Probability and Queueing Theory ①

Unit - 1
Random variables

Discrete Distribution:

Binomial Distribution functions

PMF	$P(X=x) = n_{C_x} p^x q^{n-x}$
MGF	$(q+pe^t)^n$
Mean	np
variance	npq
S.D	$\sqrt{npq}$

① Find the MGF of the binomial R.V with parameters n & p and hence find its mean & variance.

Soln

WKT, The MGF of a R.V X about origin whose probability function $p(x)$ is given by

$$M_X(t) = \sum_{x=0}^{n} e^{tx} p(x) \qquad [p(x) \text{ is a pmf}]$$

$$p(x) = n_{C_x} p^x q^{n-x}$$

$$M_X(t) = \sum_{x=0}^{n} e^{tx} n_{C_x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (e^t)^x p^x n_{C_x} q^{n-x}$$

$$= \sum_{x=0}^{n} (pe^t)^x n_{C_x} q^{n-x}$$

$$M_X(t) = (pe^t)^0 n_{C_0} q^{n-0} + (pe^t)^1 n_{C_1} q^{n-1} + \dots + (pe^t)^n n_{C_n} q^{n-n}$$

$$= q^n + (pe^t)^1 n_{C_1} q^{n-1} + \dots$$

$$\boxed{M_X(t) = (q+pe^t)^n} \quad \text{— ①}$$

$\therefore$ M.g.f of binomial distribution is $M_X(t) = (q+pe^t)^n$

mean $= M_x'(t) = n(q+pe^t)^{n-1} \cdot pe^t$

put $t=0$

mean $= M_x'(0) = n(q+pe^0)^{n-1} pe^0$ $[e^0=1]$

$= n(q+p)^{n-1} p(1)$ $[\because q+p=1]$

$= n(1)^{n-1} p$

$$\boxed{\text{mean} = M_x'(0) = np}$$

$Var(x) = E(x^2) - [E(x)]^2$

$= M_x''(0) - [M_x'(0)]^2$.

WKT

$M_x'(t) = n(q+pe^t)^{n-1} pe^t$

$M_x'(t) = np[(q+pe^t)^{n-1} e^t]$ — ②

Here $u = (q+pe^t)^{n-1}$, $v = e^t$

$u' = (n-1)(q+pe^t)^{n-2} \cdot pe^t$ $v' = e^t$

WKT
$d(uv) = u'v + uv'$

Diff w.r.to eqn ②

$M_x''(t) = np\left[(n-1)(q+pe^t)pe^t \cdot e^t + (q+pe^t)^{n-1} \cdot e^t\right]$

put $t=0$

$M_x''(0) = np\left[(n-1)(q+pe^0)^{n-1} pe^0 \cdot e^0 + (q+pe^0)^{n-1} e^0\right]$

$= np\left[(n-1)(q+p)^{n-1} p + (q+p)^{n-1}\right]$

$\because q+p=1$, $q=1-p$

$M_x''(0) = np\left[(n-1)(1)^{n-1} p + (1)^{n-1}\right]$

$= np[(n-1)p+1]$

$= np[np-p+1]$

$= np[np+1-p]$ $(\because q=1-p)$

$= np[np+q]$

$$M_x''(0) = n^2p^2 + npq$$

$$\text{Var}(x) = M_x''(0) - [M_x'(0)]^2$$

$$= n^2p^2 + npq - n^2p^2$$

$$\boxed{\text{Var}(x) = npq}$$

(2) Find the 1st 3 moments of the binomial distribution from $M_x(t) = (q + pe^t)^n$

Note
$q + p = 1$
$1 - p = q$

Soln

WKT

$$M_x'(t) = np[(q + pe^t)^{n-1} \cdot e^t]$$

$$M_x''(t) = np[(n-1)e^t(q+pe^t)^{n-2} \cdot pe^t + (q+pe^t)^{n-1}e^t]$$

$$= np[(n-1)(q+pe^t)^{n-2}pe^{2t} + (q+pe^t)^{n-1}e^t]$$

$$M_x'''(t) = np[(n-1)(n-2)(q+pe^t)^{n-3}pe^t \cdot pe^{2t} + (n-1)(q+pe^t)^{n-2} p2e^{2t} + (n-1)(q+pe^t)^{n-2}pe^te^t + (q+pe^t)^{n-1}e^t]$$

Put $t = 0$

$$M_x'''(0) = np[(n-1)(n-2)(q+p(1))^{n-3}p(1) \cdot p(1) + (n-1)(q+p(1))^{n-2} p2(1) + (n-1)(q+p(1))^{n-1}p(1)(1) + (q+p(1))^{n-1}(1)]$$

$$= np[(n^2 - 3n + 2)p^2 + (n-1)2p + (n-1)p + 1]$$

$$= np[p^2n^2 - 3p^2n + 2p^2 + 2pn - 2p + np - p + 1]$$

$$= np[p^2n^2 - 3p^2n + 2p^2 + 3np - 3p + 1]$$

$$= np[p^2n^2 + 2p^2 + 3np(1-p) - 3p + 1]$$

$$= np[p^2n^2 + 2p^2 + 3npq - 3p + 1]$$

(3) State and prove additive property of Binomial distribution.

