Team Test

Interlake Invitational Math Competition

March 14, 2015

1. **[10]** If $x^3 - 2x^2 + 5x - 7 = 0$ has three (not necessarily real) roots a, b, and c, find $a^2 + b^2 + c^2$.

2. **[10]** Determine the greatest integer that always divides $n(n + 2014)(n + 2015)$ for integer n.

3. **[10]** Find all pairs of positive integers (x, y) satisfying the equation $x^4 = y^2 + 15$. If there are multiple pairs, give a comma separated list of pairs. Otherwise, write a single pair.

4. **[12]** 4 distinct people are sitting down at a round table that has 9 distinct seats. The four people don't like each other, so no two can sit in adjacent seats. In how many ways can they do this?

5. **[12]** A cyclic quadrilateral $ABCD$, with AC perpendicular to BD, is inscribed in a circle O. Reflect point D over line AC to a point P. Suppose now that $DP = CD$. Given that $AD^2 = 10$ and $CD^2 = 36$, find AB^2.

6. **[14]** Suppose that $P(x)$ is a monic polynomial with integer coefficients such that $P(0) = 2015$. Moreover, it is given that all roots of $P(x)$ are rational and distinct (in particular, there are no complex roots). Find the maximum possible degree of P.

7. **[16]** Given a function f such that $f(\sqrt{xy}) = xyf(x - y + 2015)$ for all real x and y, find the value of $f(2015)$.

8. **[18]** Triangle ABC has lengths $AB = 4$, $AC = 5$, and $BC = 6$. Point D is on the angle bisector of $\angle BAC$ such that DC is parallel to BA. Find the length AD.

9. **[20]** n is a positive integer less than 100 such that $2^n - 1$ divisible by 77. Find the sum of all possible values of n.

10. **[24]** Three boys A, B, C and three girls D, E, F are in a magical forest. At first, all three boys love F. Every hour, a mischievous fairy picks a boy at random and sprinkles magic dust on his eyes, making him fall in love with a different girl at random. Each boy loves one girl at a time. When all the boys love a different girl, at that instant the mischievous fairy sprinkles dust on the girls' eyes to make them fall in love with the corresponding boy, and the children will leave the forest and live happily ever after. What is the expected number of hours required for this to happen?

11. **[24]** Six Alphas and Six Betas would like to pair up. An Alpha can pair with either 0 or 1 Betas, and a Beta can pair with either 0 or 1 Alphas. Moreover, the Alphas are numbered 1 through 6, and the Betas are numbered 1 through 6 as well. After pairing up, if there are two Alphas A and a such that A has a higher number than a, but is paired with a lower-numbered Beta than a, then A and a switch partners so that A is paired with the higher-numbered Beta. This continues until there are no more switches to be made. How many final configurations are possible?

12. **[30]** There are 101 lights in a circle, numbered 1 to 101, all initially off. There are 100 people numbered 1 to 100. Each person number x flips the switch numbered x, then the switch numbered $x^2 \pmod{101}$, then the switch numbered x^3, and so on until he/she flips the switch numbered 1, at which point he/she leaves. (For example, person 1 flips only the switch numbered 1, while person 100 flips switch 100 and then switch 1.) After all 100 people are done, how many switches are left on?