Algebra/Number Theory Individual Test

Interlake Invitational Math Competition

March 14, 2015

1. **[3]** When the numerator of a fraction, in lowest terms, is added to both the numerator and the denominator of the fraction, a new fraction is obtained which is $\frac{14}{11}$ of the original. Find the fraction.

2. **[3]** Find the largest integer n such that 390 divided by n, 235 divided by n, and 576 divided by n leave the same remainder.

3. **[3]** Suppose that a and r are two real numbers with $r > 0$ such that $a - ar^2 + ar^4 - ar^6 + \cdots = \frac{18}{5}$ and $a + ar^2 + ar^4 + ar^6 + \cdots = \frac{9}{2}$. Find $a + ar + ar^2 + \cdots$.

4. **[3]** Find the last three digits of $2015^{2015^{2015}}$.

5. **[4]** Consider the prime factorization of the product of the divisors of 330. What is the product of the exponents of the primes in this prime factorization?

6. **[4]** Consider the sequence of positive numbers $\{19, 28, 37, \ldots\}$, containing every positive integer which has a sum of digits divisible by 10. Find the 2015th number in this sequence.

7. **[4]** For some function f, $f(x) + f(y) = f(x + y) + 1$ for all integers x and y. If $f(1) = 3$, determine $f(0) + f(2015)$.

8. **[5]** Consider the complex number $z = \tan^2(72^\circ) + 2i\tan(72^\circ)$, and the complex number $w = 1 - z$. Given that $w = r(\cos\theta + i\sin\theta)$ for real number r and angle $0 \le \theta < 180$, find θ.

9. **[5]** A set S of 6 parabolas in the coordinate plane is given, where each parabola is of the form $f_i(x) = (6 - i)x^2 + p_i x + q_i$, for $0 \le i \le 5$. For $0 \le i \le 4$, parabolas f_i and f_{i+1} are tangent to each other at a point on the x-axis. Given that $f_0(x)$ has roots 1 and 0, let M be the sum of all 12 roots of all 6 parabolas (roots of different parabolas need not be distinct). Find the maximum possible value of M over all sets S.

10. **[6]** Let $X = \{5, 10, 13, 15, \ldots, 2010\}$, the set of all positive integers less than 2015 that are **not** relatively prime to 2015. For each $x \in X$, define g_x to be the greatest common factor of x and 2015, and define l_x to be the least common multiple of all positive integers less than $\frac{2015}{g_x}$ relatively prime to $\frac{2015}{g_x}$. Compute the remainder when the sum $\sum_{x \in X} \frac{{g_x}^2 l_x}{x}$ is divided by 2015.