1D Unsteady diffusion

$$\frac{\partial c}{\partial t} = D\left[\frac{\partial^2 c}{\partial x^2}\right]$$

Writing by taking all terms to one side.

$$\left[\frac{\partial c}{\partial t} - D\frac{\partial^2 c}{\partial x^2}\right] = 0$$

Generating the weak form by using weighted residual method

$$\int_{\Omega^e} \left(w\frac{\partial c}{\partial t} + w\frac{\partial w}{\partial x}\left(\frac{\partial c}{\partial x}\right)(-1)D\right] dx = BT$$

Boundary terms

$$w\frac{\partial c}{\partial t}\Big|_i^j - \int_{x_i}^{x_j} \frac{\partial w}{\partial x} D \frac{\partial c}{\partial x} = 0 \quad \text{——— (1)}$$

Let l_e = length of element $= |x_j - x_i|$

Let us assume $c(x,t)$ to be of the form

$$c^e(x,t) = \sum_{j=1}^{n} c_j(t)\, s_j(x)$$

(product of some function in x & t).

$\therefore$ (1) becomes

$$\int_{\Omega^e} \left[w\left(\sum \frac{dc}{dt} s_j\right) + D\frac{dw}{dx}\left(\sum c_j \frac{ds_j}{dx}\right)\right] dx = 0$$

Using Gallerkin FEM, selecting $w = s_i(x)$,

we get

$$\int_{\Omega^e} s_i\left(\sum \frac{dc}{dt} s_j\right) + D\left(\frac{ds_i}{dx}\right)\left(\sum c_j \frac{ds_j}{dx}\right)\Big] dx = 0.$$

Taking summation outside the integral

$$\sum\left[\int_{\Omega^e} s_i s_j\, dx\right]\frac{dc_j}{dt} + \sum\left[\left[D\frac{ds_i}{dx}\frac{ds_j}{dx}\right]dx\right] T_j = 0$$

The final equation is of the form,

$$[M^e][\dot{c}^e] + [K^e][c^e] = \{F^e\} \quad \text{--- ②}$$

here $F^e = 0$ given

Time Derivative term:

1) Using forward difference

$$\{\dot{c}\}_s = \frac{\{c\}_{s+1} - \{c\}_s}{\Delta t} + O(\Delta t).$$

Equation ② becomes

$$[M^e]\frac{\{c\}_{s+1} - \{c\}_s}{\Delta t} + [K^e][c^e] = 0$$

$$\boxed{\{c\}_{s+1} = [M^e]^{-1}\left[[M]\{c\}_s + \Delta t\,(-[K^e]\{c\}_s\right]} \quad \text{③}$$

To find out $[M^e]$ and $[K^e]$ for the elements:

Comparing ② with the integral equation, we get

$$M^e_{ij} = \int_{\Omega^e} S_i S_j \, dx.$$

we know from a linear trial function of Concentration,

$$S_i = \left[\frac{x_j - x}{x_j - x_i}\right] \qquad S_j = \left[\frac{x - x_i}{x_j - x_i}\right].$$

$$M^e_{ij} = \int_{x_i}^{x_j} S_i S_j \, dx$$

For a 2 element system

$$M^e_{ii} = \int_{x_i}^{x_j} S_i S_j \, dx = \left[\frac{l_e}{3}\right] = M^e_{jj}$$

$$M^e_{ij} = \int_{x_i}^{x_j} S_i S_j \, dx = \left[\frac{l_e}{6}\right] = M^e_{ji} \text{ [Symmetry]}$$

$$\therefore M^e = \frac{l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$\therefore$ M^e for the system =

$$\frac{l_e}{6}\begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & & \ddots & \\ & & & & 12 \end{bmatrix}$$

$$K_{ij} = \Sigma \int \left[\frac{dS_i}{dx}\right]\left[\frac{dS_j}{dx}\right] D \, dx$$

$$\frac{dS_i}{dx} = \frac{-1}{l_e} \qquad K_{ij} = -\int_{x_i}^{x_j} \frac{D}{l_e^2} \times \begin{bmatrix} -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix}[C]\, dx$$

$$K_{ij} = -\frac{D}{l_e} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\therefore$ $[K^e]$ for the system =

$$-\frac{D}{l_e}\begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & \ddots \\ & & -1 & 1 \end{bmatrix}$$

Solving system ③ using Gauss Seidal Method gives the desired Concentration values.