Statement

The sum of two independent binomial distribution is also binomial distribution.

Proof

Let x_1 and x_2 be two independent binomial distributions with parameters (n_1, p) and (n_2, p)

i.e, $X_1 \sim B(n_1, p)$, $X_2 \sim B(n_2, p)$

$$M_{X_1}(t) = (q + pe^t)^{n_1}, \quad M_{X_2}(t) = (q + pe^t)^{n_2}$$

$$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$$

$$= (q + pe^t)^{n_1} \cdot (q + pe^t)^{n_2}$$

$$\boxed{M_{X_1+X_2}(t) = (q + pe^t)^{n_1+n_2}}$$

$\Rightarrow X_1 + X_2 \sim B(n_1 + n_2, p)$

$\therefore X_1 + X_2$ is binomial distribution with parameters (n_1+n_2, p)

Poisson Distribution

Note

PMF	$P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, \ldots \infty$
MGF	$M_X(t) = e^{(e^t-1)\lambda}$
mean	λ
var(x)	λ
S.D	$\sqrt{\lambda}$

$x \to$ no. of success

$\lambda \to$ parameter

1. By calculating the MGF of poisson distribution with parameter λ. Prove that the mean & variance of the poisson distribution are equal.

Soln

To find MGF

$$M_X(t) = \sum_{x=0}^{\infty} e^{xt} p(x)$$

Note
* $\frac{d}{dx}(e^{ax}) = a e^{ax}$
* $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$

WKT PMF $p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$

$$M_x(t) = \sum_{x=0}^{\infty} e^{xt} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} (e^t \lambda)^x \cdot \frac{e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}$$

$$= e^{-\lambda} \left[\frac{(e^t\lambda)^0}{0!} + \frac{(e^t\lambda)^1}{1!} + \frac{(e^t\lambda)^2}{2!} + \dots \right]$$

$$= e^{-\lambda} \left[1 + \frac{(e^t\lambda)^1}{1!} + \frac{(e^t\lambda)^2}{2!} + \dots \right]$$

$$= e^{-\lambda} \left[e^{e^t \lambda} \right]$$

$$\boxed{M_x(t) = e^{-\lambda + e^t \lambda}} \quad (\text{or}) \quad \boxed{M_x(t) = e^{(e^t - 1)\lambda}}$$

To find mean:

WKT mean $= E(x) = M_x(t) = e^{(e^t-1)\lambda}$

$$M_x'(t) = e^{(e^t-1)\lambda} \cdot (e^t - 0)\lambda$$

$$= e^t \lambda \, e^{(e^t-1)\lambda}$$

$$\left[M_x'(t) \right]_{t=0} = e^0 \lambda \, e^{(e^0-1)\lambda} = \lambda$$

$$\boxed{\therefore \text{Mean} = M_x'(t) = \lambda}$$

To find variance:

$$\text{var}(x) = E(x^2) - [E(x)]^2$$

Here $M_x'(t) = e^t \lambda \, e^{(e^t-1)\lambda}$

$u = e^t \lambda$	$v = e^{(e^t-1)\lambda}$
$u' = e^t \lambda$	$v' = e^{(e^t-1)\lambda} \cdot (e^t - 0)\lambda$

$M_X''(t) = uv' + vu'$

$= (e^t\lambda)\left(e^{(e^t-1)\lambda} e^t\lambda\right) + \left(e^{(e^t-1)\lambda}\right)(e^t\lambda)$

$= \lambda^2 e^{2t} e^{(e^t-1)\lambda} + e^t\lambda e^{(e^t-1)\lambda}$

$\left[M_X''(t)\right]_{t=0} = \lambda^2 e^{2(0)} e^{(e^0-1)\lambda} + e^0\lambda\, e^{(e^0-1)\lambda}$

$\boxed{\left[M_X''(t)\right]_{t=0} = \lambda^2 + \lambda}$

$Var(x) = E(x^2) - [E(x)]^2$

$= M_X''(t) - [M_X'(t)]^2$

$= \lambda^2 + \lambda - [\lambda]^2 = \lambda^2 + \lambda - \lambda^2$

$\boxed{Var(x) = \lambda}$

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(2) State and prove additive property of poisson distribution

Statement

The sum of two independent poisson distribution is also a poisson distribution.

