Combinatorics: Mathematics of Counting

BEIT 3rd Yr 1st Sem For Class Test 1

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Topics

- Basics of counting (rule of sum, product, permutation, combination)
- Pegion-hole principal
- Inclusion-Exclusion
- Generating Function
- Reccurence Relation

Note [**Except reading these slides also read the corresponding things from the book (Alan Tucker-Applied combinatorics and Chapter 6 and 8 From Rosen) provided**]

Basics of counting

How many ways can you pick 5 numbers from 1 to 10?

How may ways ***10*** bags can be filled by *5* balls ?

Number of Ball: X

Number of Bag: B

How many ways can you pick 5 numbers from 1 to 10?

- 1 2 3 4 5 6 7 8 9 10
  - 1 2 3 4 5
  - 5 4 3 2 1

  Does order matters?
- 1 2 2 3 3

  Does the number repeat?

	Ordered	**Not Ordered**
Repeat	**Sequence:**	
Not Repeat	**Arrangement/ Permutation**	**Combination/ Selection**

Rule of sum

- If an action is performed by making **A** choices or **B** other choices, then it can be performed

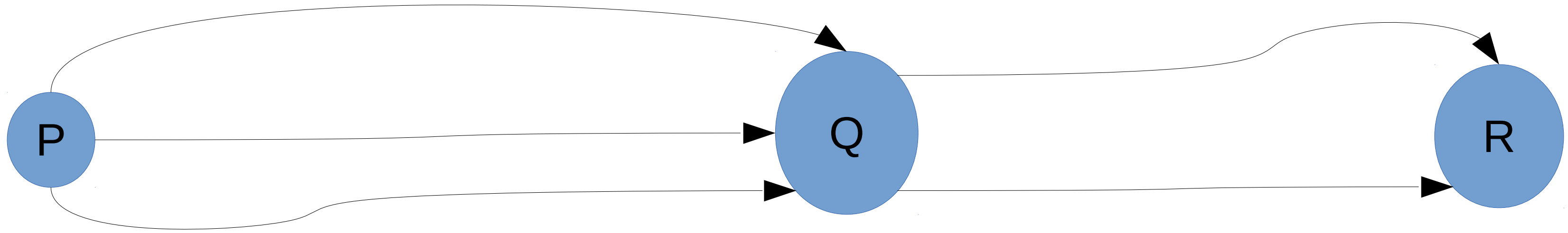
A + B ways

Q can be reached from P in (1+1+1 = 3) ways

Rule of Product

- If an action can be performed by making **A** choices followed by **B** choices, then it can be performed **AB** ways

