12 Fold - Way of Combinatorics

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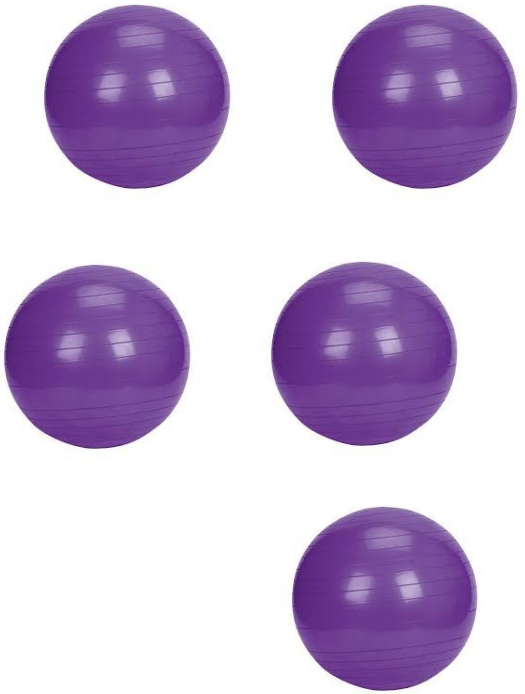
How may ways ***B*** bags can be filled by ***X*** balls ?

Number of Ball: X

Number of Bag: B

Balls : Distinct/Identical



Bags : Distinct/Identical

Bag: Capacity (any/≤1/≥1)

- Any number of balls
- At most one ball
- At least one ball

2X2X3 = 12 possible questions!

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	?	?	?
I	D	?	?	?
D	I	?	?	?
I	I	?	?	?

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D		0	
I	D		0	
D	I		0	
I	I		0	

X > B

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D			0
I	D			0
D	I			0
I	I			0