Proof

Let X_1, X_2 be two Independent poisson distribution with parameters λ_1 & λ_2 respectively.

$\Rightarrow X_1 \sim P(\lambda_1)$; $X_2 \sim P(\lambda_2)$

$M_{X_1}(t) = e^{(e^t-1)\lambda_1}$; $M_{X_2}(t) = e^{(e^t-1)\lambda_2}$

$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$ ($\because X_1$ & X_2 are independent)

$= e^{(e^t-1)\lambda_1 + (e^t-1)\lambda_2}$

$= e^{(t-1)(\lambda + \lambda_2)}$

$\Rightarrow X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$

ie $X_1 + X_2$ is also poisson distribution with parameter $\lambda_1 + \lambda_2$

Geometric Distribution:-

Note

PMF	$P(X=x) = q^{x-1} \cdot p$
MGF	$\frac{pe^t}{1-qe^t}$
mean	$1/p$
var(x)	q/p^2

① Find the MGF of Geometric Distribution R.V X and hence find its mean and variance.

Soln

To find MGF

$$\text{WKT } M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(x) \qquad [p(x) = q^{x-1} p]$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} \cdot p$$

$$= \sum_{x=1}^{\infty} e^{tx} e^{t} e^{-t} q^{x-1} p$$

$$= \sum_{x=1}^{\infty} e^{t} e^{tx-t} q^{x-1} p$$

$$= pe^{t} \sum_{x=1}^{\infty} e^{(x-1)t} q^{x-1}$$

$$= pe^{t}\left[e^{0t} q^{0} + e^{t} q^{1} + e^{2t} q^{2} + \cdots \right]$$

$$= pe^{t}\left[1 + (e^{t} q)^{1} + (e^{t} q)^{2} + \cdots \right]$$

$$= pe^{t}\left[1 - e^{t} q \right]^{-1}$$

$$\boxed{\text{MGF} = \frac{pe^{t}}{1-qe^{t}}}$$

To find mean.

$$\text{WKT } M_X(t) = \frac{pe^{t}}{1-qe^{t}}$$

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$$M_x'(t) = \frac{[(1-qe^t)pe^t - pe^t(0-qe^t)]}{(1-qe^t)^2}$$

$$= \frac{[pe^t - qe^t pe^t + qe^t pe^t]}{(1-qe^t)^2}$$

$$M_x'(t) = \frac{pe^t}{(1-qe^t)^2}$$

$$[M_x'(t)]_{t=0} = \frac{p}{(1-q)^2}$$

WKT, $p+q=1$, $1-q=p$

$$[M_x'(t)]_{t=0} = \frac{p}{(p)^2} = 1/p$$

$$\boxed{\text{mean} = 1/p}$$

To find variance

$$var(x) = E(x^2) - [E(x)]^2$$

WKT, $[M_x'(t)] = \frac{pe^t}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$

$$M_x''(t) = \frac{(1-qe^t)^2(pe^t) - 2(1-qe^t)(0-qe^t)pe^t}{(1-qe^t)^4}$$

$$[M_x''(t)]_{t=0} = \frac{(1-q)^2 p - 2(1-q)(-q)p}{(1-q)^4} = \frac{p^2 \cdot p + 2pqp}{p^4}$$

$\because p+q=1$, $1-q=p$

$$= \frac{p^2(p+2q)}{p^4} = \frac{p+2q}{p^2}$$

$$\boxed{[M_x''(t)]_{t=0} = \frac{p+2q}{p^2}}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= M_x''(t) - [M_x'(t)]^2$$

$$= \frac{p+2q}{p^2} - \frac{1}{p^2}$$

$$= \frac{1-q+2q-1}{p^2}$$

$$\boxed{Var(x) = q/p^2}$$

2) State and prove memoryless property

Statement

Let X be the ~~geme~~ geometric distribution then for any 2 positive integer m & n. $P(x > m+n / x > m) = P(x > n)$