R can be reached from P in (3X2 = 6) ways



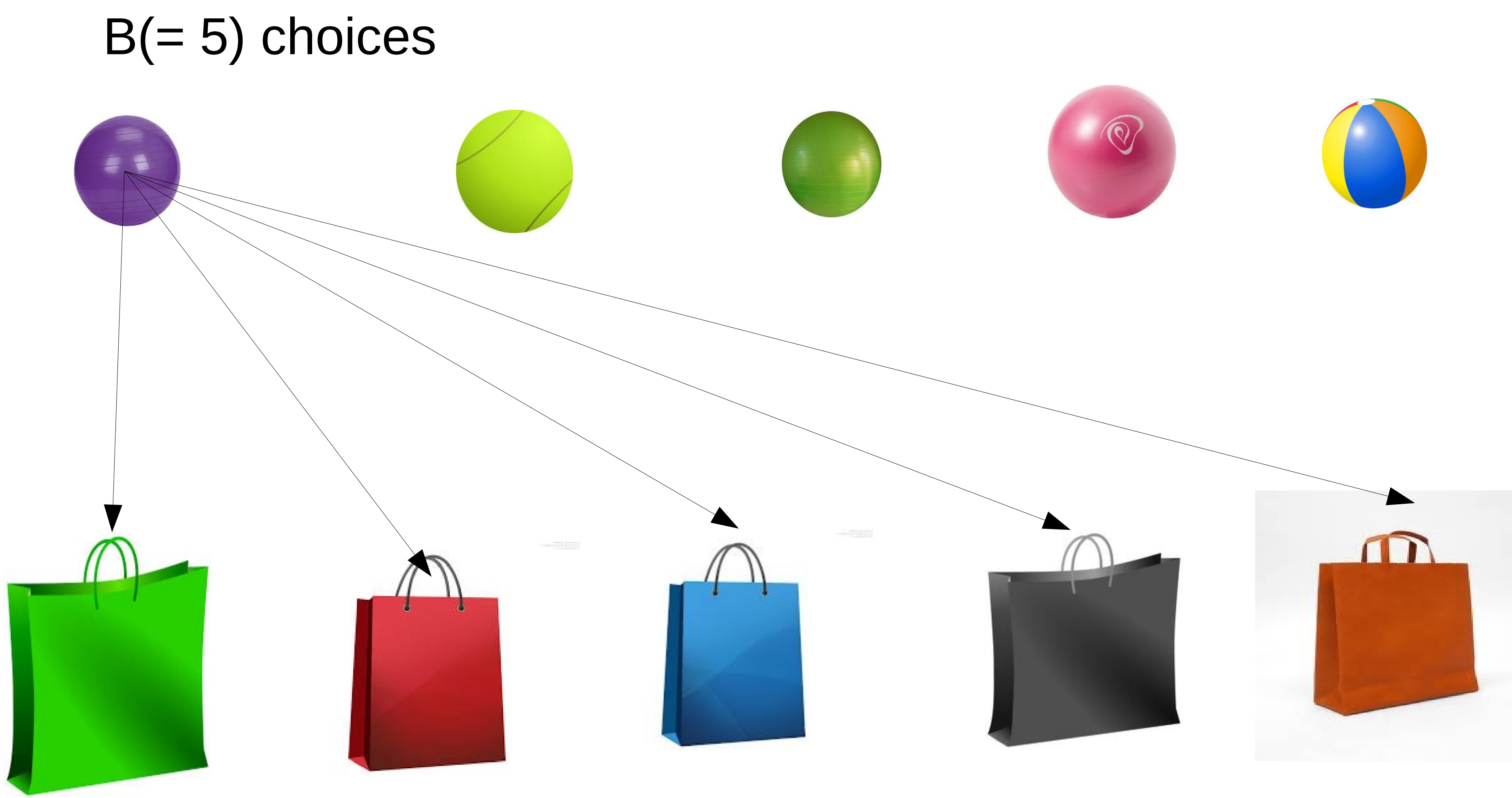
Examples: Repeatation allowed

- How many zip code are possible with 5 digits ? (10 ^5)
- How many of them uses only odd digits?(5^5)
- How many use **at least one** even digit? (10^5 -5^5). - Strategy of complements :

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	?		
I	D			
D	I			
I	I			

Distinct ball, Distinct bag, with Any Capacity



Distinct ball, Distinct bag, with Any Capacity

B choices B choices B choices B choices B choices

Ans: B ^X

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	**B^X**		
I	D			
D	I			
I	I			

Examples: Repeatation NOT allowed

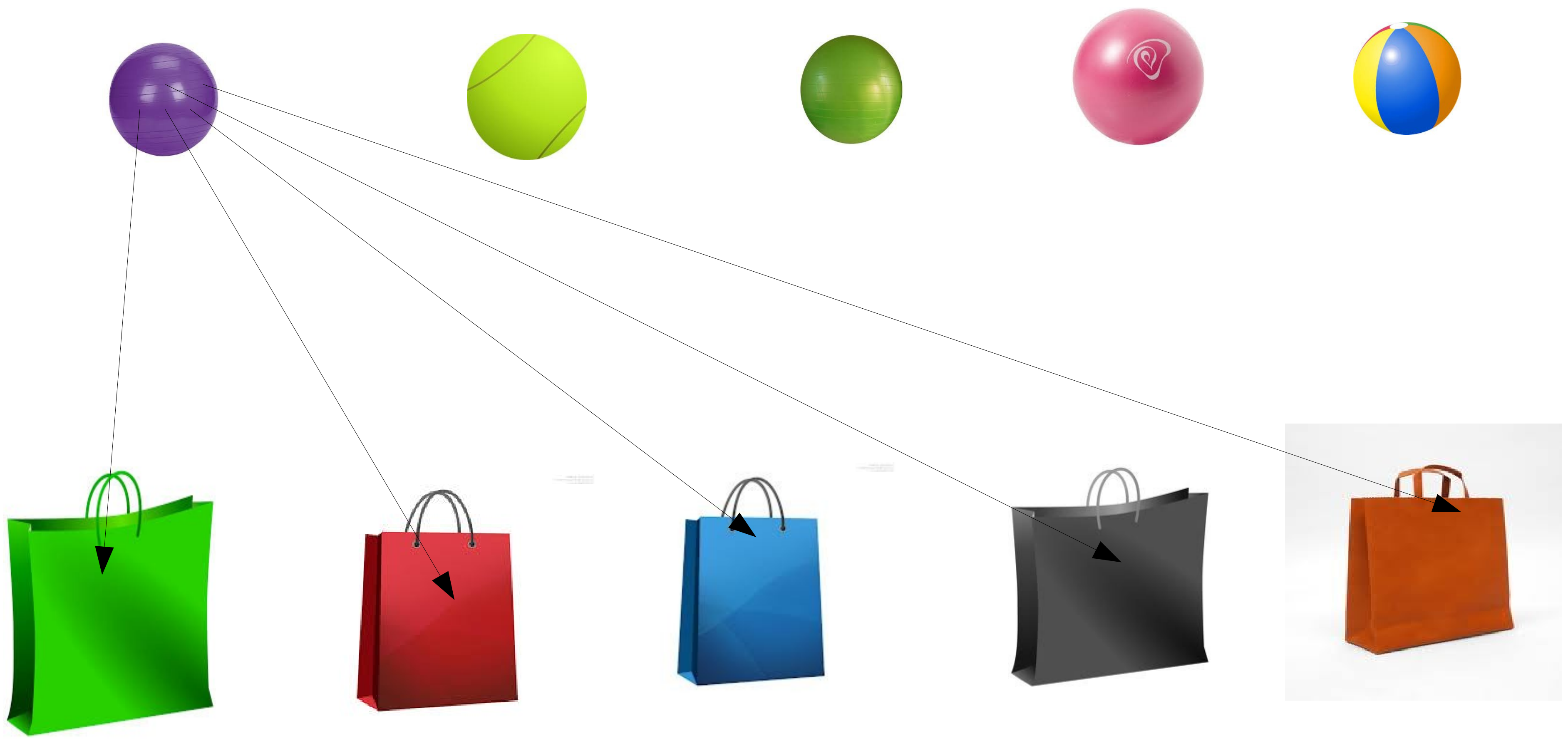
- How many zip code are possible with 5 digits ? (10 X 9 X 8 X 7 X 6)
- How many of them uses only odd digits?(5 X 4 X 3 X 2 X 1)
- How many use **at least one** even digit?
- How many zip code are possible with 10 digits ? (10!)