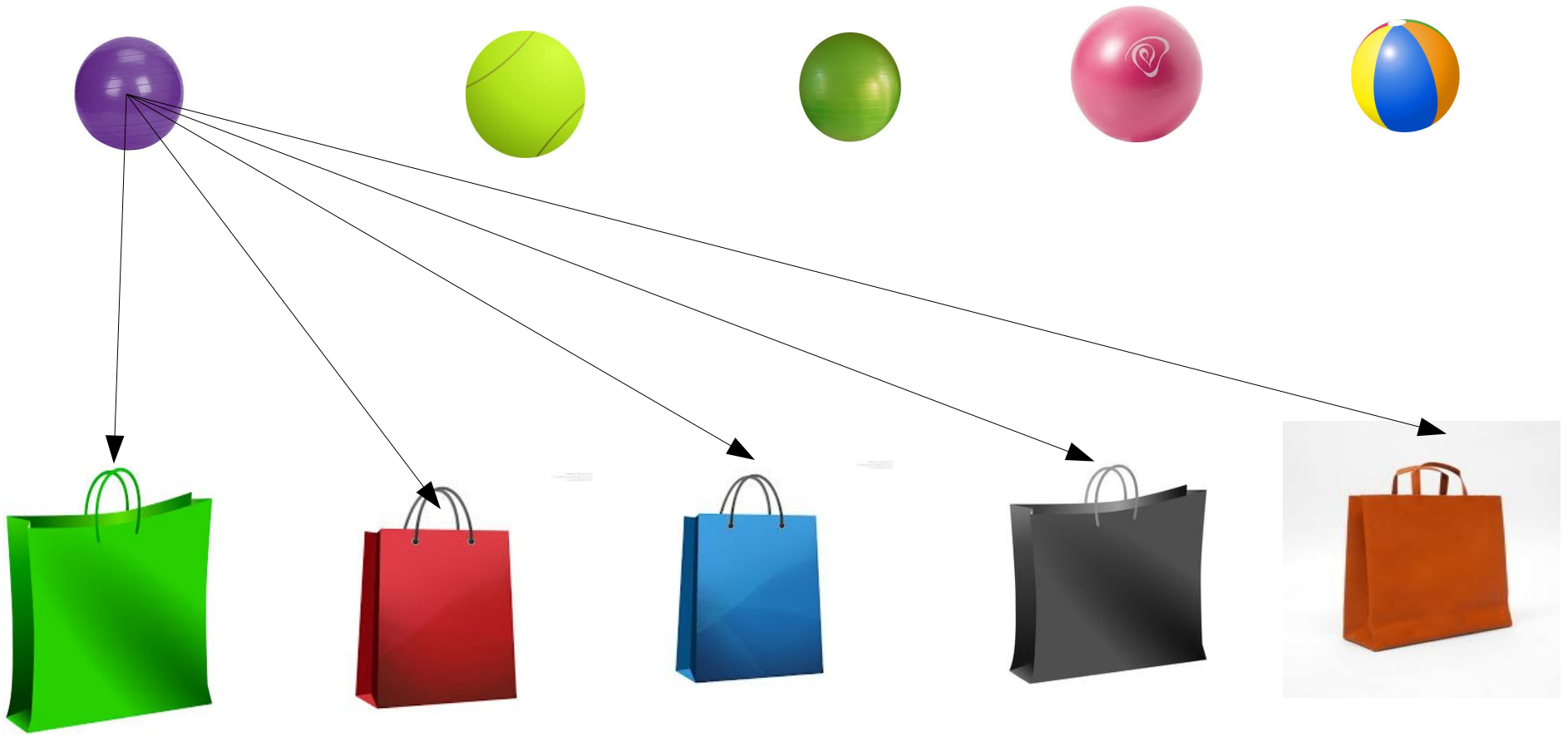
X < B

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	**?**		
I	D			
D	I			
I	I			

Distinct ball, Distinct bag, with Any Capacity

B(= 5) choices



Distinct ball, Distinct bag, with Any Capacity

B choices | B choices | B choices | B choices | B choices









Ans: B ^X

12 Fold Way Table

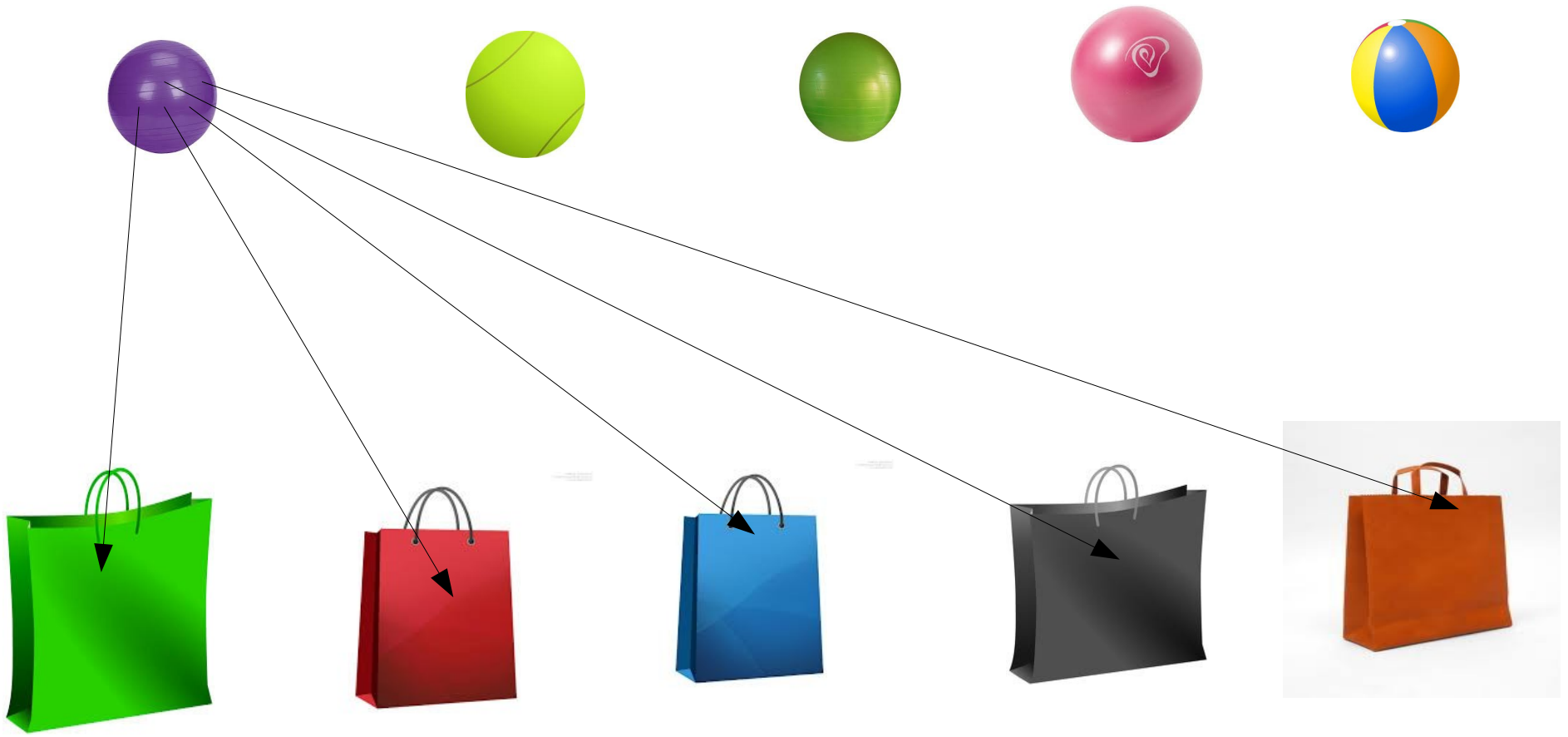
X	B	Any	<=1 At most one	>=1 At least one
D	D	**B^X**		
I	D			
D	I			
I	I			