Proof

Given, $P(x > m+n / x > m) = P(x > n)$

WKT $P(x = x) = q^{x-1} \cdot p$

RHS:-

$$P(x > n) = P(x = n+1) + P(x = n+2) + P(x = n+3) + \cdots$$

$$= q^{n+1-1} p + q^{n+2-1} p + q^{n+3-1} p + \cdots$$

$$= q^n p + q^{n-1} p + q^{n+2} p + \cdots$$

$$= q^n p [1 + q^1 + q^2 + \cdots]$$

$$= q^n p [1-q]^{-1}$$

$$= q^n p [p]^{-1}$$

$$= \frac{q^n p}{p}$$

$$\boxed{P(x > n) = q^n}$$

Similarly,

$$P(x>m) = q^m$$

$$P(x>m+n) = q^{m+n}$$

LHS:-

$$P(x>m+n/x>m) = \frac{P(x>m+n)\cap P(x>m)}{P(x>m)}$$

$$= \frac{P(x>m+n)}{P(x>m)} = \frac{q^{m+n}}{q^m}$$

$$= \frac{q^m \cdot q^n}{q^m}$$

$$\boxed{P(x>m+n/x>m) = q^n}$$

Continuous Distribution:

Let x be a uniform distribution in the interval (a,b)

with PDF, $f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$

To find MGF

WKT $M_x(t) = \int_{-\infty}^{\infty} e^{xt} f(x)\, dx.$

$f(x) = \frac{1}{b-a}, \quad a<x<b$

$$M_x(t) = \int_a^b e^{xt} \frac{1}{b-a}\, dx.$$

$$= \frac{1}{b-a} \int_a^b e^{tx}\, dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

$$= \frac{1}{t(b-a)} \left[e^{bt} - e^{at} \right]$$

To find mean

$$\text{Mean } E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_a^b \frac{1}{b-a}\, x\, dx$$

$$= \frac{1}{b-a}\int_a^b x\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{1}{\cancel{b-a}}\left[\frac{(b+a)\cancel{(b-a)}}{2}\right]$$

$$\boxed{E(x) = \frac{b+a}{2} \text{ (or) } \frac{a+b}{2}}$$

Find variance:

$$\text{var}(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right] = \frac{1}{b-a}\left[\frac{(b-a)(b^2+ab+a^2)}{3}\right]$$

$$E(x^2) = \frac{b^2+ab+a^2}{3}$$

$$\text{var}(x) = \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4(b^2+ab+a^2) - 3(a^2+b^2+2ab)}{12}$$

$$= \frac{4b^2+4ab+4a^2-3a^2-3b^2-6ab}{12} = \frac{a^2+b^2-2ab}{12}$$

$$\boxed{\text{var}(x) = \frac{(a-b)^2}{12}}$$

Exponential Distribution

Memory less property

Statement:

If x is a Exponential distribution, let m, n be two positive integers then $P(x > m+n / x > m) = P(x > n)$.

Proof

Let x be a ED $\therefore f(x) = \lambda e^{-\lambda x}, x \geq 0$

RHS $P(x > n) = \int_n^\infty f(x)\,dx$

$$= \int_n^\infty \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_n^\infty e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_n^\infty$$

$$= -\left[e^{-\infty} - e^{-\lambda n} \right] = e^{-\lambda n}$$

$$P(x > n) = e^{-\lambda n}$$

Similarly, $P(x > m) = e^{-\lambda m}$

$$P(x > m+n) = e^{-\lambda(m+n)}$$

LHS $P(x > m+n / x > m) = \dfrac{P(x > m+n \cap x > m)}{P(x > m)}$.

$$= \frac{P(x > m+n)}{P(x > m)}$$

$$= \frac{e^{-\lambda(m+n)}}{e^{-\lambda m}}$$

$$= \frac{\cancel{e^{-\lambda m}} \cdot e^{-\lambda n}}{\cancel{e^{-\lambda m}}}$$

$$= e^{-\lambda n}$$

LHS = RHS

Exponential Distribution

Let x be a exponential distribution with PDF

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

To find MGF:

$$M_x(t) = \int_{-\infty}^{\infty} e^{xt} f(x)\, dx$$

WKT ED is

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \ (0, \infty)$$

$$M_x(t) = \int_0^{\infty} \lambda e^{-\lambda x} e^{xt}\, dx.$$

$$= \lambda \int_0^{\infty} e^{-\lambda x} e^{xt}\, dx$$

$$= \lambda \int_0^{\infty} e^{-\lambda x + xt}\, dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x}$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \lambda \left[\frac{e^{-\infty}}{-(\lambda - t)} + \frac{e^{-0}}{\lambda - t} \right]$$

$$\boxed{M_x(t) = \frac{\lambda}{\lambda - t}}$$

To find mean

WKT $M_x(t) = \dfrac{\lambda}{\lambda - t}$

$$M_x'(t) = \lambda \left[-\frac{1}{(\lambda - t)^2} (-1) \right]$$

$$\left[M_x'(t) \right]_{t=0} = \lambda \left[+ \frac{1}{\lambda^2} \right]$$

$$\boxed{\left[M_x'(t) \right]_{t=0} = 1/\lambda}$$

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To find variance

$\text{var}(x) = E(x^2) - [E(x)]^2$

$M_x'(t) = \dfrac{\lambda}{(\lambda - t)^2}$, $M_x''(t) = \dfrac{-2\lambda}{(\lambda - t)^3}(-1) = \dfrac{2\lambda}{(\lambda - t)^3}$