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	**?**	
I	D			
D	I			
I	I			

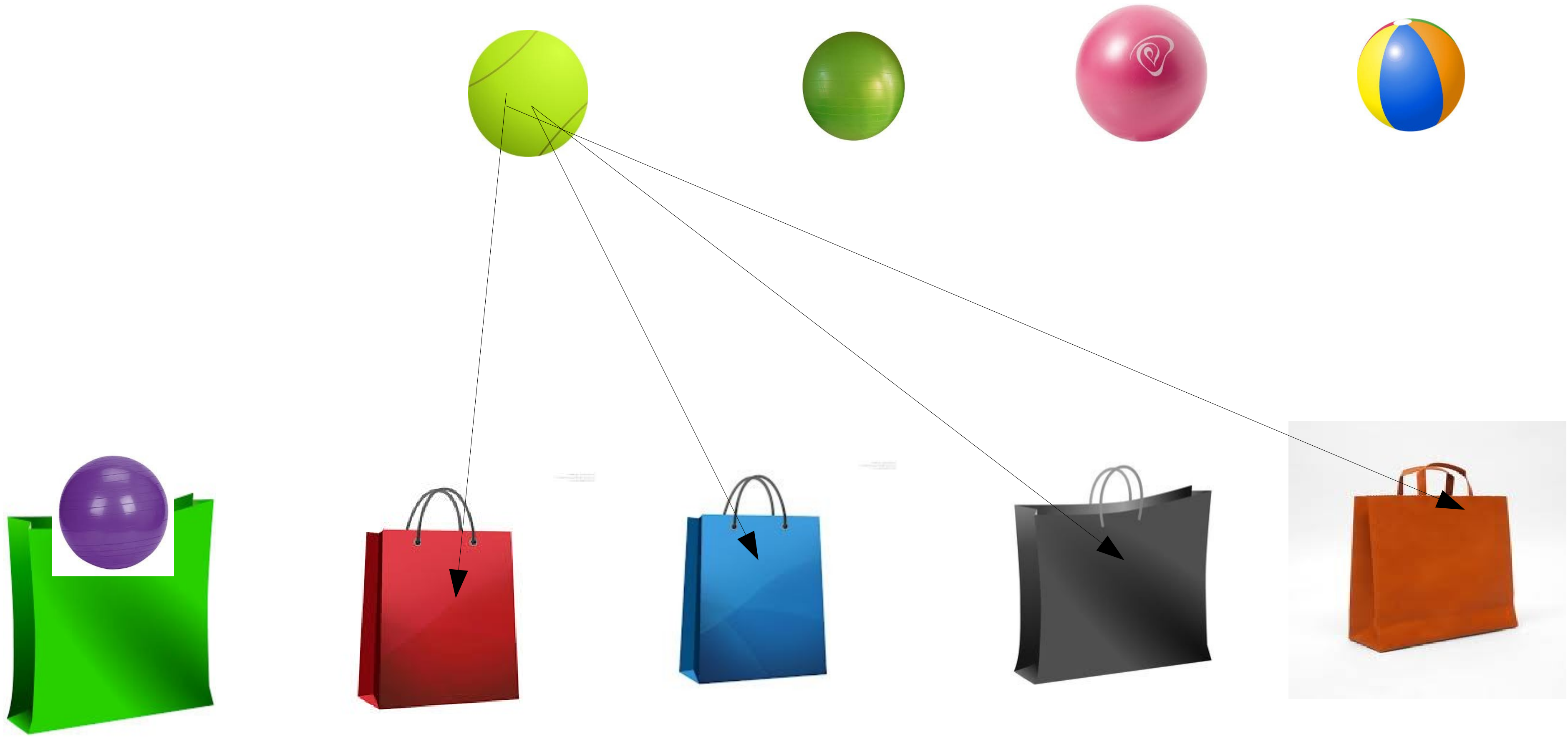
Distinct ball, Distinct bag, with Capacity at most One

