 2x_2$
subject to $2x_1 - x_2 \ge 0$
$2x_1 + 3x_2 \ge -6$
$x_1, x_2 \ge 0$

8.49 Maximize $z = 2x_1 + 2x_2 + x_3$
subject to $10x_1 + 9x_3 \le 375$
$x_1 + 3x_2 + x_3 \le 33$
$2 \ge x_3$
$x_1, x_2, x_3 \ge 0$

8.50 Maximize $z = x_1 + 2x_2$
subject to $-2x_1 - x_2 \ge -5$
$3x_1 + 4x_2 \le 10$
$x_1 \le 2$
$x_1, x_2 \ge 0$

8.51 Minimize $f = -2x_1 - x_2$
subject to $-2x_1 - x_2 \ge -5$
$3x_1 + 4x_2 \le 10$
$x_1 \le 3$
$x_1, x_2 \ge 0$

8.52 Maximize $z = 12x_1 + 7x_2$
subject to $2x_1 + x_2 \le 5$
$3x_1 + 4x_2 \le 10$
$x_1 \le 2$
$x_2 \le 3$
$x_1, x_2 \ge 0$

8.53 Maximize $z = 10x_1 + 8x_2 + 5x_3$
subject to $10x_1 + 9x_2 \le 375$
$5x_1 + 15x_2 + 3x_3 \le 35$