Algorithms and Data Structures

Sorting:
Simple Methods and a Lower Bound

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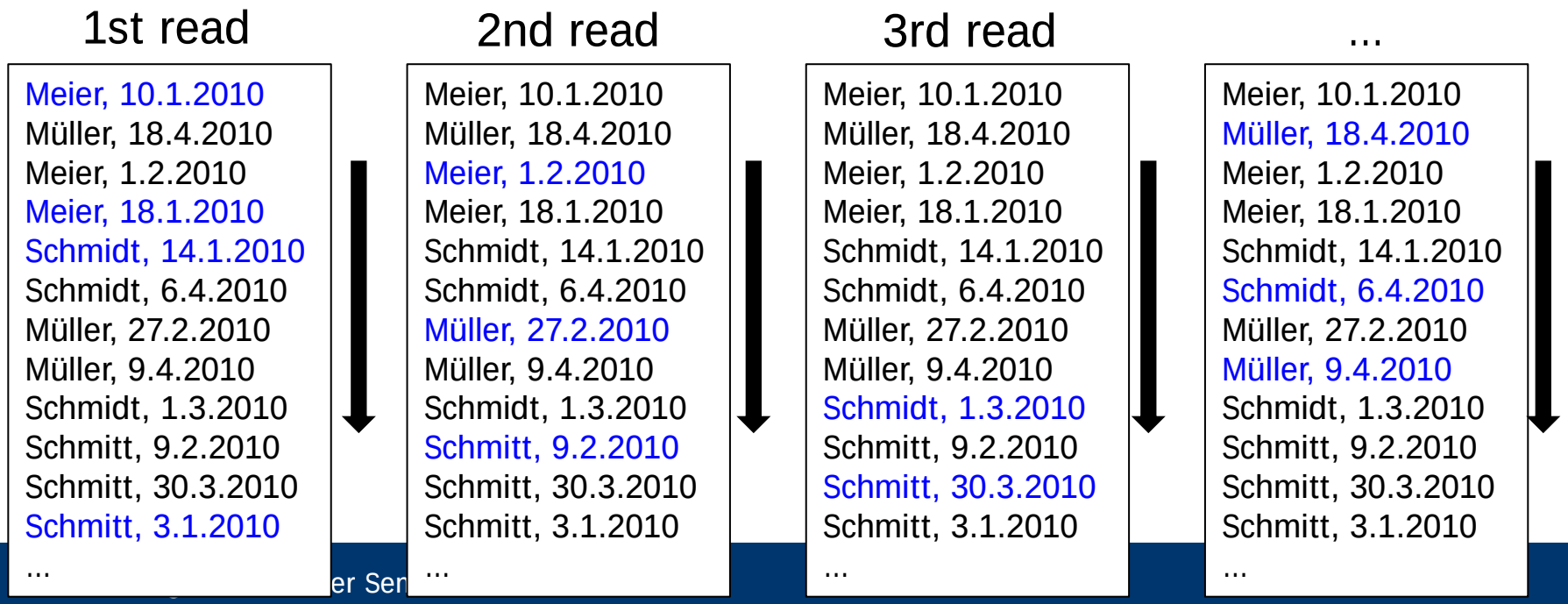
Large-Scale Sorting

- Imagine you are the IT head of a telco-company
- You have 30.000.000 customers each performing ~100 telephone calls per months, each call creating 200 bytes
  - That's 30M*100*12*200=7.200.000.000.000 bytes per year
  - Imagine the data is in a file, one line per call
- At the end of the year, management wants list of all customers with aggregated revenue per day
  - That's ~30M*12*30 ~ 10.000.000.000 real numbers
- Problem: How can we compute these 10.000.000.000 numbers?