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	**?**	
I	D			
D	I			
I	I			

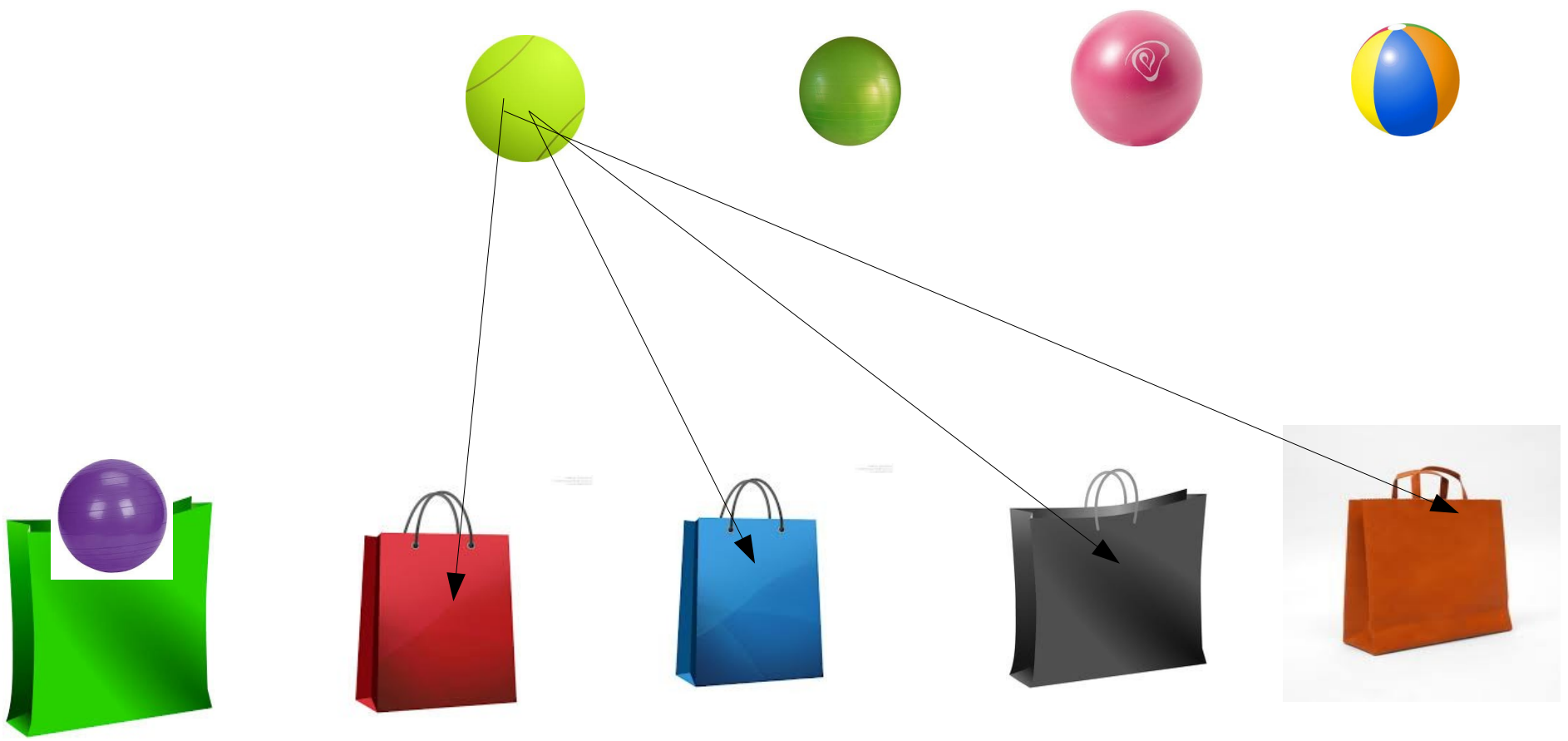
Distinct ball, Distinct bag, with Capacity at most One