$[M_x''(t)]_{t=0} = \dfrac{2\lambda}{(\lambda - 0)^3} = \dfrac{2\cancel{\lambda}}{\cancel{\lambda^3}} = 2/\lambda^2$

$E(x^2) = 2/\lambda^2$

$\text{var}(x) = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$

$\boxed{\text{var}(x) = 1/\lambda^2}$

Normal Distribution

A continuous R.V to the parameters μ and σ^2 is said to follow a normal distribution, if its PDF is given by

$$f(x) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, & -\infty < x < \infty, \ \sigma > 0 \\ 0, & \text{otherwise} \end{cases}$$

x is called normal R.V.

To find MGF

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx$$

$\boxed{x = \sigma z + \mu}$, $z = \frac{x-\mu}{\sigma} = x/\sigma - \mu/\sigma$

when $x = \infty$, $z = \infty$

when $x = -\infty$, $z = -\infty$

$$\frac{dz}{dx} = 1/\sigma$$

$$dz = \frac{dx}{\sigma}$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - z^2/2} \cancel{\sigma}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - z^2/2} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z - z^2/2} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z^2/2 - t\sigma z)} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2}{2} - t\sigma z + \frac{t^2\sigma^2}{2} - \frac{t^2\sigma^2}{2}\right)} dz$$

$$= \frac{e^{t\mu} e^{t^2\sigma^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(z/\sqrt{2} - \frac{t\sigma}{\sqrt{2}}\right)^2} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} e^{\frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} e^{-1/2 (z - t\sigma)^2} dz$$

let $u = z - \sigma t$ | $z = -\infty, u = -\infty$

$du = dz$ | $z = \infty, u = \infty$

$$M_x(t) = \frac{e^{\mu t}}{\sqrt{2\pi}} e^{\frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} e^{-1/2 u^2} du$$

$$\boxed{\int_0^{\infty} e^{-u^2/2} du = \sqrt{\pi/2}}$$

$$M_x(t) = \frac{e^{\mu t}}{\sqrt{2\pi}} e^{\frac{\sigma^2 t^2}{2}} 2\int_0^{\infty} e^{-1/2 u^2} du$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} e^{\frac{\sigma^2 t^2}{2}} 2\left(\sqrt{\pi/2}\right) = \frac{e^{\mu t}}{\cancel{\sqrt{2}}\cdot\cancel{\sqrt{\pi}}} e^{\frac{\sigma^2 t^2}{2}} \cancel{\sqrt{2}} \cdot \cancel{\sqrt{2}} \cancel{\frac{\sqrt{\pi}}{\sqrt{2}}}$$

$$\boxed{M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}}$$

<u>To find mean</u>

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$E(x) = \text{mean} = M_x'(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \left(\mu + \frac{2\sigma^2 t}{2}\right)$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} (\mu + \sigma^2 t)$$

$$M_x'(t) = (\mu + \sigma^2 t)\left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)$$

$u = \mu + \sigma^2 t$	$v = e^{\mu t + \frac{\sigma^2 t^2}{2}}$
$du = \sigma^2$	$dv = e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot (\mu + \sigma^2 t)$

$$M_x''(t) = uv' + vu'$$

$$= (\mu + \sigma^2 t)\left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)(\mu + \sigma^2 t) + \sigma^2\left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)$$

$$M_x''(t) = (\mu+\sigma^2 t)^2\left(e^{\mu t+\frac{\sigma^2 t^2}{2}}\right) + (\sigma^2)\left(e^{\mu t+\frac{\sigma^2 t^2}{2}}\right)$$

$$\text{mean} = \left[M_x'(t)\right]_{t=0} = \mu$$

$$\boxed{M_x'(t) = \mu}$$

<u>To find var(x)</u>

$$\left[M_x''(t)\right]_{t=0} = \mu^2+\sigma^2$$

$$\text{var}(x) = E(x^2) - [E(x)]^2$$

$$= \mu^2+\sigma^2-\mu^2$$

$$\boxed{\text{var}(x) = \sigma^2}$$

<u>To find S.D</u>

$$S.D = \sqrt{\text{var}(x)}$$

$$= \sqrt{\sigma^2}$$

$$\boxed{S.D = \sigma}$$