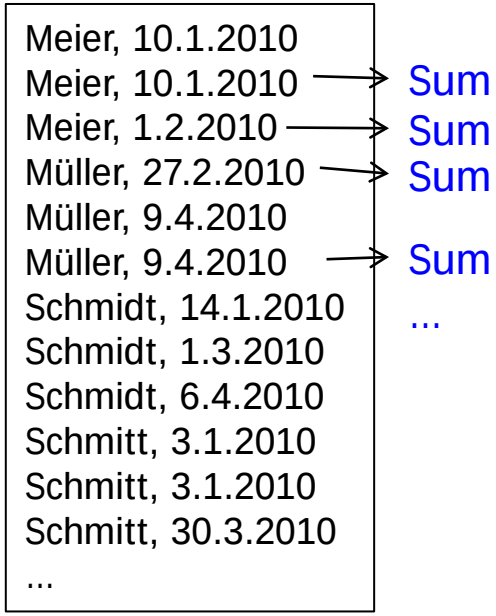
Approach: Multiple Reads

- Assume we can keep 30M*30 ~ 1E9 numbers in memory
  - Solve the problem month-by-month
  - Read the call-file 12 times, each time computing aggregates for all customers and the days of one month
  - This will be slow



Approach: Sorting

- Alternative?
  - Sort the file by customer and day
  - Read sorted file once and compute aggregates on the fly
  - Whenever a pair (day, customer) is finished (i.e., new ID values appear), sum can be written out and next day/customer starts
  - This will be very fast
  - Needs virtually no memory during counting
- But: Can we sort ~3 billion records using less than 12 reads?



Content of this Lecture

- Sorting
- Simple Methods
- Lower Bound

Sorting

- Assumptions
  - We have n values (integer) that should be sorted
  - Values are stored in an array S (i.e., O(1) access to i'th element)
  - Comparing two values costs O(1)
  - We usually count # of comparisons; sometimes also # of swaps
  - Values are not interpreted
    - We do not know what a "big" value is or how many percent of all values are probably smaller than a given value
  - All we can do is comparing two values
- We seek a permutation π of the indexes of S such that

$$\forall i,j \leq n \text{ with } \pi(i) < \pi(j) : S[\pi(i)] \leq S[\pi(j)]$$

Variations

- External versus internal sorting
  - Internal sorting: S fits into main memory
  - External sorting: There are too many records to fit in memory
  - We only loot at internal sorting
- In-place or with additional memory dependent on n
  - In-place sorting only requires a constant (independent of n) amount of additional memory on top of S
  - We will look at both
- Pre-Sorting
  - Some algorithms can take advantage of an existing (incomplete, erroneous) order in the data, some not
  - We will not exploit pre-sorting

Applications

- Sorting is a ubiquitous task in computer science
  - [OW93] claims that 25% of all computing times is spent in sorting
- Second example: Information Retrieval
  - Imagine you want to build g******++
  - Fundamental operation: In a very large set of documents, find those that contain a given set of keywords
  - Popular way of doing this: Build an inverted index

Inverted Index

ID	Text
1	Baseball is played during summer months.
2	Summer is the time for picnics here.
3	Months later we found out why.
4	Why is summer so hot here?

Term	Freq	Document ids
baseball	1	[1]
during	1	[1]
found	1	[3]
here	2	[2], [4]
hot	1	[4]
is	3	[1], [2], [4]
months	2	[1], [3]
summer	3	[1], [2], [4]
the	1	[2]
why	2	[3], [4]