B(= 5) choices



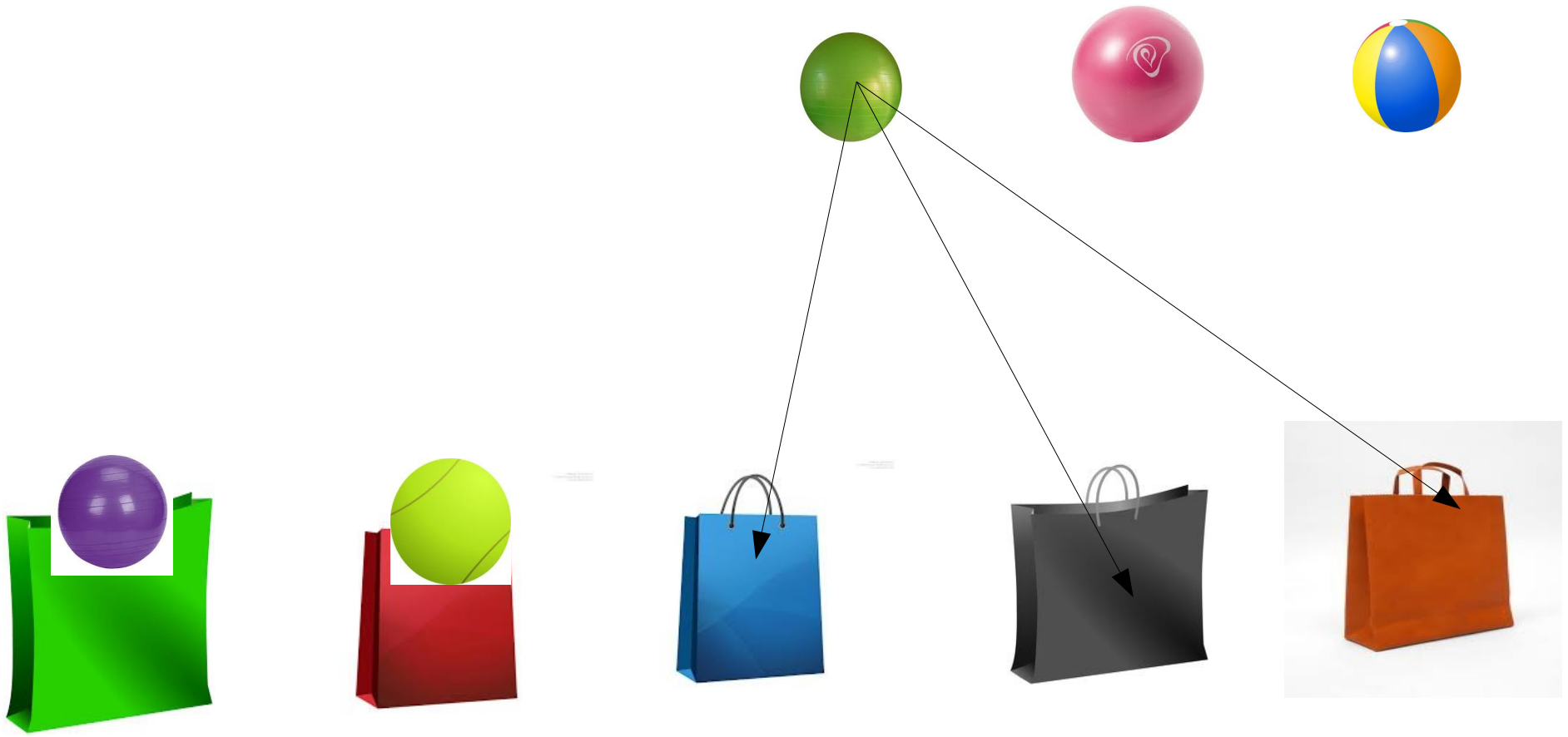
Distinct ball, Distinct bag, with Capacity at most One

