(1)

(3) (a) Suppose that the matrix $A \in \mathbb{R}^{n\times n}$ is diagonalizable that is $A = T\Lambda T^{-1}$ for an invertible matrix $T \in \mathbb{R}^{n\times n}$, where $\Lambda = \mathrm{diag}(\lambda_1 \dots \lambda_n)$ is diagonal, use the notation $t^{(i)}$ for all columns of T so that $T = [t^{(1)} \dots t^{(n)}]$ where $t^{(i)} \in \mathbb{R}^n$. Show that $At^{(i)} = \lambda_i t^{(i)}$ so that the eigen values/eigenvectors pairs of A are $(t^{(i)}, \lambda_i)$.

Soln:— Let
$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & a_{nn} \\ a_{n1} & a_{n2} & & \end{bmatrix}_{n\times n} = T\Lambda T^{-1}.$$

where $T \in \mathbb{R}^{n\times n}$

$\because$ $T_{n\times n}$ so T^{-1} also $n\times n$ matrix —

an A can decompose in term of $(T\Lambda T^{-1})$ only when all in same order as is given y_3, after that.

If we know that A & B similar matrix only when $B = TAT^{-1}$ $(B = A)$ when A & B — semi positive definite matrix

Similar matrix have similar eigen values.

$\left.\begin{array}{l} B = (TAT^{-1}) \\ \quad = (T^{-1}AT) \end{array}\right\}$

So, eigen value of B = $Ax = \lambda x$

$ATT^{-1}x = \lambda x$

multiply T^{-1} $\quad (T^{-1}AT)T^{-1}x = \lambda T^{-1}x$

$B(T^{-1}x) = \lambda(T^{-1}x)$.

② So, the eigen vector of $B = T^{-1}$ of eigen vector of A.

So, we can write in

$AT = TA$.

$$\left[\quad\right]\left[\quad\right] = \left[\quad\right]\left[\quad\right]$$

as $A = T \lambda T^{-1}$

diagonal matrix

$\therefore T^{-1} A T = \lambda$.

So, A is similar to λ _

So, (eigenvalue & eigen vector pair) $(\underline{t}^{(i)}, \lambda_i)$

where $t^{(i)}$ = columns. with eigen value

If not same λ_i it is not happens.

Any matrix can be decomposed of

$L = M^{T} \lambda M$

$= M \lambda M^{T}$

λ = diagonal matrix _

(5) (c) show that if A is PSD, then $\lambda_i(A) \geq 0$ for each i.

Sol^n :-

A is positive semi definite (PSD), for n-order. $A = A^T$

As we know that $x^T A x \geq 0$ &

Given $\lambda = \lambda_i(A)$ denote the i^{th} eigenvalues of A.

$$A X = A[x_1 x_2 \ldots x_n] = [\lambda_1 x_1 \ldots \lambda_n x_n]$$

$$= [x_1 \ldots x_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & & \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

$=$ diagonal $\lambda_1, \lambda_2 \ldots \lambda_n$

$\therefore$

$A X = X \lambda$. multiply

$X^T A X = X^T X \lambda$. $\because X^T X = 1$

$= \lambda$. when $\lambda_i \geq 0$ & PSD

$[u^i]^2 \geq 0$

So, all eigenvalue ≥ 0

So, $\lambda_i(A) \geq 0$ for each i $\Rightarrow$ square function

or

$A X = X \lambda$

$X^T A X = \lambda$.

$A = X \lambda X^T$

as $v^{(i)}$ its eigenvalue $= \lambda_i(A)$. $\Rightarrow$ $x = v^{(i)}$

$(v^i)^T A v^{(i)} \Rightarrow$

$= (v^i)^T (v^i) \lambda$

$= \lambda_i |v^i|^2$

$\because$ diagonal always positive.

if $U = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}$, $U^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

$\therefore$

$$U^T U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ 0 & 0 & 1 & \ddots & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I.$$

So.

$$AU = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & & \vdots \\ 0 & & \ddots & x_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{n \times n}$$

$$= \begin{bmatrix} x_1 & 0 & & \\ 0 & x_2 & & \\ 0 & & \ddots & \\ 0 & & & x_n \\ & & & & \ddots \\ 0 & \cdots & & & x_n \end{bmatrix} = U\Sigma$$

$$U^T U = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

③ (b) Let A be symmetric, show that if $U = [u^{(1)} \dots u^{(n)}]$ is orthogonal where $u^{(i)} \in \mathbb{R}^n$ and $A = U\Lambda U^T$, then $u^{(i)}$ is a eigenvector of A and $Au^{(i)} = \lambda_i u^{(i)}$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots \lambda_n)$.

Sol:- Since,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix}.$$

& Λ is symm.

$\underline{A_{n\times n}}$ a $\underline{\Lambda_{n\times n}}$ & $\Lambda = \Lambda^T$

So,

$A = U\Lambda U^T$

$A^T = (U\Lambda U^T)^T = (U^T)^T . \Lambda^T . U^T = U\Lambda^T . U^T$.

So, $\boxed{U\Lambda U^T = U\Lambda^T U^T}$.

So, $AU = U\Lambda U^T . U$ $\quad$ $U^T U = I$

orthon., [Identity matrix]

$\boxed{AU = U\Lambda}$

Simil. Can. $AB = BA$ only when A & B have similar matrix has same eigenvalue. & eigenvector.

same eigenvalue, further eigen value

So $\begin{bmatrix} \lambda_1 & 0 & \\ 0 & \lambda_2 & \\ \vdots & 0 & \ddots \; \lambda_n \\ \vdots & & \cdots \end{bmatrix} \begin{bmatrix} & \end{bmatrix} = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ \vdots & \lambda_2 \\ \vdots & \ddots \\ \vdots & \lambda_n \end{bmatrix}$