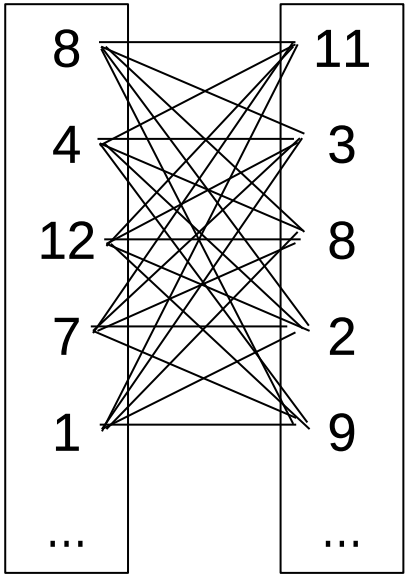
Source: http://docs.lucidworks.com

Answering a IR-style Query

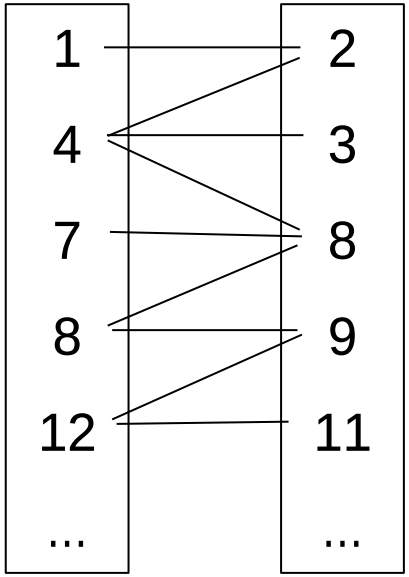
- A query is a set of keywords
- Finding the answer
  - For each keyword k_i of the query, load list d_i of docs containing k_i from inverted index
  - Build intersection of all d_i
  - Docs in this list are your answer
- Imagine the query “the man eats a bread” on the Web
  - Doc-list for “the” and “a” will contain >10 billion documents
- How do we compute the intersection of two sets of 10 billion IDs?

Intersection of Two Sets

With non-sorted sets:
O(m*n)



With sorted sets:
O(n+m)



Content of this Lecture

- Sorting
- Simple Methods
  - Selection sort
  - Insertion sort
  - Bubble sort
- Lower Bound

Recall: Selection Sort

```
S: array_of_names;
n := |S|
for i = 1..n-1 do
  for j = i+1..n do
    if S[i]>S[j] then
      tmp := S[j];
      S[j] := S[i];
      S[i] := tmp;
    end if;
  end for;
end for;
```

- Analysis showed that selection sort is in $O(n^2)$
- It is easy to see that selection sort also is in $\Omega(n^2)$
- How often do we swap values?
  - That depends a lot on the pre-sortedness of the array
  - But actually we can do a bit better

Selection Sort Improved

```
S: array_of_names;
n := |S|
for i = 1..n-1 do
  min_pos := i;
  for j = i+1..n do
    if S[min_pos]>S[j] then
      min_pos := j;
    end if;
   end for;
   tmp := S[i];
   S[i] := S[min_pos];
   S[min_pos] := tmp;
end for;
```

- How often do we swap values?
  - Once for every position
  - Thus: O(n)

Analogy

- Let‘s assume you keep your cards sorted
- How to get this order?
  - Selection sort: Take up all cards at once and building sorted prefixes of increasing length
  - Insertion sort: Take up cards one by one and sort every new card into the sorted subset at your hand
  - Bubble sort: Take up all cards at once and swap neighbors until everything is fine

Insertion Sort

```
S: array_of_names;
n := |S|
for i = 2..n do
  j := i;
  key := S[j];
  while (S[j-1]>tkey) and (j>1) do
    S[j] := S[j-1];
    j := j-1;
  end while;
  S[j] := key;
end for;
```

- After each loop of i, the prefix S[1..i] of S is sorted
- While-loop runs backwards from current position (to be inserted) until values get too small (smaller than S[j])
- Example: 5 4 8 1 6
- One problem is the required movement of many values until correct place is found
  - Could be implemented much better with a double-linked list

Complexity (Worst Case)

```
S: array_of_names;
n := |S|
for i = 2..n do
  j := i;
  key := S[j];
  while (S[j-1]>tkey) and (j>1) do
    S[j] := S[j-1];
    j := j-1;
  end while;
  S[j] := key;
end for;
```

