(2) (2). Let $z \in \mathbb{R}^n$ be an n-vector. Show that $A = zz^T$ is positive semidefinite. (1)

Sol^n. A square matrix $A = zz^T$ is positive semidefinite if it is symmetric and

the A symm. matrix

Let λ be an eigen value of A and x be the corresponding eigen vector.

$$Ax = \lambda x$$

$$x^T A x = \lambda x^T x$$

$$\text{or } \lambda = \frac{x^T A x}{x^T x}$$

$$\lambda \geq 0 \quad \text{so,} \quad \frac{x^T A x}{x^T x} \geq 0 \Rightarrow x^T A x \geq 0 \ \forall x \in \mathbb{R}^n, \text{ nonzero } x.$$

for positive semidefin..

Given $z \in \mathbb{R}^n$.

So. $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}_{n \times 1}$, $z^T = [z_1, z_2 \ldots z_n]_{1 \times n}$

$$\text{so } A = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} [z_1, z_2 \ldots z_n] = \begin{bmatrix} z_1^2 & z_1 z_2 & \ldots & z_1 z_n \\ z_2 z_1 & z_2^2 & \ldots & z_2 z_n \\ z_3 z_1 & z_3 z_2 & z_3^2 & \ldots \\ \vdots & & & \\ z_n z_1 & \ldots & \ldots & z_n^2 \end{bmatrix}$$

All leading diagonal elements are positive or (pivots ≥ 0) so:

$A = zz^T$ positive semi defin matrix

it also symmetric matrix

So, it's eigen value $\lambda_1 \ldots \lambda_n \geq 0$

$$A = \begin{bmatrix} z_1^2 & z_1 z_2 & \cdots & z_1 z_n \\ z_2 z_1 & z_2^2 & \cdots & z_2 z_n \\ \vdots & \vdots & \ddots & \\ z_n z_1 & z_n z_2 & \cdots & z_n^2 \end{bmatrix}$$

we Take.

So, $x^T A x$.

when $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

$$\Rightarrow \underset{1 \times n}{\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} z_1^2 & z_1 z_2 & \cdots & z_1 z_n \\ z_2 z_1 & z_2^2 & \cdots & z_2 z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_n z_1 & z_n z_2 & \cdots & z_n^2 \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} z_1^2 x_1 + z_1 z_2 x_2 + z_1 z_3 x_3 + \cdots + z_1 z_n x_n \\ z_2 z_1 x_1 + z_2^2 x_2 + \cdots + z_2 z_n x_n \\ \vdots \\ z_n z_1 x_1 + z_n z_2 x_2 + \cdots + z_n^2 x_n \end{bmatrix}$$

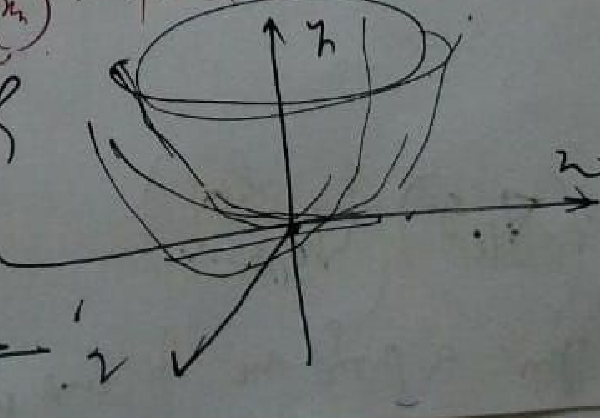
$$\Rightarrow \Big[z_1^2 x_1^2 + z_1 z_2 x_2 x_1 + z_1 z_3 x_3 x_1 + \cdots + z_1 z_n x_n^2 + z_2 z_1 x_1 x_2 + z_2^2 x_2^2 + \cdots$$
$$\cdots + z_2 z_n x_n x_2 + \cdots$$
$$\cdots z_n z_1 x_1 x_n + z_n z_2 x_2 x_n + \cdots \quad z_n^2 x_n^2 \Big]$$

$x^T A x \geq 0 \quad \forall x \in \mathbb{R}^n$. quadratic fun. —

$(z_1 x_1)^2 + z_1 z_2 x_2 x_1 + \cdots \quad z_n z_n (x_n)^2$ — form) In general so.

Convex Bowl graph

Positive Semi Definite



(2) (b) Let $z \in \mathbb{R}^n$ be a nonzero n-vector. Let $A = zz^T$. What is the null space of A? What is the rank of A.

:- $z \in \mathbb{R}^n$ be a nonzero n-vector ∴ $Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$.

$A = zz^T$.

$$A = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix} \begin{bmatrix} z_1, z_2, \dots z_n \end{bmatrix}$$

$$= \begin{bmatrix} z_1^2 & z_1 z_2 & \dots & z_1 z_n \\ z_2 z_1 & z_2^2 & \dots & z_2 z_n \\ z_3 z_1 & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \\ z_n z_1 & z_n z_2 & \dots & z_n^2 \end{bmatrix}$$

Symmetric matrix.

ii) $x^T A x$ is a $1 \times n$ matrix.

$$\left[z_1^2 x_1^2 + z_1 z_2 x_2 x_1 + \dots z_1 z_n x_n, \; z_2 z_1 x_2 x_1 + z_2^2 x_2^2 + \dots \dots + z_n z_1 x_n x_1 + \dots z_n^2 x_n^2 \right]$$

iii) null space of A is at origin mean only zero vector. ii's of Rank matrix

null space = zero vector only.