IMO True/False Test

Interlake Invitational Math Competition

March 14, 2015

To the left of each problem, answer "Yes," "No," "Open," or "None." Write "Open" if you believe the problem is unsolved and that no one knows the solution. Write "None" if you believe the answer is neither "Yes," "No," or "Open."

1. In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied? (ISL 2010 C1)

2. Let $\mathbb{Q}$ be the set of rationals. Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B, C such that the sets $A+B, B+C, C+A$ are disjoint?

   Here $X+Y$ denotes the set $\{x+y : x \in X, y \in Y\}$, for $X, Y \subseteq \mathbb{Z}$ and for $X, Y \subseteq \mathbb{Q}$. (ISL 2012 A2)

3. Does there exist a function f from reals to reals such that

   $$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x-y|$$

   holds for all real (x, y)? (USAMO 2000 #1)

4. Consider three real numbers a, b, c with product 1, and the expression

   $$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)$$

   Write "Yes" if this expression is at most 1, and "No" if this expression is at least 1. (IMO 2000 #2)

5. Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair (f, g) of functions from S into S is a [i]Spanish Couple[/i] on S, if they satisfy the following conditions: (i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all x, $y \in S$ with $x < y$;

   (ii) The inequality $f(g(g(x))) < g(f(x))$ holds for all $x \in S$.

   Decide whether there exists a Spanish Couple on the set $S = \mathbb{N}$ of positive integers. (ISL 2008 A3)

6. An acute-angled triangle ABC has orthocenter H. The circle passing through H with center the midpoint of BC intersects the line BC at A_1 and A_2. The points B_1 and B_2 are defined similarly. Does there exist a circle always passing through A_1, A_2, A_3, and A_4? (IMO 2008 #1)

7. Does the system of equations

   $$x^6 + x^3 + x^3y + y = 147^{157}$$
   $$x^3 + x^3y + y^2 + y + z^9 = 157^{147}$$

   have any integer solutions (x, y, z)? (USAMO 2005 #2)

8. In a triangle ABC, let D be the foot of the angle bisector from A to BC, and let E be the foot of the angle bisector from B to AC. A rhombus is inscribed in quadrilateral $AEDB$ such that one vertex of the rhombus is on each side of the quadrilateral. Does the obtuse angle of the rhombus always equal the obtuse angle formed by the intersection of AD and BE? (ISL 2013 G3)

9. Does there exist a set larger than the set of integers but smaller than the set of reals? (a set A is larger than a set B if there exists no pairing taking every element of set A to a distinct element of set B)

10. Does there exist, for every positive integer n, an n-digit number divisible by 5^n with only odd digits?

11. For every integer k, does there exist an arithmetic sequence of prime numbers of length k?

12. Does there exist an infinitely long arithmetic sequence of prime numbers?

13. Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0? (Putnam 2011 B3)

14. Determine whether, for every positive integer $n \geq 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. (USAMO 2011 #4)

15. Two players A and B play a game, given an integer N. A writes down 1 first. After this, on each player's turn, the player reads the integer n which his opponent most recently wrote down and then writes either $n+1$ or $2n$, as long as his number is not larger than N. The player who writes N wins. N is called type-A if A has a winning strategy, and type B if B has a winning strategy. Is the number 2004 type A? (ISL 2004 C5)

16. Call a number n a *primitive root* of a prime number p if and only if $p-1$ is the smallest integer such that n^{p-1} is congruent to 1 mod p. Is the number 10 a primitive root of infinitely many primes?

17. Let $f(x, y)$ be a continuous, real-valued function on $\mathbb{R}^2$. Suppose that, for every rectangular region R of area 1, the double integral of $f(x, y)$ over R equals 0. Must $f(x, y)$ be identically 0? (Putnam 2012 A6)

18. Let $f : \mathbb{R} \to \mathbb{N}$ be a function which satisfies $f\left(x + \dfrac{1}{f(y)}\right) = f\left(y + \dfrac{1}{f(x)}\right)$ for all x, $y \in \mathbb{R}$. Is f surjective over $\mathbb{N}$? (ISL 2008 A6)

19. Does there exist a strictly increasing function f from reals to reals which is continuous nowhere?

20. Let H be an arbitrary subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(M)$ of a differentiable manifold M. We say that an $\mathcal{C}^\infty$-vector field X is weakly tangent to the group H, if there exists a positive integer k and a $\mathcal{C}^\infty$-differentiable map $\varphi :]-\varepsilon, \varepsilon[^k \times M \to M$ such that (i) for fixed $t_1, \ldots, t_k$ the map

$$\varphi_{t_1,\ldots,t_k} : x \in M \mapsto \varphi(t_1, \ldots, t_k, x)$$

is a diffeomorphism of M, and $\varphi_{t_1,\ldots,t_k} \in H$; (ii) $\varphi_{t_1,\ldots,t_k} \in H = \mathsf{Id}$ whenever $t_j = 0$ for some $1 \leq j \leq k$; (iii) for any $\mathcal{C}^\infty$-function $f : M \to \mathbb{R}$

$$Xf = \left.\frac{\partial^k (f \circ \varphi_{t_1,\ldots,t_k})}{\partial t_1 \ldots \partial t_k}\right|_{(t_1,\ldots,t_k)=(0,\ldots,0)}.$$

Is every commutator of $\mathcal{C}^\infty$-vector fields that is weakly tangent to $H \subset \mathsf{Diff}^\infty(M)$ also weakly tangent to H? (Miklos Schweitzer 2009 7)