B(= 5) choices



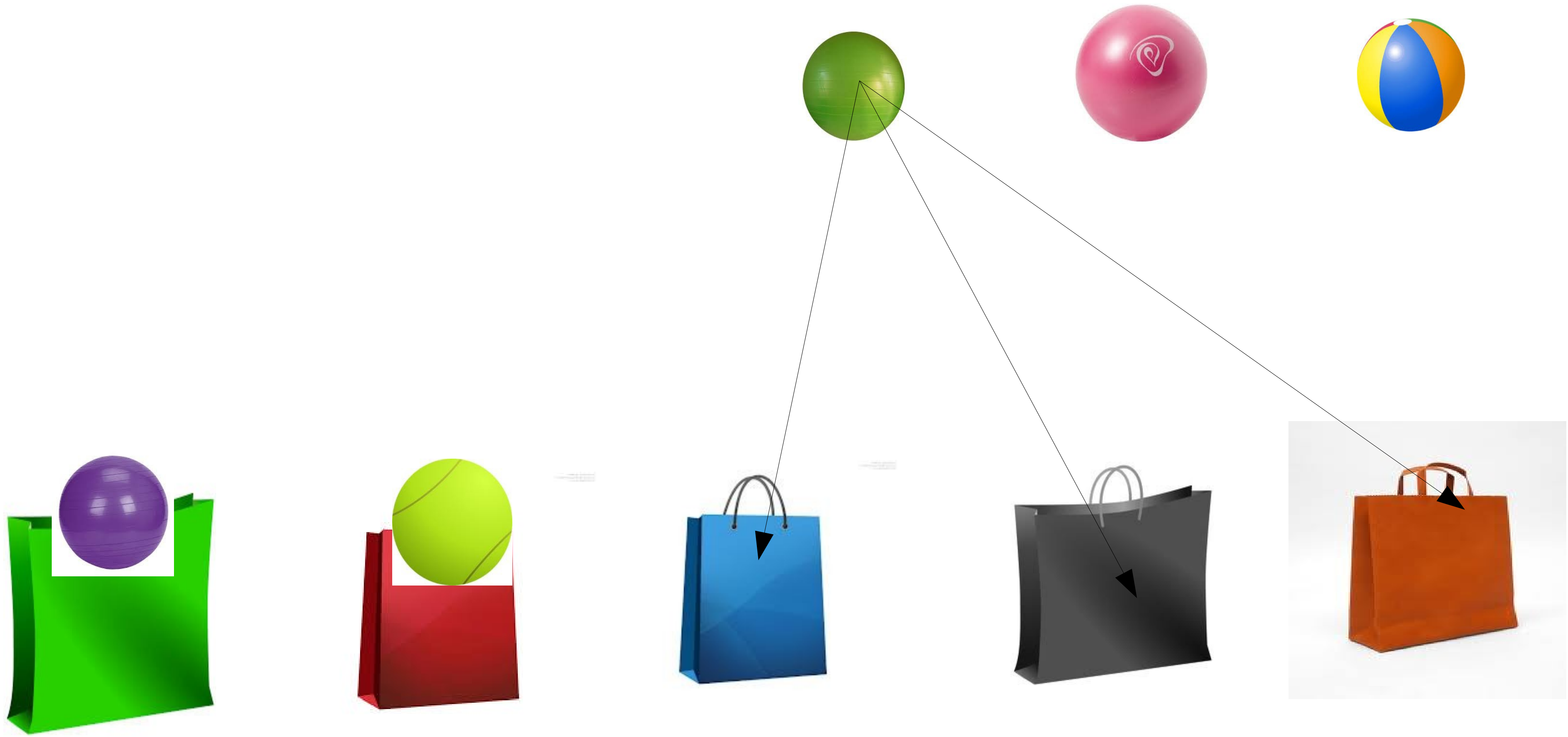
Distinct ball, Distinct bag, with Capacity at most One

