Energy flux in a conducting wire

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1 Introduction

Recently, high reach youtubers have explained how electromagnetic energy is flowing in electrical circuits composed of simple elements. While the general idea is correct, a particularly popular video[2] contains the claim that "energy doesn't flow in wires", which, as we will see, is incorrect. This document is about the verification of this claim, not about how the energy flows in a circuit (and the space around it).

From now and on, the "system" consists of "the wire" which conducts electricity. We will assume that the wire carries a uniform current in its interior, which is a reasonable assumption for DC (or low frequency AC), and this occurs when the charge distribution on the surface of the wire is linear along the wire (yet another reason why the Youtube video(s) are misleading, since they show a uniform charge density along the wire instead of a linearly increasing/decreasing one.).

2 The quick way

From thermodynamics, we know that $dU = TdS + \overline{\mu}dN$ in the system. We have neglected thermal expansion for simplicity (no volume change), but this does not lose generality. This relation expresses a change in the internal energy U in terms of thermodynamics variables. T is the absolute temperature, S is the entropy, N is the number of particles and $\overline{\mu}$ is the electrochemical potential, which is worth $\mu - eV$ where μ is the chemical potential, e is the electron charge and V is the electrostatics potential (related to the electric field via $\vec{E} = -\nabla V$).

In out of equilibrium thermodynamics (which is our case, since there is an electric current, and, as we will see, a thermal gradient), time makes an appearance, and we obtain the relation

$$\vec{J_U} = T\vec{J_S} + \overline{\mu}\vec{J_e} \tag{1}$$

which deals with fluxes. The left side represent the internal energy flux (its units are that of energy per surface area per unit time, or simply power per unit area), the right side contains the entropy flux and the current density (which is proportional to the particle flux). These quantities are defined at every single point in the wire. Using a thermodynamics relation between heat and entropy, we can write $\vec{J_S} = \vec{J_Q}/T$ (yes, it's an equality rather than an inequality despite the fact that we are dealing with irreversible processes). And we can see that as long as there is an electric current, there is an associated energy flux, which exists everywhere the current density is defined. In particular, in a current-carrying wire, there is a non zero energy flux that goes along the wire, inside the wire. And that's it, this is enough to conclude that energy does flow in wires when there is a current.

3 The in-depth way

The Youtube videos focus on the Poynting vector, which does not catch the whole energy flux inside the current-carrying wire. As we will see and as the videos correctly point out, the Poynting vector is radial and points inward the wire. We will, however, see that it makes up for a part of the internal energy flux of the wire.

3.1 Détour with the Poynting vector

The Poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$ is the energy flux coming from the EM fields. The continuity equation applied to the EM energy density u yields

$$\frac{\partial u}{\partial t} = -\nabla \cdot \vec{S} - \vec{J_e} \cdot \vec{E}.$$

So that in steady state, $\nabla \cdot \vec{S} = -\vec{J_e} \cdot \vec{E}$. We can rewrite $\vec{E}$ using Ohm's law $\vec{J_e} = -\sigma \nabla V$. By doing so, one finds that

$$\nabla \cdot \vec{S} = -\rho |\vec{J_e}|^2. \tag{2}$$

From this expression, it is evident that the Poynting vector does not catch the whole energy involved in the system, because in steady state, the divergence of the (total) energy flux must vanish. Anyway, let's ignore this for now. In the wire, $\vec{E}$ points along the wire, because $\vec{E}$ is directly proportional to $\vec{J_e}$, and it is constant everywhere in the wire. $\vec{B}$ on the other hand has direction $\hat{\theta}$ (the unit vector in cylindrical coordinates), and goes like r because it is proportional to the current enclosed by a cross section of radius r divided by the perimeter of this cross section. Mathematically, $\vec{B} = \frac{\mu_0 I_0 r}{2\pi R^2}\hat{\theta}$. This means the magnetic field vanishes at the center of the wire, and so does the Poynting vector. The direction of the Poynting vector is radially inward the wire, i.e. it points towards the center of the wire ($-\hat{r}$), and its magnitude goes like r, i.e. its magnitude decreases from the surface of the wire towards its center. In the case of zero resistivity (or infinite conductivity), the electric field vanishes inside the wire, and so does the Poynting vector.

3.2 Back to thermodynamics

We go back to where we were, that is eq. (1).

