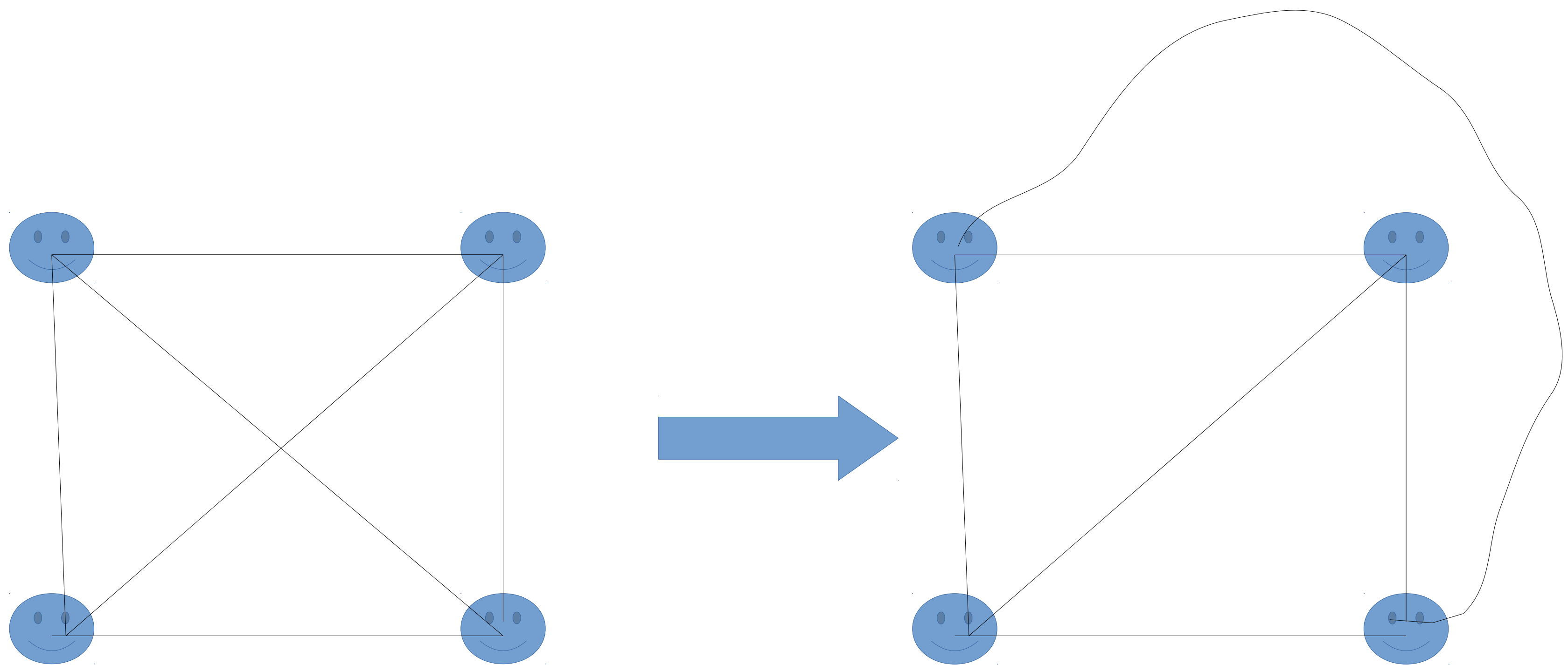
Planner Graph
&
Graph Coloring(Basic)

SGC

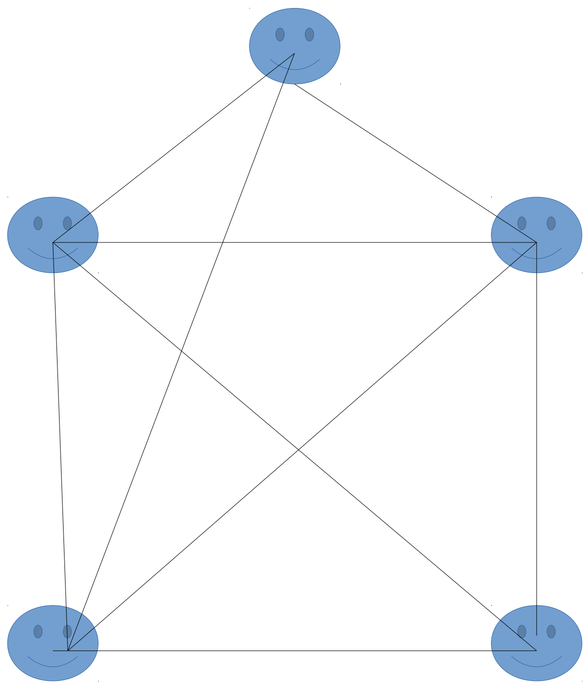
Planner Graph

- A planar graph is a graph that can be drawn in such a way that no edges cross.