B(= 5) choices (B-1) (= 4) choices



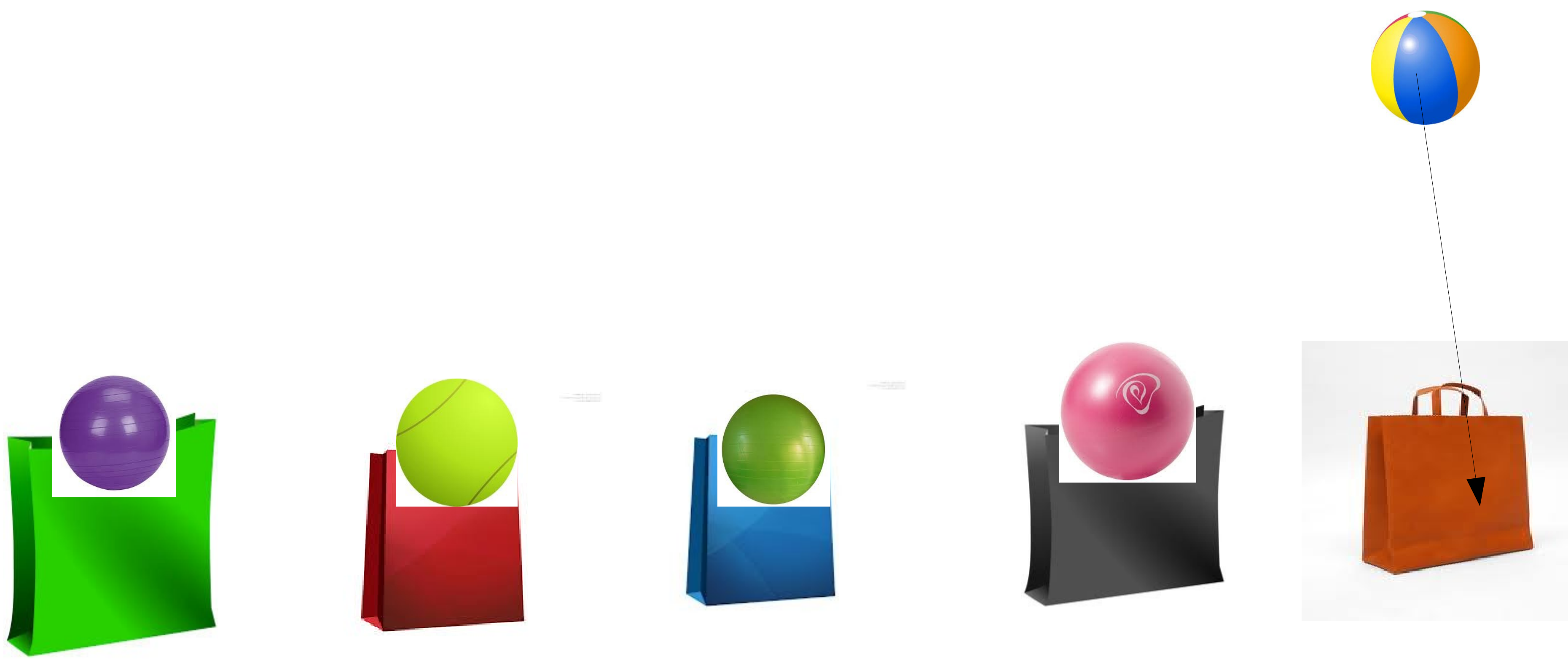
Distinct ball, Distinct bag, with Capacity at most One

B(= 5) choices (B-1) (= 4) choices (B-2) (= 3) choices



Distinct ball, Distinct bag, with Capacity at most One

B(= 5) choices (B-1) (= 4) choices (B-2) (= 3) choices (B-x+1) (= 1) choices



Ans: B(B-1)(B-2)....(B-X+1) = B! /(B-X)!

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	**B!/(B-X)!**	
I	D			
D	I			
I	I			

Combination

- When order does not matter and pepeatation is allowed.
- Example: How many subset of cardinality **k** is possible with **n** numbers ? (n choose k)

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D		?	
D	I			
I	I			

Indentical ball, Distinct bag, with Capacity at most One

$X \leq B$

Indentical ball, Distinct bag, with Capacity at most One

2 ball can be put **(5 choose 2)** ways into 5 bags

Indentical ball, Distinct bag, with Capacity at most One

Ans: B choose X

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D		**B choose X**	
D	I			
I	I			

How many ways k objects can be selected from n distincet object?

	Ordered	Not Ordered
Repeat	**Sequence (n^k)**	***Multi Subsets***
Not Repeat	**Arrangement/ Permutation (P(n,k))**	**Combination (n choose k)**

s

Problem 1

- In how many ways can you choose two books in different language among 5 books in latin, 7 books in greek and 10 books in french ?
- Hint: Choices: Latin & Greek = 5X7 = 35

  +Greek & French = 7X10=70

  +French & Latin = 10 X 5 = 50

  --Total: 155

Problem 2

- Explain combinatorially,

  $P(n,r)=C(n,r) \times P(r,r)$

- Hint: What is the number of ways of selecting r objects out of n objects($C(n,r)$) and arranging them ($P(r,r)$) ?

Problem 3

- Explain combinatorially,

$$C(n,r)=C(n-1,r-1)+C(n-1,r)$$

- Hint:
- Put an object aside, select r-1 objects out of n-1 objects.
- Now there are two case, either the object you took will be with your selected object (C(n-1,r-1)) or it will not be(C(n-1,r))

Problem 4

- How many ways n people can stand to form a ring?

•Hint:

A B C D E are on straight line => number of arrangement = 5!

•If circular => ABCDE/BCDEA/CDEAB/DEABC/EABCD ....same.

So for 5 linear arrangements there are one circular arrangement .

•For 5! linear arrangments there are 5!/5 (4!) circular arrangements.

Permutation of n objects when q1 objects among n are of 1st kind, q2 objects among n are of 2nd kind......qk objects are of kth kind.

Ans: n!/q!.q2....qk. Argue wky?

Problem 5

- How many ways 5 dots and 8 dashes can be arranged?
- Hint1: from slide 31.

Problem 6

- Show that (k!)! is divisible by (k!)^(k-1)!
- Hint: Consider there are k! Objects.
- Out of that k! are of 1st kind, k! are of 2nd kind......k! are of (k-1) kind
- => Compute this = this is not fraction.

