Algorithms and Data Structures

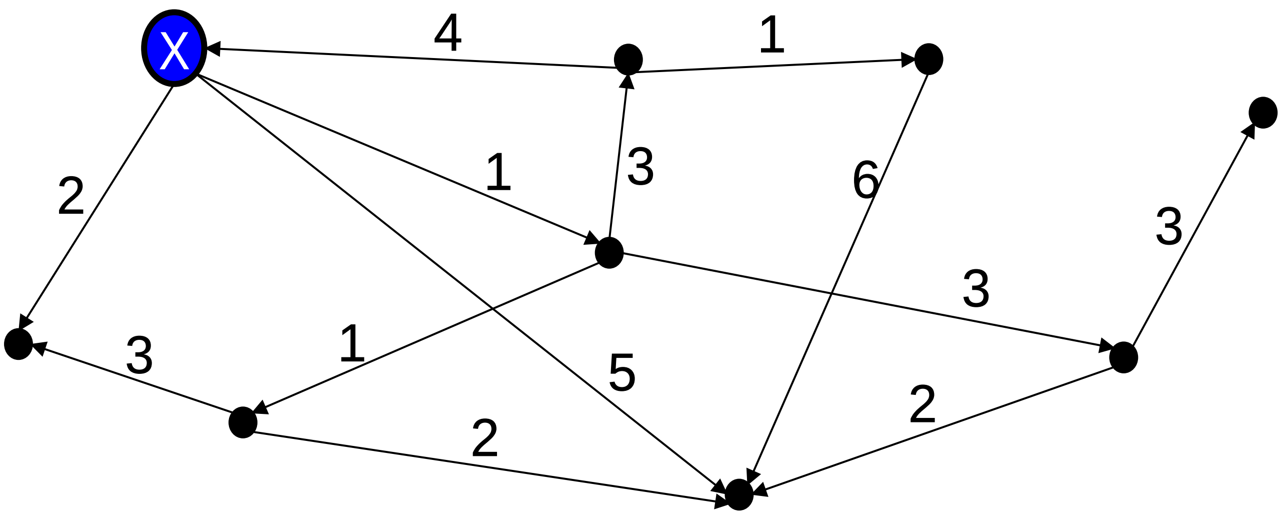
Priority Queues

Ulf Leser

Special Scenarios for Searching

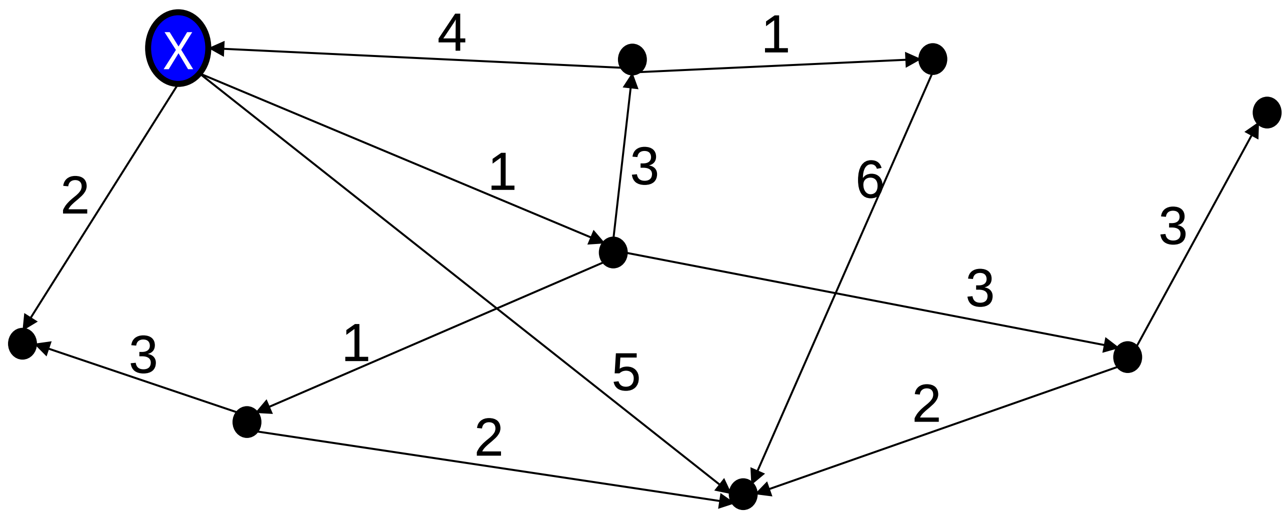
- Up to now, we assumed that all elements of a list are equally important and that any of them could be searched next (with varying probability)
- What if some elements are more important than others?
  - There is a (maybe partial) order on list elements
  - The most important elements are always retrieved next
  - Priority Queues
- Difference to Self-Organizing Lists
  - Most important element is always retrieved next – should be O(1)
  - List most be kept ordered by importance
  - We look at a scenario where new elements are inserted all the time and the most important element is removed regularly

