Individual Final Round

Interlake Invitational Math Competition

March 14, 2015

This test contains three problems, to be attempted in 40 minutes. It is not intended that contestants will be able to solve all three problems within the allotted time. Write your solutions neatly on sheets of blank paper, and do not write within 1 inch of the edges of your paper. In the upper left hand corner of every page you submit, write your name, your school name, and your team number. In the upper right hand corner, write the problem number, page number and total pages for that problem (for example, "Problem 2, Page 2/3" would mean that you submitted a total of 3 pages for problem 2, and the current sheet is your second sheet for that problem). Do not write on both sides of any sheet submitted.

Each problem is scored out of 7 points. Partial credit is possible for nontrivial progress submitted.

For any geometry solution, a neat, large, labeled, and relatively in-scale diagram is required for full credit. Failure to include a diagram will result in an automatic one-point deduction from any solution earning at least one point.

1. (a) Let x be a positive number. Prove that
$$x^2 + 2 \geq 2\sqrt{x^3 + 1}$$
.
(b) Let a, b, c be positive real numbers. Prove that
$$\left(\frac{a+b+c}{3}\right)^2 + 2 \geq 2\sqrt{abc + 1}.$$

2. Prove that for all positive integers n, the number $5^{2^n} - 3^{2^n}$ has at least n distinct prime factors.

3. In $\triangle ABC$, point D is given on BC such that $\angle BAD = 2\angle ACD$ and $\angle CAD = 2\angle ABD$. Consider the circumcircles Γ_1 and Γ_2 of $\triangle ABD$ and $\triangle ACD$, respectively. Let E be the midpoint of arc BD on Γ_1, and let F be the midpoint of arc CD on Γ_2. Suppose that EA intersects the perpendicular bisector of line CD at F', and that FA intersects the perpendicular bisector of line BD at E'. Prove that $E'F' = EF$.