When the number of objects are not all distinct the number of ways to select **one or more objects** from them is $(q1+1)(q2+1)...(qt+1)$ **-1**,
(q_i : number of objects of type i)

Hint: (we can either take q1 type of object or no object)(we can either take q2 type of object or no object))...... - if we do not take any object.

Problem 7

- How many divisors does the numer 1400 have ?
- Hint: 1400=2^**3**X5^**2**X7^**1**

  Ans: (3+1)(2+1)(1+1)

Pegion hole principal

Theorem

- If n+1 ball is to be placed into n bags , then there exist at least one bag having 2 balls.
- Ex: among 13 people two of them must have there birthday in same month.

Generalized Theorem

- If pn + 1 or more objects are placed into n boxes, then some box has at least p + 1 objects.

  Proof by contradiction: Suppose that each box has at most pobjects. Then the total number of objects would be at most pn. But we have pn+1 or more objects...Contradiction!!

Consider m intergers a1....am. There exist k and l,1<=k<=l<=m, such that a(k+1)+a(k+2)+....+al is divisible by m.

Hint:

- Consider a1, a1+a2,.....,a1+a2+....am.
- If any of this is divisible by m we are done!
- Suppose each of the sum has non zero reminder, i.e., 1....(m-1) => m sums/ m-1 reminder
- =>two sums have same reminder
- =>a1+...+ak=ma+r (1), a1+....+al=mb+r(2)
- => (1-2)= **ak+a(k+1)....+al= (a-b)m**.

2. A man hiked for 10 hours and covered a total distance of 45 miles. It is known that he hiked 6 miles in the first hour and only 3 miles in the last hour. Show that he must have hiked at least 9 miles within a certain period of two consecutive hours.

Hint:

a1 + a2 ...+a10 = 45/ a1 = 6 and a10 =3.

Proof by contradiction !!. suppose...

a(i)+a(i+1) <9

3. The circumference of a roulette wheel is divided into 36 sectors to which the numbers 1, 2, to 36 are assigned in some arbitrary manner. Show that there are three consecutive sectors such that the sum of their assigned numbers is at least 56.

- Hint: $a1 + \ldots + a36 = 1 + \ldots + 36$.
- Proof by contradiction, suppose,

  $a(i)+a(i+1)+a(i+2)<56/<=55$.

4. From the integers 1 – 200, 101 of them are chosen arbitrarily. Show that among the chosen numbers there exist two such that one divides another.

- Hint:
- Take 100 box as b1, b3......,b199.
- Represnt a number by a = r X 2^k. (k>=0)
- Put it in rth box.
- One box will contain 2 elements !
- m = rX2^p, n = rX2^q
- m/n =? (Suppose m>n)

5. select 100 integers from the integers 1, 2, to 200 such that no one of them is divisible by another.

- Hint: Problem 4!

6. Show that given any 52 integers there exist two of them whose sum or else whose difference is divisible by 100.

Hint:

- a1,......a52.
- ai=100k+bi (1<=bi<=99)
- Consider 51 box, b0....b50
- Put ai in j box. Such that bi = j/100-j
- One box contain two numbers.
- am=100k1+b1/an =100k2+b2
- b1 = j /100-j, b2=j/100-j
- If b1 =b2 .....?
- If b1 =j/ b2 =100-j?

7. A student has 37 days to prepare for an examination. From past experience she knows that she will require no more than 60 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she schedules her study time a whole number of hours per day however, there is a succession of days during which she will have studied exactly 13 hours.

- Home Work

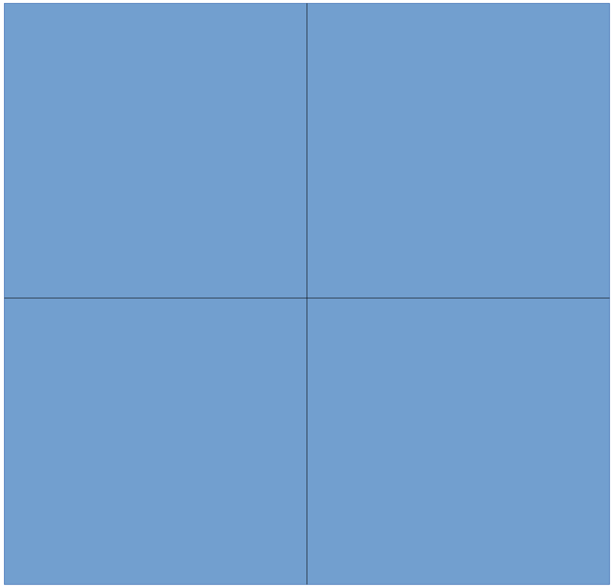
8. Suppose that in a group of 6 persons each pair are either friends or enemies. Prove that among 6 either there are 3 persons who are mutual friends each is a friend of the other or else there are 3 persons who are mutual enemies each is an enemy of the others. Also show that this is not true for 5 persons.

Homework

Problem 9

- If 5 points are chosen inside a 2 × 2 square, then there must be 2 points within 2^(1/2) of each other.