In steady state, $\nabla \cdot \vec{J_U} = 0$, i.e. the heat that enters any part of the wire must leave it (no accumulation of energy anywhere). Then, using $\vec{J_S} = \vec{J_Q}/T$ where $\vec{J_Q}$ is the heat flux, given by Fourier's law (the analog of Ohm's law for the thermal flux rather than electric flux): $\vec{J_Q} = -\kappa \nabla T$, one finds the heat equation

$$\nabla \cdot (-\kappa \nabla T) - \rho |\vec{J_e}|^2 = 0 \tag{3}$$

which is the equation temperature must satisfy in the wire (regardless of the boundary conditions that we choose to apply). This equation tells us that in any volume element, the quantity of energy generated by Joule heat must be conducted away via heat conduction, in steady state. It also shows that as long as there is a Joule heat, $\nabla T \neq \vec{0}$, i.e. there must exist a thermal gradient at every single point in the wire. It is impossible to keep the wire in isothermal conditions as long as Joule heat is present. It turns out that the thermal gradient must points inward the wire, in the radial direction. Furthermore, due to the symmetry, it must vanish at the center of the wire.

Looking back at equations (2) and (3), we can see that $\nabla \cdot (-\kappa \nabla T) + \nabla \cdot \vec{S} = 0$, from which we can infer that $\vec{S} = \kappa \nabla T$. As a checkup, we see that these expressions share the same direction, and magnitude (in particular they vanish at the center of the wire). Their magnitude is maximum at the boundaries of the wire. To complete the proof, see appendix A. Solving the heat equation in the particular case of the temperature kept fixed at the surface of the wire yields $T(r) = \frac{\rho |\vec{J_e}|^2}{4\kappa}(R^2 - r^2) + T_0$, which implies $\nabla T = -\frac{\rho |\vec{J_e}|^2 r}{2\kappa}\hat{r}$. I would conjecture that imposing Dirichlet boundary conditions at the surface of the wire does not lose generality in that if, for example, radiation was taken into account (or Neumann/Robin boundary conditions), then $T(\vec{r})$ would be shifted by a certain amount everywhere in the material. In particular ∇T would remain unchanged, which is what matters.

4 Expression of the energy flux in the wire

We have all the elements to write down the expression for the energy flux in the wire, for both the case of finite and vanishing resistivity. In the case of finite ρ, the condition that the current density is uniform implies that $\overline{\mu} = \overline{\mu_0} + \mu_1 z$, in other words that the potential drop is linear along the wire. We can go a little bit further, $\nabla \overline{\mu} = \mu_1 \hat{z} = -\rho |\vec{J_e}| \hat{z}$

Referring back to eq 1, we finally reach

$$\boxed{\vec{J_U} = \frac{\rho |\vec{J_e}|^2 r}{2}\hat{r} + (\overline{\mu_0} - \rho |\vec{J_e}| z)|\vec{J_e}|\hat{z}} \tag{4}$$

This expression can be used to sketch the total energy flux in the wire. In the case of zero resistivity, only a constant $\hat{z}$ component survives. As the resistivity is increased, there appear a radial component whose magnitude increases the further we are from the center, as well as a decrease/increase in the longitudinal component, which reflects the potential drop due to the resistitivity.

A Proof that the Poynting vector is equal to the radially conducted heat

I use cylindrical coordinates (r, θ, z). $\vec{S} = |\vec{S}|\hat{r}$. The divergence in cylindrical coordinate yields $\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + ...$ So that $\nabla \cdot \vec{S} = \frac{1}{r}\left(S_r + r\frac{\partial S}{\partial r}\right)$. But we know that $S_r \propto r$, so that $S_r = ar$, and so $\nabla \cdot \vec{S} =$ constant, and that makes sense, since we know that it should give Joule heat (which does not depend on r). If we look at the expression for ∇T obtained by solving the heat equation, from it we can conclude that $\kappa \nabla T \propto r$, which makes sense, as this quantity is equal to the Poynting vector which we know is proportional to r.

There is a condition that must hold if $\vec{S} = \kappa \nabla T$:

$$-\frac{\rho|\vec{J}_e|^2 r\hat{r}}{2} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \mu_0 \rho |\vec{J}_e| \hat{z} \times \frac{\mu_0 I_0 r}{2\pi R^2}\hat{\theta}$$

and indeed, it holds (do the math if you want.). This fully justifies the equality.

References

[1] Charles A Domenicali. "Irreversible thermodynamics of thermoelectricity". In: *Reviews of Modern Physics* 26.2 (1954), p. 237.

[2] *The Big Misconception About Electricity.* Youtube. 2021. URL: `https://www.youtube.com/watch?v=bHIhgxav9LY`.