Equations for N-body Simulations with Quintessence

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1 Introduction

Here we write down the ingredients of Einsteins equations for case where we consider following metric involving two scalar potentials and also have quintessence as dark energy. We derive field dynamics equations along with Einstein equations. We employ linear perturbation scheme in potentials($\psi(x,y,z,t)$ and $\xi(x,y,z,t)$) dropping second order and higher order terms, while we do not have any approximation in quintessence field($\Phi(x,y,z,t)$). This is a step towards building more relativistic N-body simulations. General Relativity and modifications are theories based on relativistic fields and hence for a truly consistent treatment, we need to incorporate relativistic fields in N-body approaches. Here, we are consider a scalar field(quintessence) representing dark energy. This field is evolved along with metric field represented by two scalar components of metric. At current stage, to make equations tractable, we only consider first order terms in metric. But for quintessence field, we write different codes for different levels of approximation. But all codes in this context have only upto linear order terms in metric. Similarly for particle equations of motion, we write one code with relativistic equations, another with non-relativistic terms only. For codes, please have a look at https://github.com/manu0x/nbody

Notation:i,j,k are spatial indices and go from 1 to 3 and Δ is defined as:

$$\Delta = \sum_{k=1}^{3} \frac{\partial^2}{\partial {x^k}^2} \tag{1}$$

The metric is

$$ds^2 = -a(t)^2(1-2\xi)dx^2 - a(t)^2(1-2\xi)dy^2 - a(t)^2(1-2\xi)dz^2 + (1+2\psi)dt^2 \tag{2}$$

2 Einstein Tensor

Components of Einstein tensor are:

$$G^k_k = \frac{\dot{a}^2}{a^2} + \frac{2\ddot{a}}{a} + \frac{1}{a^2}\left\{-2\psi\dot{a}^2 - 4a\psi\ddot{a} - 6a\dot{a}\dot{\xi} - 2a\dot{a}\dot{\psi} - 2a^2\ddot{\xi} + \sum_{i\neq k}\frac{\partial^2}{\partial x^{i2}}(\xi-\psi)\right\} \tag{3}$$

$$G^t_k = 2\left[\frac{\dot{a}}{a}\frac{\partial\psi}{\partial x^k} + \frac{\partial^2\xi}{\partial x^k\partial t}\right] \tag{4}$$

$$G^k_t = -\frac{2}{a^3}\left\{\dot{a}\frac{\partial\psi}{\partial x^k} + a\frac{\partial^2\xi}{\partial x^k\partial t}\right\} \tag{5}$$

$$G^j_k = \frac{1}{a^2}\left\{-\frac{\partial^2\xi}{\partial x^k\partial x^j} + \frac{\partial^2\psi}{\partial x^k\partial x^j}\right\} \tag{6}$$

$$G^t_t = \frac{3\dot{a}^2}{a^2} - \frac{2}{a^2}\left\{3\psi\dot{a}^2 + 3a\dot{a}\dot{\xi} - \Delta\xi\right\} \tag{7}$$

3 T^μ_ν for Scalar field Quintessence

$$T^k_k = V - \frac{\dot{\Phi}^2}{2}(1-2\psi) + \left[\sum_{i\neq k}\left(\frac{\partial\Phi}{\partial x^i}\right)^2 - \left(\frac{\partial\Phi}{\partial x^k}\right)^2\right]\frac{(1+2\xi)}{2a^2} \tag{8}$$

$$T^j_k = -\frac{\frac{\partial\Phi}{\partial x^k}\frac{\partial\Phi}{\partial x^j}}{a^2}(1+2\xi) \tag{9}$$

$$T^t_k = \frac{\partial\Phi}{\partial x^k}\frac{\partial\Phi}{\partial t}(1-2\psi) \tag{10}$$

$$T^k_t = -\frac{\partial\Phi}{\partial x^k}\frac{\partial\Phi}{\partial t}\frac{(1+2\xi)}{a^2} \tag{11}$$

$$T^t_t = V + \frac{\dot{\Phi}^2}{2}(1-2\psi) + \frac{\vec{\nabla}\Phi.\vec{\nabla}\Phi}{2a^2}(1+2\xi) \tag{12}$$

4 Field Dynamics

$$\ddot{\Phi} = \frac{1}{a^2(-1+2\psi)}\left\{a^2V_{,\Phi} + 3a\dot{a}\dot{\Phi} - 3a^2\dot{\Phi}\dot{\xi} - 6a\psi\dot{a}\dot{\Phi} - a^2\dot{\Phi}\dot{\psi} + \sum_{k=1}^{3}\left[\frac{\partial\Phi}{\partial x^k}\frac{\partial}{\partial x^k}(\xi-\psi)\right] - \Delta\Phi(1+2\xi)\right\} \tag{13}$$

5 Decomposition of field into background + perturbation

Here we decompose dark energy field(Φ) as

$$\Phi(x,y,z,t) = \phi(t) + \epsilon\chi(x,y,z,t) \tag{14}$$

then we get following quantities in first order approximation($O[\epsilon]^1$). Please have a look at equations 6.1 to 6.8 of reference [1] to verify that these are indeed correct apart from an issue of gauge invariance of χ(Compare, from [1], eqn 6.6 and eqn 6.7 using eqn 6.8. Our equations correspond to 6.6).