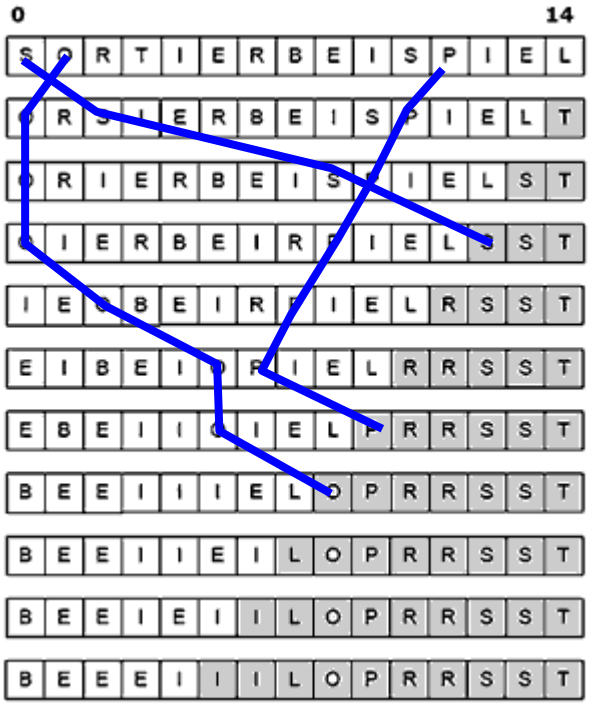
- Comparisons (worst-case)
  - Outer loop: n times
  - Inner-loop: n-i times
  - Thus, $O(n^2)$
- How many swaps?
  - (We move and don't swap, but both are in O(1))
  - In worst-case, every comparison incurs a move
  - Thus: $O(n^2)$
- We got worse?

Complexity (Best Case)

```
S: array_of_names;
n := |S|
for i = 2..n do
  j := i;
  key := S[j];
  while (S[j-1]>tkey) and (j>1) do
    S[j] := S[j-1];
    j := j-1;
  end while;
  S[j] := key;
end for;
```

- Assume the best case
  - Array is already sorted
- Comparisons
  - Outer loop: n times
  - Inner-loop: 1 time
  - Thus, O(n)
- Moves
  - None
  - (But **key** is assigned O(n) times)
- We might be better!

Bubble Sort



Source: HKI, Köln

- Go through array again and again
- Compare all direct neighbors
- Swap if in wrong order
- Repeat until first loop without swaps
- Intuitive algorithm
- About as good/bad as the others so far
  - Worst case $O(n^2)$ comparisons and $O(n^2)$ swaps
  - Best case $O(n)$ comparisons and 0 moves / swaps

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Lower Bound

- We found three algorithms with WC-complexity $O(n^2)$
- Maybe there is no better algorithm?
- Maybe the problem is $\Omega(n^2)$?