B(= 5) choices (B-1) (= 4) choices



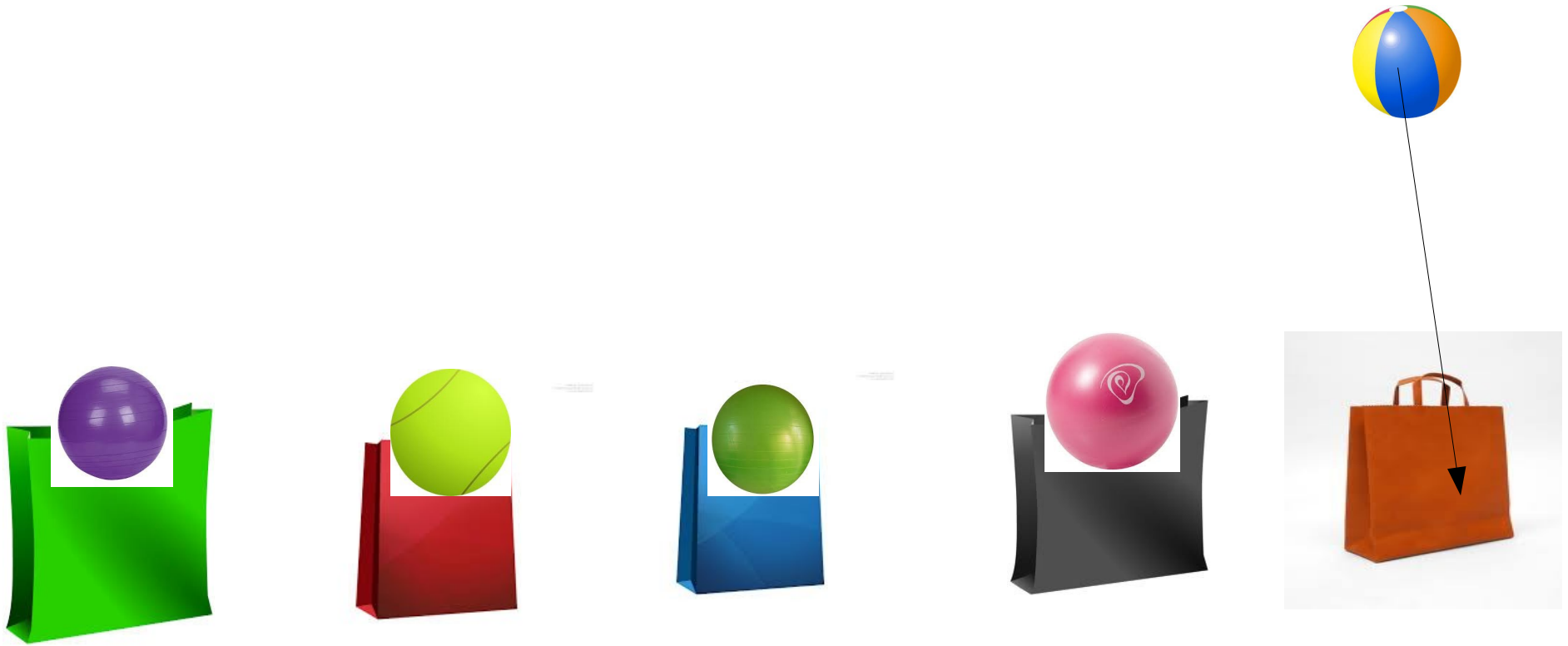
Distinct ball, Distinct bag, with Capacity at most One

B(= 5) choices (B-1) (= 4) choices (B-2) (= 3) choices



Distinct ball, Distinct bag, with Capacity at most One

B(= 5) choices (B-1) (= 4) choices (B-2) (= 3) choices (B-x+1) (= 1) choices



Ans: B(B-1)(B-2)....(B-X+1) = B! /(B-X)!

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	**B!/(B-X)!**	
I	D			
D	I			
I	I			

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	**Hold!**
I	D			
D	I			
I	I			

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D		**?**	
D	I			
I	I			

Indentical ball, Distinct bag, with Capacity at most One

X≤B

Indentical ball, Distinct bag, with Capacity at most One






2 ball can be put **(5 choose 2)** ways into 5 bags

Indentical ball, Distinct bag, with Capacity at most One

Ans: B choose X

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D		**B choose X**	
D	I			
I	I			

12 Fold Way Table

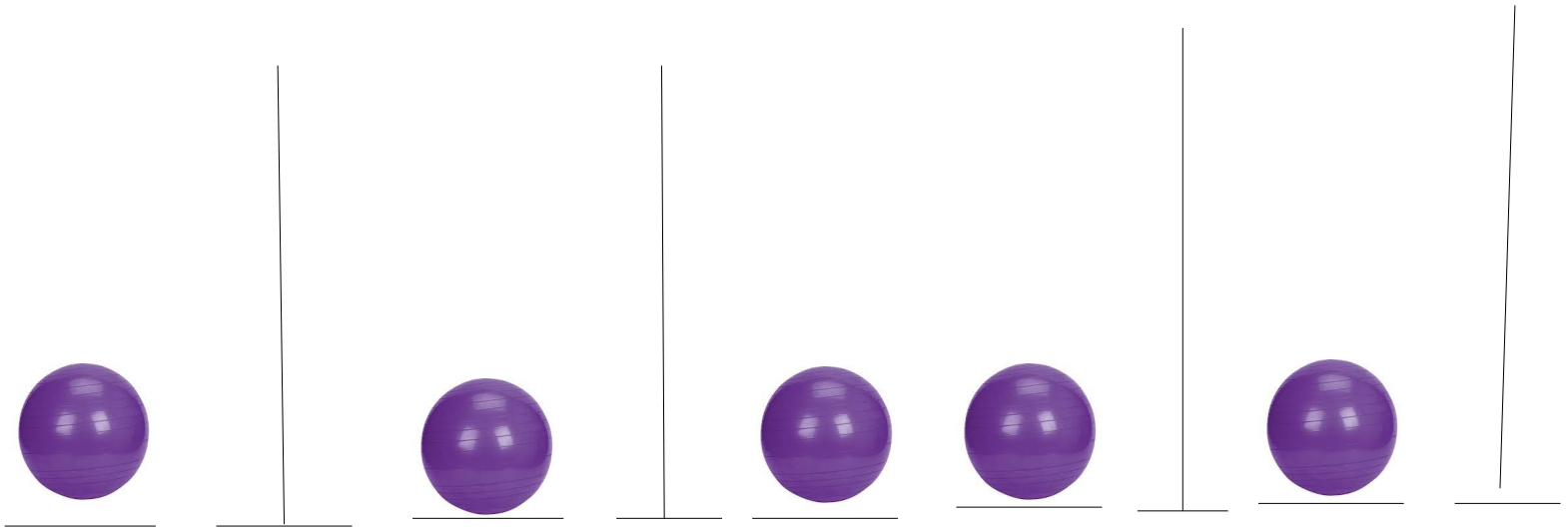
X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D	**?**	B choose X	
D	I			
I	I			