Shortest Paths in a Graph



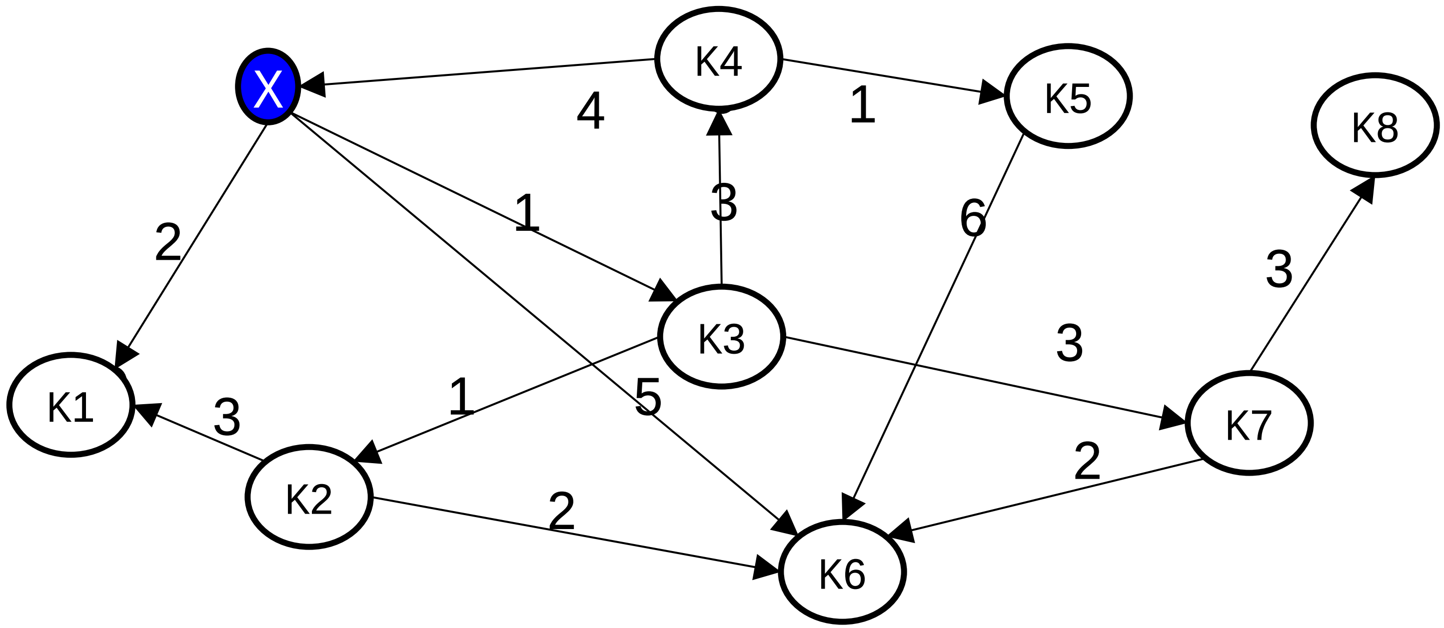
- Task: Find the distance between X and all other nodes
  - Classical problem: Single-Sink-Shortest-Paths
  - Famous solution: Dijkstra's algorithm