- Let's see if we can find a lower bound on the number of comparisons

Lemma

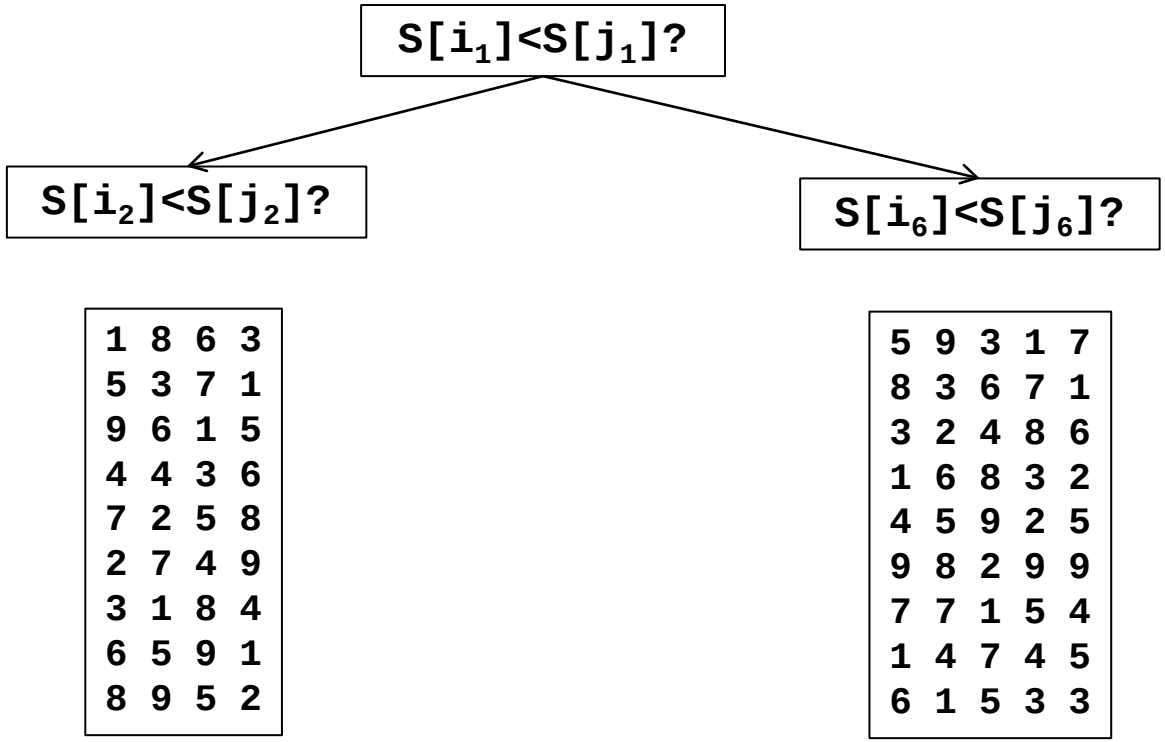
- Lemma
  To sort a list of n distinct values, every algorithm using only value comparisons will need at least $\Omega(n*log(n))$ comparisons
- Proof structure
  - We argue about all possible ways to find the right permutation π
  - Observe that there are n! different permutations
  - Each could be the right one (and there is only one right one)
  - To decide which, we are only allowed to compare two values
  - Every comparison splits the group of all permutations into two disjoint partitions
  - How often do we need to compare such that every partition has size 1 – in the best of all worlds?