Indentical ball, Distinct bag, with Any Capacity

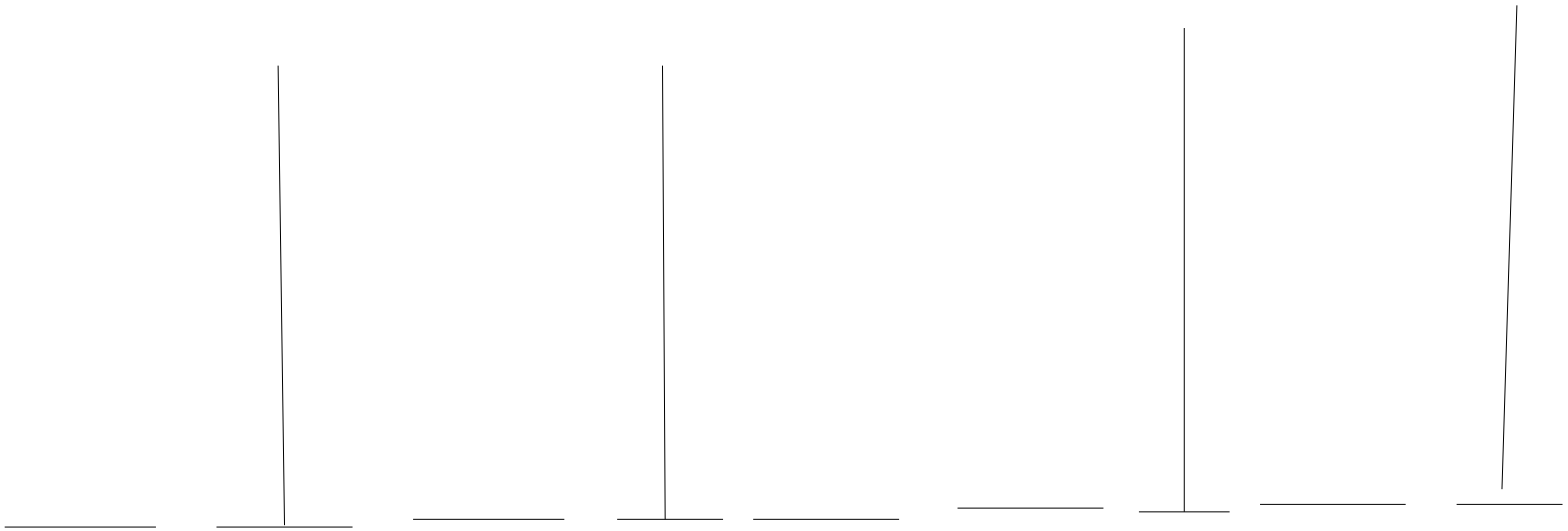
Indentical ball, Distinct bag, with Any Capacity

Indentical ball, Distinct bag, with Any Capacity

How many ways X balls and B-1 bars can be arranged?

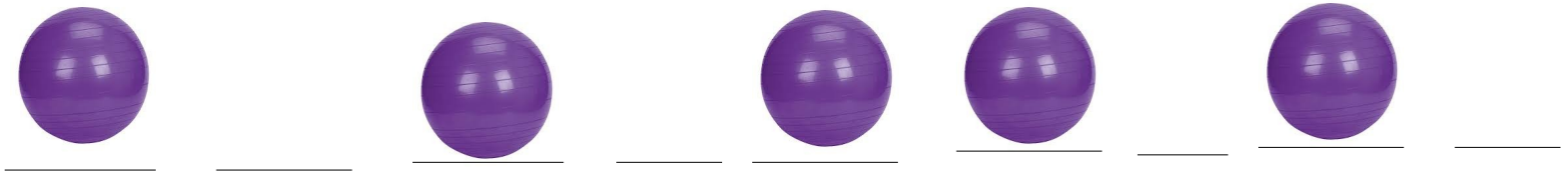


How many ways B-1 bars can be arranged in (X+B-1) places?



Ans: (X+B-1)choose(B-1)

How many ways X balls can be arranged in (X+B-1) places?