Assumptions



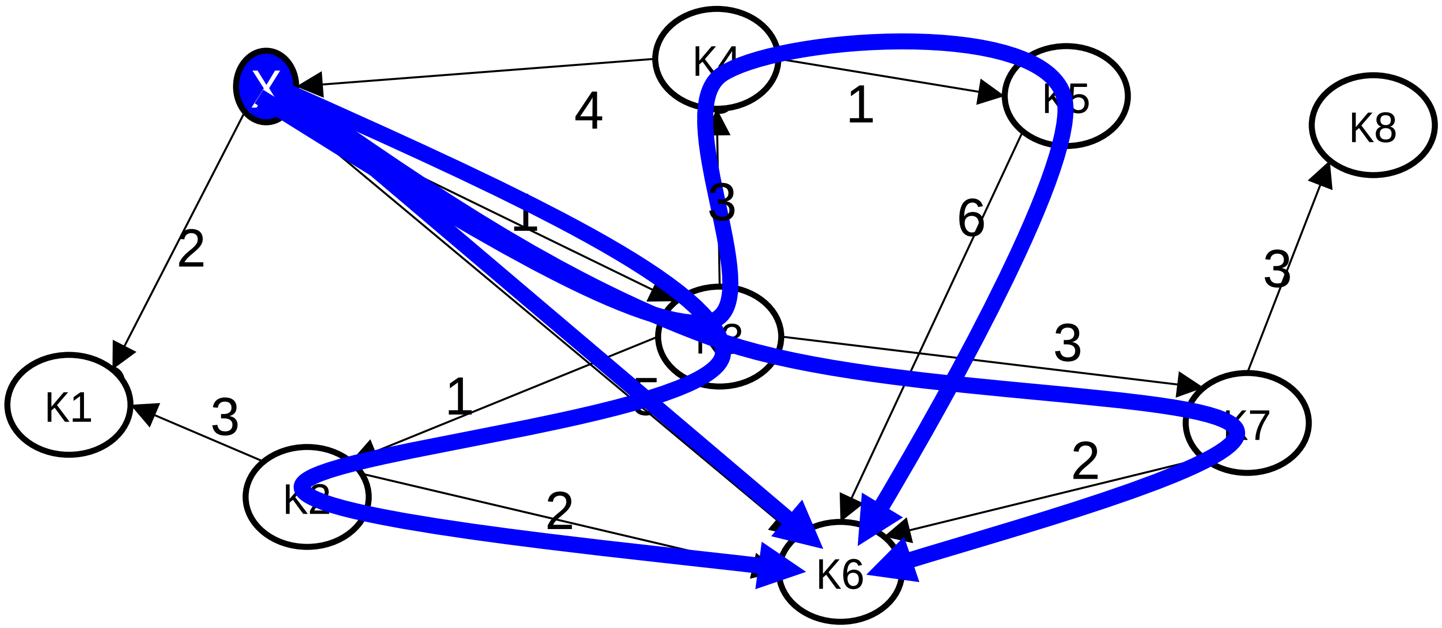
- We assume that there is at least one path between X and any other node (every node is reachable from X)
- We assume strictly positive edge weights
- Distance is the length (=sum of weights) of the shortest path
- There might be many shortest paths, but distance is unique
- We only want the distance and need no "witness path"

Exhaustive Solution



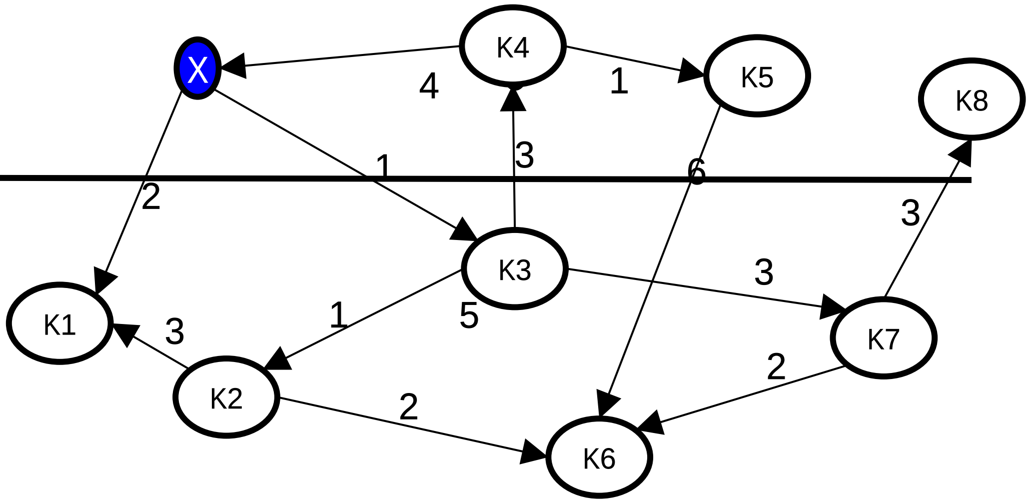
- First approach: Enumerate all paths
  - Need to break cycles (e.g. X – K3 – K4 – X – K3 - ...)
  - Using DFS: X – K3 – K4 – X [BT-K4] – K5 – K6 [BT-K5] [BT-K4] [BT-K3] – K7 – K8 [BT-K7] – K6 [BT-K7] [BT-K3] – K2 – K6 [BT-K2] – K1 [BT-K2] [BT-K3] [BT-X] K6 - ...