5.1 T_μ^ν

$$T_k^k = \left(V - \frac{\dot\phi^2}{2}\right) + \left(\chi V_{,\phi} + \psi\dot\phi^2 - \dot\phi\dot\chi\right)\epsilon + O[\epsilon]^2 \tag{15}$$

$$T_t^t = \left(V + \frac{\dot\phi^2}{2}\right) + \left(\chi V_{,\phi} - \psi\dot\phi^2 + \dot\phi\dot\chi\right)\epsilon + O[\epsilon]^2 \tag{16}$$

$$T_k^t = \left(\dot\phi\frac{\partial\chi}{\partial x^k}\right)\epsilon + O[\epsilon]^2 \tag{17}$$

$$T_t^k = -\left(\frac{\dot\phi}{a^2}\frac{\partial\chi}{\partial x^k}\right)\epsilon + O[\epsilon]^2 \tag{18}$$

$$T_k^j = O[\epsilon]^2 \tag{19}$$

5.2 Field dynamics equation

$$\begin{aligned} T_0{}_{,\nu}^{\nu} &= V_{,\phi}\,\dot\phi + 3\frac{\dot a}{a}\dot\phi^2 + \dot\phi\ddot\phi \\ &+ \frac{\epsilon}{a^2}\left(-6a\psi\dot a\dot\phi^2 + a^2\chi\dot\phi V_{,\phi\phi} - 2a^2\psi\dot\phi\ddot\phi - 3a^2\dot\phi^2\dot\xi + a^2V_{,\phi}\,\dot\chi + 6a\dot a\dot\phi\dot\chi + a^2\ddot\phi\dot\chi - a^2\dot\phi^2\dot\psi + a^2\dot\phi\ddot\chi - \dot\phi\Delta\chi\right) \end{aligned} \tag{20}$$

$$\begin{aligned} \ddot\chi = \frac{1}{a^2\dot\phi}\Big\{&-a^2V_{,\phi}\,\dot\phi - 3a\dot a\dot\phi^2 + 6a\psi\dot a\dot\phi^2 - a^2\chi\dot\phi V_{,\phi\phi} - a^2\ddot\phi\dot\phi \\ &+2a^2\psi\dot\phi\ddot\phi + 3a^2\dot\phi^2\dot\xi - a^2V_{,\phi}\,\dot\chi - 6a\dot a\dot\phi\dot\chi - a^2\ddot\phi\dot\chi + a^2\dot\phi^2\dot\psi + \dot\phi\Delta\chi\Big\} \end{aligned} \tag{21}$$

6 Einstein's equations zeroth order

Energy density for matter fluid is $\rho_m = \rho_{m0} + \epsilon(\delta\rho_m)$, pressure is $P = P_0 + \epsilon(\delta P)$ and velocity is $u^\alpha = u_0^\alpha + \epsilon(\delta u^\alpha)$

$$G_k^k = \frac{\dot{a}^2}{a^2} + \frac{2\ddot{a}}{a} = 8\pi\left\{-P_0 - \left(\frac{\dot{\phi}^2}{2} - V\right)\right\} \tag{22}$$

$$G_k^t = 0 \tag{23}$$

$$G_k^j = 0 = 0 \tag{24}$$

$$G_t^t = \frac{3\dot{a}^2}{a^2} = 8\pi\left\{\rho_{m0} + \frac{\dot{\phi}^2}{2} + V\right\} \tag{25}$$

7 Einstein's equations 1st order

$$G_k^k = \frac{1}{a^2}\left\{-2\psi\dot{a}^2 - 4a\psi\ddot{a} - 6a\dot{a}\dot{\xi} - 2a\dot{a}\dot{\psi} - 2a^2\ddot{\xi} + \sum_{i\neq k}\frac{\partial^2}{\partial x^{i2}}(\xi - \psi)\right\} = 8\pi\left\{\chi V,_\phi + \psi\dot{\phi}^2 - \dot{\phi}\dot{\chi} - (\delta P)\right\} \tag{26}$$

$$G_k^t = 2\left[\frac{\dot{a}}{a}\frac{\partial\psi}{\partial x^k} + \frac{\partial^2\xi}{\partial x^k\partial t}\right] = 8\pi\left\{\dot{\phi}\frac{\partial\chi}{\partial x^k} + (\rho_{m0} + P_0)(\delta u_k)\right\} \tag{27}$$

$$G_k^j = \frac{1}{a^2}\left\{-\frac{\partial^2\xi}{\partial x^k\partial x^j} + \frac{\partial^2\psi}{\partial x^k\partial x^j}\right\} = 0 \tag{28}$$

$$G_t^t = -\frac{2}{a^2}\left\{3\psi\dot{a}^2 + 3a\dot{a}\dot{\xi} - \Delta\xi\right\} = 8\pi\left\{\chi V,_\phi - \psi\dot{\phi}^2 + \dot{\phi}\dot{\chi} + (\delta\rho_m)\right\} \tag{29}$$