Ans: (X+B-1)choose(X)

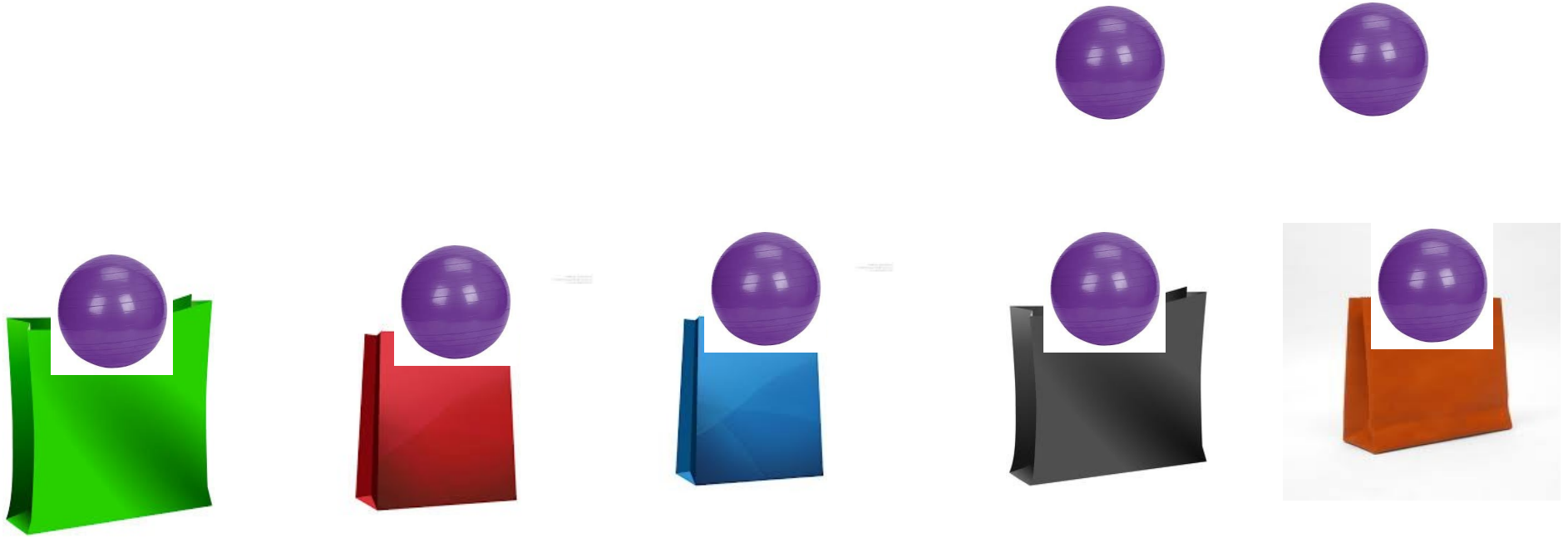
12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D	**(X+B-1)choose (B-1) = (X+B-1)choose(X)**	B choose X	
D	I			
I	I			

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D	**(X+B-1)choose (B-1) = (X+B-1)choose(X)**	B choose X	**?**
D	I			
I	I			

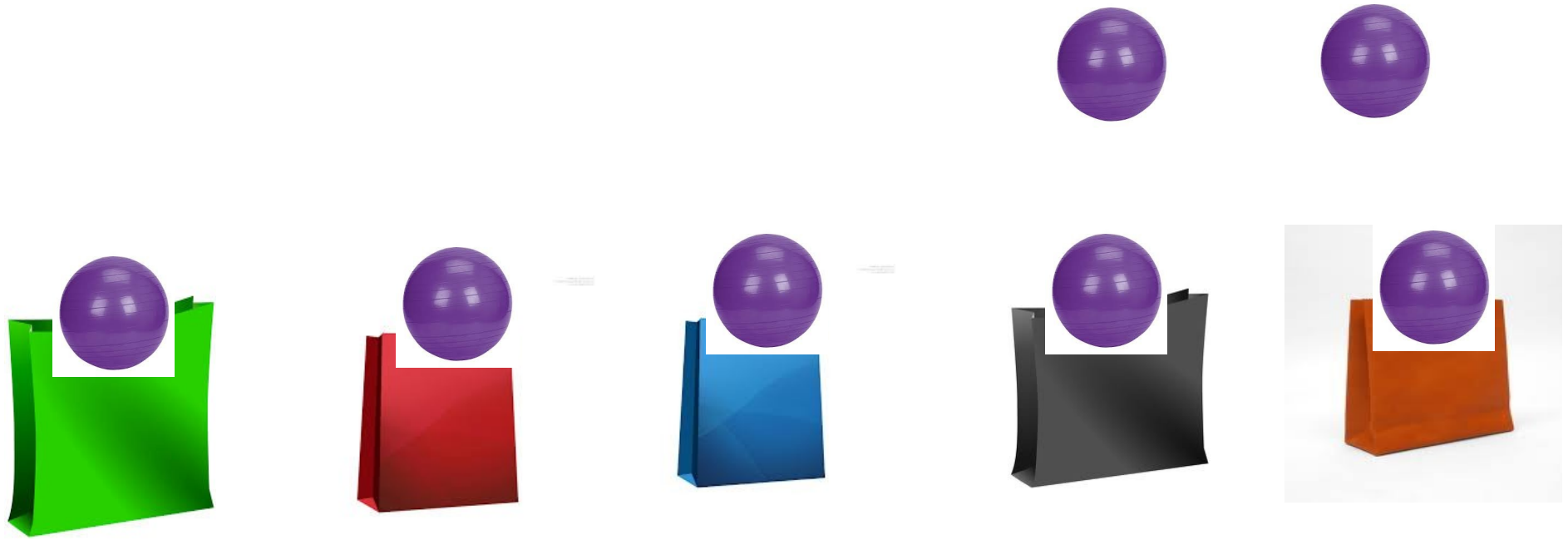
Indentical ball, Distinct bag, with Capacity at least One