  Example: Tree, K4

- 

Draw Planner Graph



Euler’s planar graph theorem:

- If G is a connected planer graph with n vertices, e edges, and f faces, then $n - e + f = 2$.

  Proof: By Induction on e.
- When $e = 0$, $n=1$, $f=1$, hence, $1 -0 +1 =2$.
- Induction Hypothesis: true for $e = k$
- We have to prove that it is true for $e = k+1$

Euler Theorem Proof

- Case 1: For Tree: $e = n-1$, $f = 1$, $n-e+f = n - n + 1 + 1 = 2$
- Case 2: Not Tree (but connected) => the graph must has a cycle!
- Let $e = k+1$
- Delete an edge from the cycle => $e = k$
- It will reduce f by 1, $f = f-1$
- In new graph: $n-k+(f-1) = 2 \Rightarrow n-(k+1) + f = 2$.

Properties of Tree

- Connected
- No cycle
- Planner
- n-1 edges

If G is a planar graph with $n \geq 3$ vertices and e edges, then $e \leq 3n - 6$.

- Proof :
- Case 1 : G is connected :
- Construct edge – face matrix (M).
- m(ij) =1 ; if edge i is adjecent to face j / otherwise 0.
- Compute the number of 1s in M: Let it is x.
- If we compute by row, $x \leq 2e$
- If we compute by column, $x \geq 3f$
- $3f \leq x \leq 2e \Rightarrow 3f \leq 2e \Rightarrow 3(2-n+e) \leq 2e \Rightarrow e \leq 3n - 6$.

If G is a planar graph with $n \geq 3$ vertices and e edges, then $e \leq 3n - 6$.

- Proof :
- Case 2 : G (e,n) is disconnected :
- Make G connected by adding edges => new graph G'
- The theorem is true of G' (e',n)
- Hence, $e' \leq 3n - 6$
- But $e < e'$
- Hence, $e < 3n - 6$
- Hence proved

- $e = 3n - 6$ is achievable, for example, on the graphs K_3 ($n = 3$, $e = 3$) and K_4 ($n = 4$, $e = 6$)
- **Theorem:** K_5 is nonplanar.
- Proof : K_5 has $n = 5$ vertices and $e = 10$ edges. Since $e > 3n - 6 = 9$, K_5 cannot be planar.

"if $e \leq 3n - 6$, then G is nonplanar" - is false.

- The complete bipartite graph K (3,3) has $n = 6$ vertices and $e = 9 < 12 = 3n - 6$ edges. Nevertheless, K (3,3) is nonplaner
- Excercise: Prove that K (3,3) is non planner.

  [Follows from the theorem: If a connected planar simple graph has e edges and n vertices with $v \geq 3$ and **no circuits of length three**, then $e \leq 2v - 4$. (Proof is similar to previous one only one change, $x \geq 4f$)]

Bipartite Graph

- No cycle of odd degree
- Tree is a bibartite graph

•Kuratowski's theorem

- Every nonplanar graph contains inside it K5 or K3,3 or a subdivision of 5 or K3,3.

If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

- $e \leq 3v - 6$, so $2e \leq 6v - 12$.
- If the degree of every vertex were at least six, then

  $2e \geq 6v$ (From handshaking theorem),

  contradicts the inequality $2e \leq 6v - 12$.

Overview

- Graph Coloring-Chromatic Number:

  -minimum color required to color a graph so that no two adjecent nodes will be of same color.
- Four Color Theorem Statement
- Bipartite graph, Tree: two colorable.
- Find out the chormatic number for following graphs: Complete graph, Circuit Graphs.

References

- Rosen, Chapter 10