8 Geodesic Equations

$$\frac{d^2x^k}{d\tau^2} = -2u^t u^k\left(\frac{\dot{a}}{a}\right) + \epsilon\left[2u^t u^k\dot{\xi} + 2u^k\left(\sum_{j=1}^{3}u^j\frac{\partial\xi}{\partial x^j}\right) - \frac{\partial\xi}{\partial x^k}\left\{\sum_{j=1}^{3}(u^j)^2\right\} - \frac{u^{t2}}{a^2}\frac{\partial\psi}{\partial x^k}\right] \tag{30}$$

$$\frac{d^2t}{d\tau^2} = -\left\{\sum_{j=1}^{3}(u^j)^2\right\}a\dot{a} + \epsilon\left[2a\dot{a}\left(\sum_{j=1}^{3}(u^j)^2\right)(\xi + \psi) + \left(\sum_{j=1}^{3}(u^j)^2\right)a^2\frac{\partial\xi}{\partial t} - 2u^t\sum_{j=1}^{3}\left(u^j\frac{\partial\psi}{\partial x^j}\right) - u^{t2}\frac{\partial\psi}{\partial t}\right] \tag{31}$$

9 Equations Linear in Metric Potentials but non-linear in source fields

Lets define for matter

$$(\delta T^{\nu}_{\mu})_m = T^{\nu}_{\mu\,m} - \bar{T}^{\nu}_{\mu\,m} \tag{32}$$

This means δ_m is not assumed to be small and χ is also not truncated in higher orders while we do truncate higher order terms of metric potentials on both sides.

$$\begin{aligned} G^k_k &= \frac{1}{a^2}\left\{-2\psi\dot{a}^2 - 4a\psi\ddot{a} - 6a\dot{a}\dot{\xi} - 2a\dot{a}\dot{\psi} - 2a^2\ddot{\xi} + \sum_{i\neq k}\frac{\partial^2}{\partial x^{i2}}(\xi-\psi)\right\} \\ &= 8\pi\left\{V(\Phi) - V(\phi) + \psi\dot{\phi}^2 - \frac{\dot{\chi}^2(1-2\psi)}{2} - (1-2\psi)\dot{\phi}\dot{\chi} + \left[\sum_{i\neq k}\left(\frac{\partial\chi}{\partial x^i}\right)^2 - \left(\frac{\partial\chi}{\partial x^k}\right)^2\right]\frac{(1+2\xi)}{2a^2}\right\} \\ &+ 8\pi(\delta T^k_k)_m \end{aligned} \tag{33}$$

$$G^t_k = 2\left[\frac{\dot{a}}{a}\frac{\partial\psi}{\partial x^k} + \frac{\partial^2\xi}{\partial x^k\partial t}\right] = 8\pi\left\{(\dot{\phi}+\dot{\chi})\frac{\partial\chi}{\partial x^k}(1-2\psi)\right\} + 8\pi(\delta T^t_k)_m \tag{34}$$

$$G^j_k = \frac{1}{a^2}\left\{-\frac{\partial^2\xi}{\partial x^k\partial x^j} + \frac{\partial^2\psi}{\partial x^k\partial x^j}\right\} = 8\pi\left\{-\frac{\frac{\partial\chi}{\partial x^k}\frac{\partial\chi}{\partial x^j}}{a^2}(1+2\xi)\right\} + 8\pi(\delta T^j_k)_m \tag{35}$$

$$\begin{aligned} G^t_t &= -\frac{2}{a^2}\left\{3\psi\dot{a}^2 + 3a\dot{a}\dot{\xi} - \Delta\xi\right\} \\ &= 8\pi\left\{V(\Phi) - V(\phi) - \psi\dot{\phi}^2 + \chi\dot{\phi}(1-2\psi) + \frac{\dot{\chi}^2}{2}(1-2\psi) + \frac{\vec{\nabla}\chi.\vec{\nabla}\chi}{2a^2}(1+2\xi)\right\} \\ &\quad + 8\pi(\delta T^t_t)_m \end{aligned} \tag{36}$$

References

[1] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, "Theory of cosmological perturbations," , vol. 215, pp. 203–333, June 1992.