Redundant work



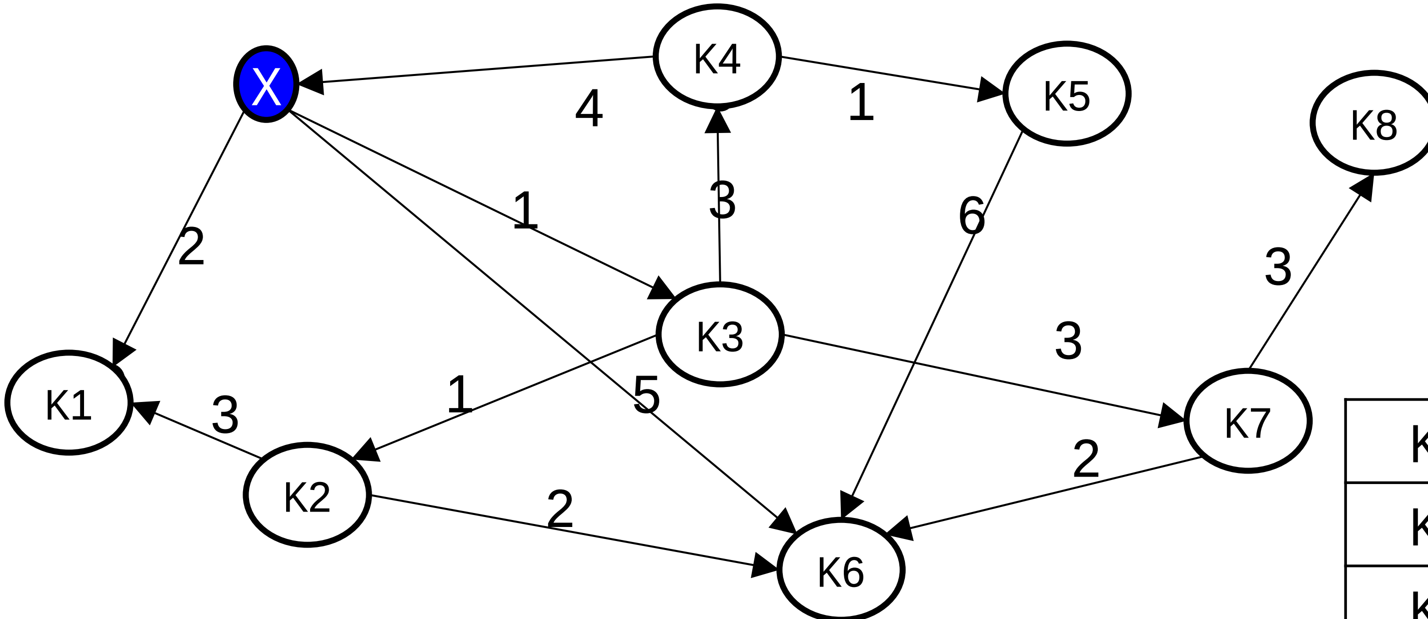
- First approach: Enumerate all paths
  - Need to break cycles (e.g. X – K3 – K4 – X – K3 - ...)
  - Using DFS: X – K3 – K4 – X [BT-K4] – K5 – K6 [BT-K5] [BT-K4] [BT-K3] – K7 – K8 [BT-K7] – K6 [BT-K7] [BT-K3] – K2 – K6 [BT-K2] – K1 [BT-K2] [BT-K3] [BT-X] K6 - ...