Each Bag gets at least one ball

Indentical ball, Distinct bag, with Capacity at least One

Problem reduces to => Previous problem by replacing X by (X-B)



Ans: (X -B+B-1)choose(X-B)=(X-1)choose(X-B)

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D	(X+B-1)choose (B-1) = (X+B-1)choose(X)	B choose X	**(X-1)choose(B-1)= (X-1)choose(X-B)**
D	I			
I	I			

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D	(X+B-1)choose (B-1) = (X+B-1)choose(X)	B choose X	**(X-1)choose(B-1)= (X-1)choose(X-B)**
D	I			**?**
I	I			

Distinct ball, Indentical bag, with Capacity at least One

Ans: Stirling Number of 2nd kind = S(X,B)
[Ref: Principal of Inclusion Exclusion]

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I			**S(X,B)**
I	I			

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	**?**		S(X,B)
I	I			

Distinct ball, Indentical bag, with Any Capacity

Ans: S(X,1) + S(X,2)+.....+S(X,B)

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Hold!
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	**S(X,1) +S(X,2)+... +S(X,B)**		S(X,B)
I	I			

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	**?**
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	S(X,1) +S(X,2)+... +S(X,B)		S(X,B)
I	I			

Distinct ball, Distinct bag, with Capacity at least One

Ans: S(X,B)

Distinct ball, Distinct bag, with Capacity at least One

Ans: B! S(X,B)

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	**B!S(X,B)**
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	S(X,1) +S(X,2)+... +S(X,B)		S(X,B)
I	I			

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	B!S(X,B)
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	S(X,1) +S(X,2)+... +S(X,B)		S(X,B)
I	I			**?**

Identical ball, Indentical bag, with Capacity at least One

Ans: P(B,X) : Partitioning X into B part
[Ref: Interger Partitioning Problem]

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	B!S(X,B)
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	S(X,1) +S(X,2)+... +S(X,B)		S(X,B)
I	I			**P(B,X)**

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	B!S(X,B)
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	S(X,1) +S(X,2)+... +S(X,B)		S(X,B)
I	I	**?**		P(B,X)

Indentical ball, Indentical bag, with Any Capacity

Ans: P(1,X) + P(2,X) + ...+P(B,X)

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	B!S(X,B)
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	S(X,1) +S(X,2)+... +S(X,B)		S(X,B)
I	I	**P(1,X) + P(2,X) + ...+P(B,X)**		P(B,X)

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	B!S(X,B)
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	S(X,1) +S(X,2)+... +S(X,B)	**?**	S(X,B)
I	I	P(1,X) + P(2,X) + ...+P(B,X)		P(B,X)

Distinct ball, Indentical bag, with Capacity at most One

Ans: 1 [X ≤ B]

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	**B!S(X,B)**
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	S(X,1) +S(X,2)+... +S(X,B)	**1 (X≤B)**	S(X,B)
I	I			P(B,X)

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Stirling Number (Principal of Inclusion -Exclusion)
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	Function of S(B,X)	1 (X≤B)	S(B,X)
I	I	Function of P(B,X)	**?**	P(B,X)

Identical ball, Indentical bag, with Capacity at most One

Ans: 1 [X ≤ B]

12 Fold Way Table

X	B	Any	<=1 At most one	>=1 At least one
D	D	B^x	B!/(B-X)!	Stirling Number (Principal of Inclusion -Exclusion)
I	D	(X+B-1)choose(B-1) = (X+B-1)choose(X)	B choose X	(X-1)choose(B-1) = (X-1)choose(X-B)
D	I	Function of S(B,X)	1	S(B,X)
I	I	Function of P(B,X)	**1 (X≤B)**	P(B,X)

Reference

- Discrete Mathematics Lechures by

  Arthur T. Benjamin, Harvey Mudd College.