Decision Tree

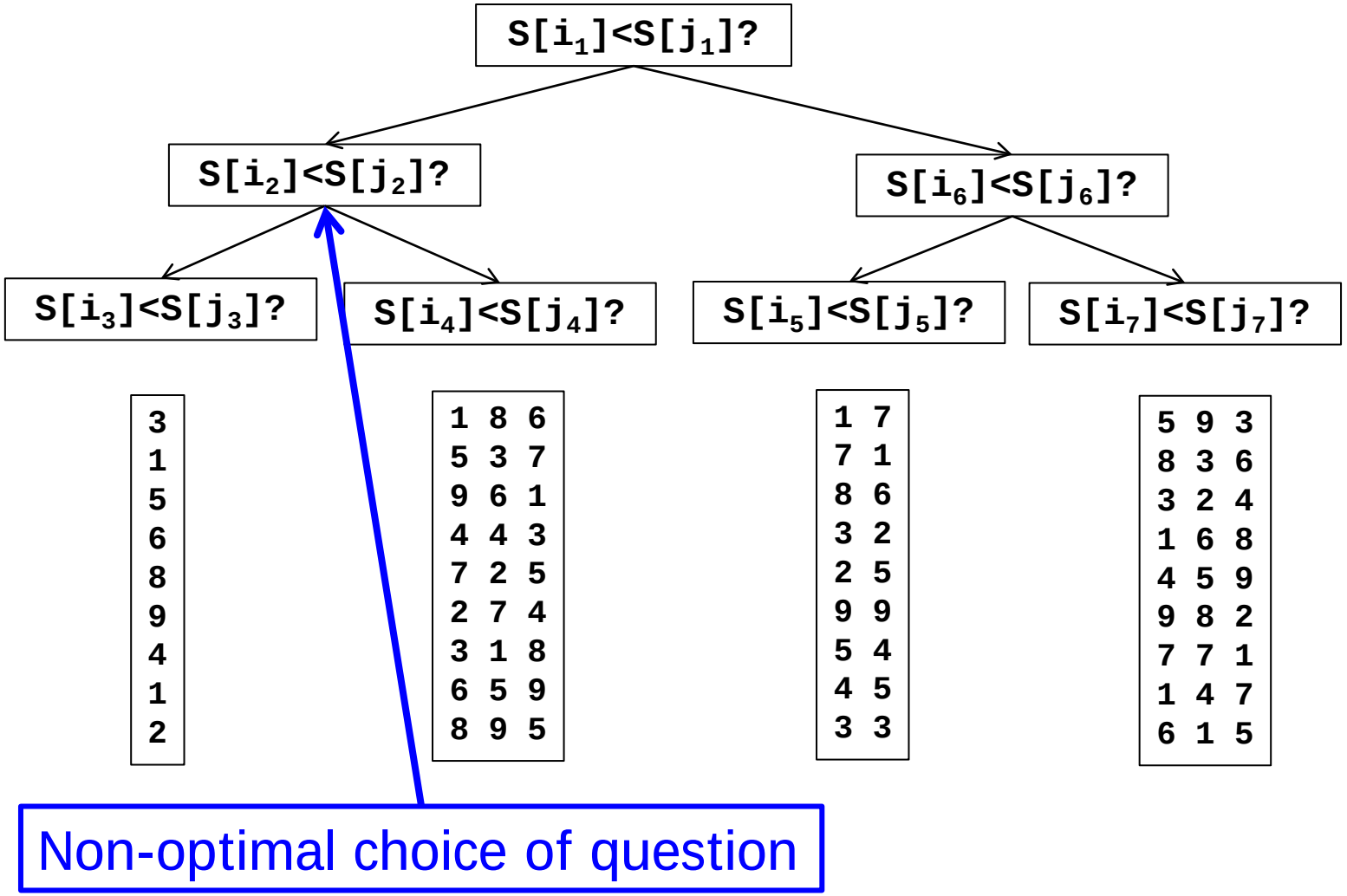
$S[i_1]<S[j_1]?$

1	8	6	3	5	9	3	1	7
5	3	7	1	8	3	6	7	1
9	6	1	5	3	2	4	8	6
4	4	3	6	1	6	8	3	2
7	2	5	8	4	5	9	2	5
2	7	4	9	9	8	2	9	9
3	1	8	4	7	7	1	5	4
6	5	9	1	1	4	7	4	5
8	9	5	2	6	1	5	3	3