  Hint: 4 sectors/ 5 points => at least 2 points in one sector!!



Problem 10

- Claim: There are two powers of 3 whose difference is divisible by 2009.
- Hint:

Consider the 2010 numbers $3^1, 3^2, 3^3, \ldots, 3^{2010}$.

Divide each number by 2009, and look at the remainder.

2009 possible remainders (0 through 2008), 2 of them must have the same remainder r.

$3^x = 2009q1 + r$, and $3^y = 2009q2 + r$.

$3^x - 3^y = 2009(q1 – q2)$.

Problem 11

There is a positive power of 3 that ends in 001.

Hint :

Consider the 1001 numbers $3^0, 3^1, 3^2, \ldots, 3^{1000}$.

Divide each number by 1000, and 1000 divides $3^x - 3^y = 3^y(3^{(x-y)} - 1)$.

Note that 3^y is not divisible by 1000, so 1000 divides $\{3(x-y) - 1\}$.

$1000q = \{3(x-y) - 1\}$.

Thus $3(x-y) = 1000q + 1$ ends in 001.

Principal of Inclusion-Exclusion

Problem 1

- How many strings of 8 bits are possible, such that, they start with 1 or end with 00 ?
- Hint :

(total no. of possible strings)-(no. of strings starting with 1 + no. of strings ending with 00)+ (no. of strings starting with 1 & ending with 00)

Problem 2

- Among the numbers 1 through 100, how many of them are multiples of 2, 3, or 5?
- Hint: (multiples of 2 + multiples of 3 + multiples of 5)-(multiples of 2 & 3 + multiples of 3 & 5 + multiples of 5 & 2) + (multiples of 2 & 3 & 5)

Problem 3

- How many of the numbers 1 through 10 are odd or a multiple of 3?
- Hint: (no. of odd numbers 1 through 10) + (no. of numbers multiple of 3, 1 through 10) -

  (no. of numbers which are odd and multiple of 3, 1 through 10)

Problem 4

- How many ways are there to give 11 distinct candies to 4 children so that each child gets at least 1 candy?
- Hint: (Without the restriction, there would be 4^11 allocations)

  -(3^11 ways where child 1 gets none. same for children 2,3,

  and 4)

  +(2^11 ways where children 1 and 2 get none. same for all other pairs of children)

  - (1 way where children 1, 2, and 3 get none. Same for all other triples of children)

  = 4^11 -C(4,1)3^11+C(4,2)2^11-C(4,3)1^11(Let this number be N)
- Stirling number of second kind = N/4! [Ref: Match this problem Twelve-fold way problem]
- Excercise: Find general equation of Stirling number of 2nd kind.

Problem 5

- How many numbers between 1 and 1000 are not divisible by 2, 3, or 5?
- Hint: 1000-(multiple of 2 + multiple of 3 + multiple of 5)+(multiple of 2 & 3 + multiple of 3 & 5 + multiple of 5 & 2)-(multiple of 2 & 3 & 5)

Problem 6

- How many ways can n homework assignments be returned to n students such that no student gets her own homework back? (Dearrangement Problem)
- Hint : (If there were no restrictions, then there would be n! ways)

  - (1 student gets her own homework = C(n,1) (n – 1)!) +

  (2 students get their own homeworks = C(n,2) (n – 2)!)-........

  = n! - C(n,1) (n – 1)! +C(n,2) (n – 2)!)-......

12 Fold Way Table (Complete it)

X	B	Any	<=1 At most one	>=1 At least one
D	D			
I	D			
D	I			
I	I			

Generating Function

Problem

Bob is allowed to choose two items from a tray containing an apple, an orange, a pear, a banana, and a plum. In how many ways can she choose?

Problem

Bob is allowed to choose two items from a tray containing an apple, an orange, a pear, a banana, and a plum. In how many ways can she choose?

$$\binom{5}{2}$$

Counting with Generating Functions

(0 apple + 1 apple)(0 orange + 1 orange)
(0 pear + 1 pear)(0 banana + 1 banana)
(0 plum + 1 plum)

In this notation, apple^2 stand for choosing 2 apples., and + stands for an exclusive or.

$$(1+x)^5 = (1+x)(1+x)(1+x)(1+x)(1+x)$$

Take the coefficient of x^2.

Problem

Bob is allowed to choose two items from a tray containing TWO apples, an orange, a pear, a banana, and a plum. In how many ways can she choose?

The two apples are identical.

Counting with Generating Functions

(0 apple + 1 apple + + 2 apple)
(0 orange + 1 orange)(0 pear + 1 pear)
(0 banana + 1 banana)(0 plum + 1 plum)

$$(1+x+x^2)(1+x)(1+x)(1+x)(1+x)$$

Take the coefficient of x^2.

The function $f(x)$ that has a polynomial expansion

$f(x)=a0 + a1\ x + \ldots + an\ x^n$

is the generating function for the sequence

$a0, a1, \ldots, an$

If the polynomial $(1+x+x^2)^4$

is the generating polynomial

for ak, what is the combinatorial meaning of ak?