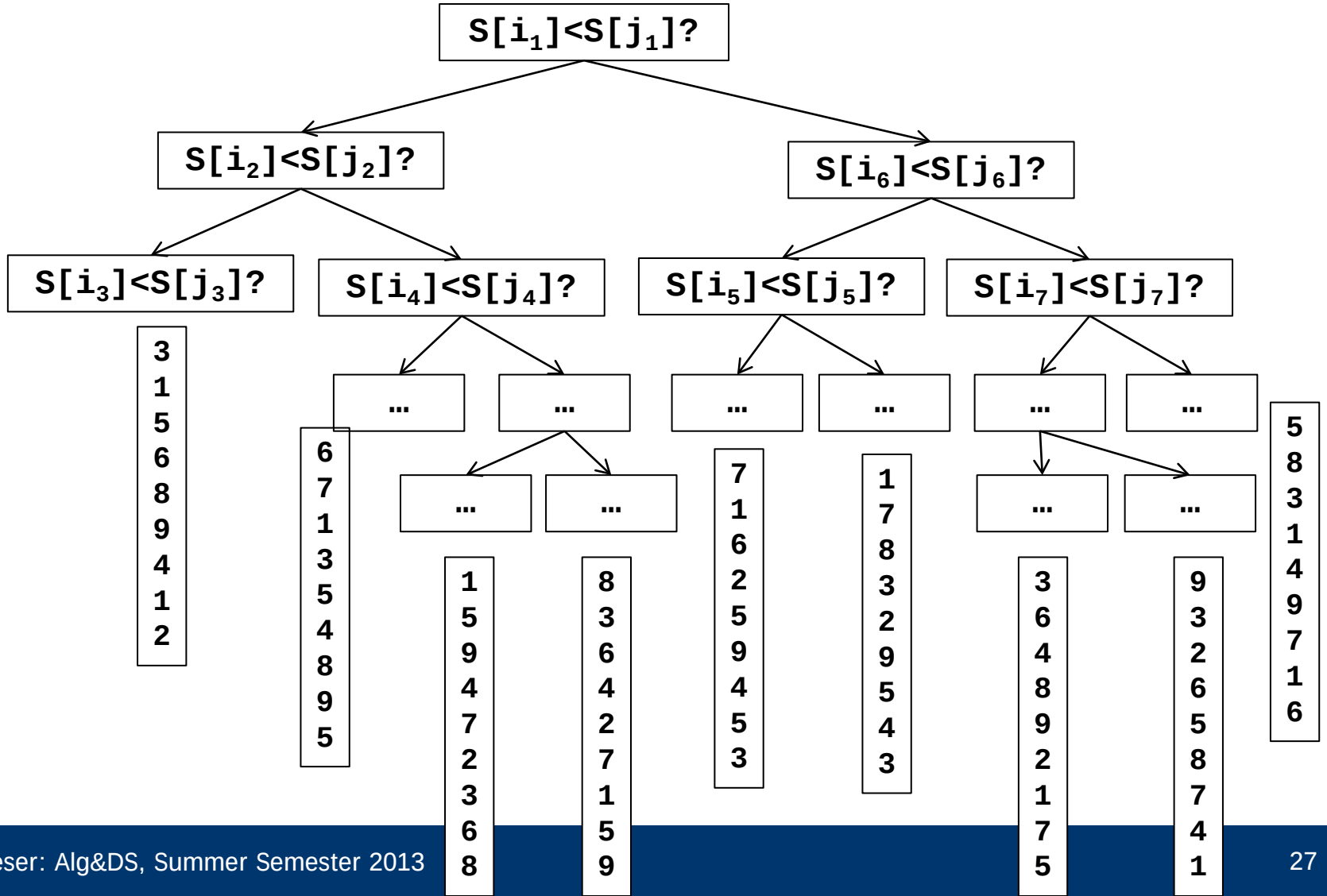
Decision Tree



Decision Tree



Full Decision Tree

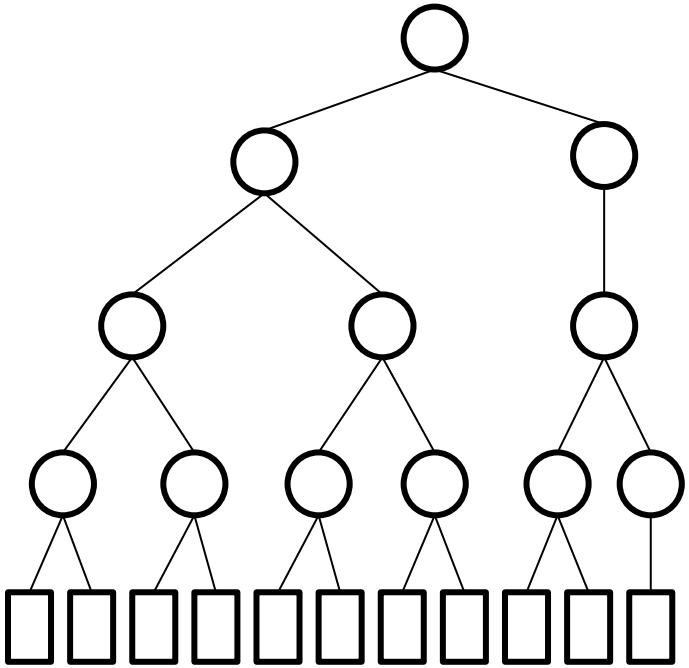


Optimal Set of Comparison

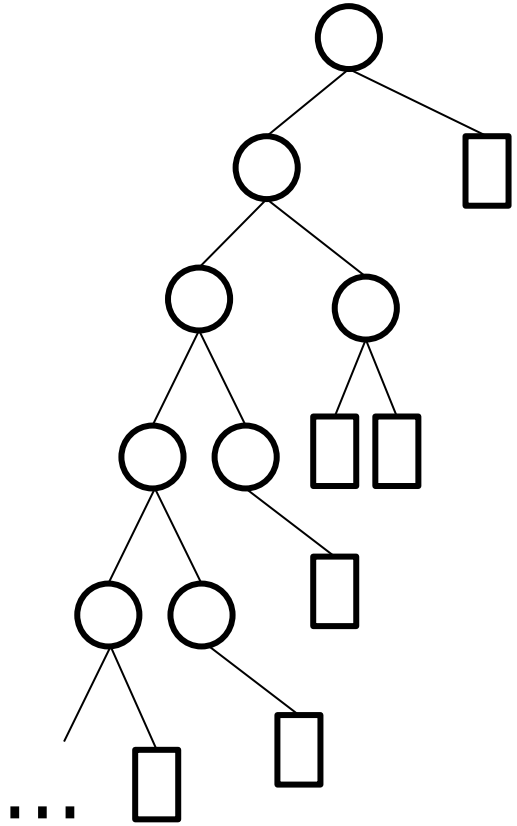
- We have no clue about which concrete series of comparisons is optimal for a given list
- This doesn't matter; we are looking for a lower bound
  - We may always assume to take the best choice
- Best choice means: Creating only 1-partitions with as few comparisons as possible
- Thus, we want to know the length of the longest path through the optimal decision tree
  - Even in the best of all worlds we may need to make this number of comparisons to find the correct permutation
- Other way round: The optimal tree is the one with the shortest longest path

Intuition

Good



Bad

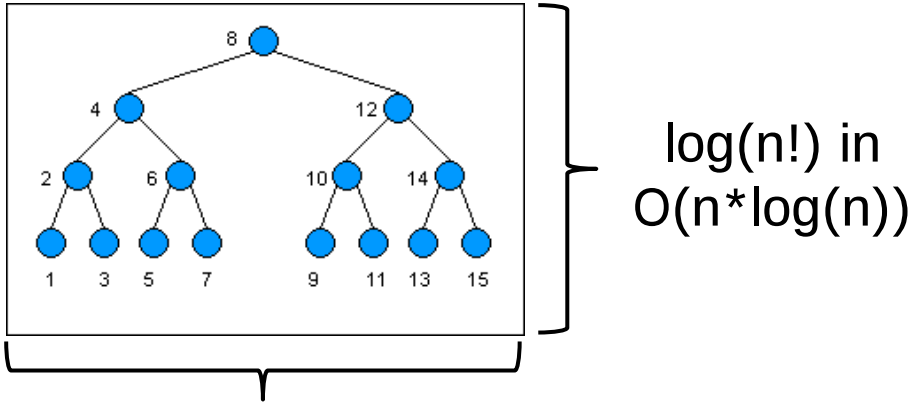


Shortest Longest Path

- Definition
  The height of a binary tree is the length of its longest path.
- Theorem
  A binary tree with k leaves has at least height log(k).
- Proof
  - Every inner node has at most two children
  - To cover as many leaves as possible in the level above the leaves, we need ceil(k/2) nodes
  - In the second level, we need ceil(k/2/2) nodes, etc.
  - After log(k) levels, only one node remains (root)
  - Qed.

Putting it all together

- Our decision tree has n! leaves (all permutations)
- The height of a binary tree with n! leaves is at least log(n!)
- Thus, the longest path in the optimal tree has at least log(n!) comparisons
- Since $n! \geq (n/2)^{n/2}$: $\log(n!) \geq \log((n/2)^{n/2}) = n/2 * \log(n/2)$
- This gives the overall lower bound $\Omega(n*\log(n))$
- qed.



Summary

	Comparisons worst case	Comparisons best case	Additional space	Moves worst/best
Selection Sort	$O(n^2)$	$O(n^2)$	O(1)	O(n)
Insertion Sort	$O(n^2)$	O(n)	O(1)	$O(n^2)$ / O(n)
Bubble Sort	$O(n^2)$	O(n)	O(1)	$O(n^2)$ / O(1)
Merge Sort	O(n*log(n))	O(n*log(n))	O(n)	O(n*log(n))