The number of ways to select k

object from 4 types with at most 2 of

each type.

Bob wants to order pastry. The shop only has 2 apple, 3 cheese, and 4 raspberry pastries left. What the number of possible orders?

Counting with Generating Functions

$$(1+x+x^2)\ (1+x+x^2+x^3)\ (1+x+x^2+x^3+x^4)$$

apple *cheese* *raspberry*

$$= 1+3x+6x^2+9x^3+11x^4+11x^5+9x^6+6x^7+3x^8+x^9$$

The coefficient by x^8 shows that there are only 3 orders for 8 pastries.

Bob wants to order pastry. The shop only has 2 apple, 3 cheese and 4 raspberry pastries left. What the number of possible orders? Raspberry pastries come in multiples of two.

Counting with Generating Functions

$$(1+x+x^2)\ (1+x+x^2+x^3)\ (1+x^2+x^4)$$

apple *cheese* *raspberry*

$$= 1+2x+4x^2+5x^3+6x^4+6x^5+5x^6+4x^7+2x^8+x^9$$

The coefficient by x^8 shows that there are only 2 possible orders.

Problem

Find the number of ways to select R balls from a pile of 2 red, 2 green and 2 blue balls

Coefficient by x^R in $(1+x+x^2)^3$

Exercise

Find the number of integer solutions to

x1+x2+x3+x4=21

Take the coefficient by x^{21} in $(1+x+x^2+x^3+x^4+x^5+x^6+x^7)^4$

Bob wants to order pastry.

Unfortunately, still only 3 cheese and 4 raspberry pastries are available. Raspberry pastries come in multiples of two. But the availability of apple is unlimited ! What the number of possible orders?

Counting with Generating Functions

$$(1+x+x^2+x^3)\ (1+x^2+x^4)\ (1+x+x^2+x^3+\ \ldots)$$

cheese *raspberry* *apple*

There is a problem (…) with the above
Expression!

Counting with Generating Functions

$$(1+x+x^2+x^3)\ (1+x^2+x^4)\ (1+x+x^2+x^3+\ \ldots)=$$

cheese *raspberry* *apple*

$$(1+x+2x^2+2x^3+2x^4+2x^5+x^6+x^7)\ (1+x+x^2+x^3+\ \ldots)$$

$$=1+2x+4x^2+6x^3+8x^4+\ldots$$

The power series

$a0 + a1\ x + a2\ x^2 + \ldots$

is the generating function for the *infinite* sequence

a0, a1, a2, …,

$$\sum_{k=0}^{\infty} a_k x^k$$

Exercise

Count the number of N-letter combinations of MATH in which M and A appear with repetitions and T and H only once.

Coefficient by x^N in

$$(1+x+x^2+\ldots)^2(1+x)^2$$

What is the coefficient of X^k in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + ...)^n ?$$

A solution to:

$$e1 + e2 + . . . + en = k$$

$$ek \geq 0$$

What is the coefficient of X^k in the expansion of:

$$(1 + X + X^2 + X^3 + X^4 + \ldots)^n$$

$$\binom{n+k-1}{k}$$

$$(1+x+x^2+\ldots)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

But what is the LHS?

$$1+x+x^2+\ldots = \frac{1}{1-x}$$

$$\frac{1}{(1-x)^{n}} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^{k}$$

Power Series = Generating Function

Given a sequence of integers a0, a1, …, an

We will associate with it a function

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

Generating Functions

Sequence: 1, 1, 1, 1, …

Generating function:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Find a generating function f(x)

1, 2, 3, 4, 5, …

$$f(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$f(x) = \frac{1}{(1-x)^2}$$

Reccurence Relation

Creation &
Solving by Charecteristics equations

Problem Example 1

An elf has a staircase of n stairs to climb. Each step it takes can cover either one stair or two stairs. Find a recurrence relation for the number of different ways for the elf to ascend the n-stair staircase.

Ans: F(n)=F(n-1) + F(n-2)

[1. Let the answer be F(n).

2. If it takes 1 stair in the first step, n-1 stairs remaining and the question hold same for (n-1) stairs, whose answer is F(n-1).

3. If it takes 2 stairs in the first step, n-2 stairs remaining and the question hold same for (n-2) stairs, whose answer is F(n-1).

4. Since it can take either one or two step at the first, we can write F(n)=F(n-1) + F(n-2)]

Example Problem 2

- Suppose we draw n straight lines on a piece of paper so that every pair of lines intersect (but no three lines intersect at a common point). Into how many regions do these n lines divide the plane?
- Answer: $F(n) = F(n-1)+n$
- [Note that evey i^{th} new line drawing is adding i number of new regions.]

Solving Reccurence Relation

- Take F(n) = a^n
- Put into the equation, e.g.,
- F(n) = F(n-1)+F(n-2) => a^n =a^(n-1) + a^(n-2)
- Charecteristics equation => a^2 -a-1 =0
- Solve for a , let a = r1, r2
- Solution F(n) = C1(r1)^n + C2(r2)^n
- C1 & C2 is solved by initial condition.