Dijkstra’s Idea



- Enumerate paths by their length (neither DFS nor BFS)
- Assume we reach a node Y by a path p of length l and we have already explored all paths with length l’ ≤ l and that Y was not reached yet
  - We always mean “all paths starting from X”
- Then p must be the shortest path between X and Y
  - Because any p’ between X and Y would have a prefix of length at least l and (a) a continuation with length>0 or (b) would not need a continuation (then p is as short as p’)

Example for Idea



K3	1
K2	2
K1	2
K6	4
K4	4
K7	4
K5	5
K8	7

- 1: X – K3
- 2: X – K3 – K2
- 2: X – K1
- 4: X – K3 – K2 – K6
- 4: X – K3 – K4
- 4: X – K3 – K7
- 5: X – K3 – K4 – K5
- 7: X – K3 – K7 – K8
- Stop (all nodes found)

A Further Trick

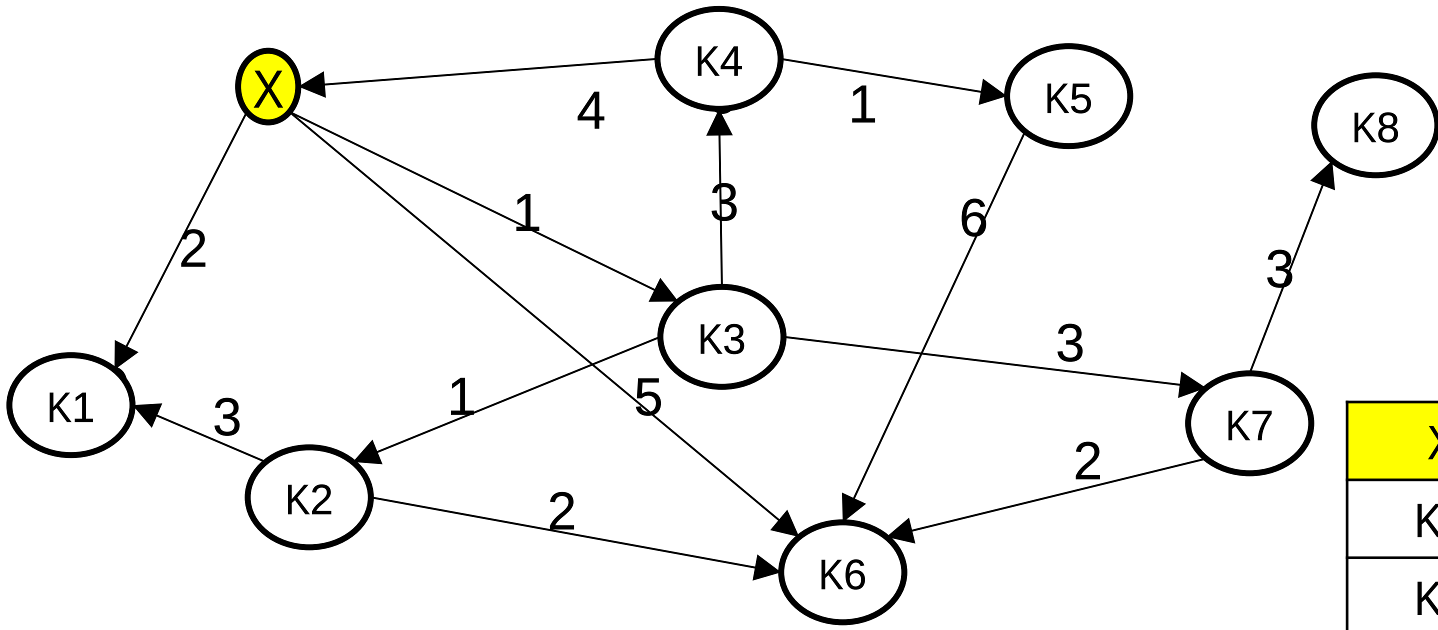
- We enumerate paths by length by iteratively extending short paths by all possible next edges
  - I.e., by looking at all edges outgoing from the end node of a short path
- These extensions
  - ... either lead to a node which we didn’t reach yet – then we found a path, but cannot yet be sure it is the shortest
  - ... or lead to a node which we already reached but we are not yet sure of we found the shortest path to it – update current best distance
  - ... or lead to a node which we already reached and for which we also surely found the a shortest path already – these can be ignored
- Eventually, we can enumerate nodes by their distance

Algorithm

```
G = (V, E);
x : start_node;    # x∈V
A : array_of_distances;
∀i: A[i]:= ∞;
L := V;
A[x] := 0;
while L≠∅
  k := L.get_closest_node();
  L := L \ k;
  forall (k,f,w)∈E do
    if f∈L then
      new_dist := A[k]+w;
      if new_dist < A[f] then
        A[f] := new_dist;
      end if;
    end if;
  end for;
end while;
```

- Assumptions
  - Nodes have IDs between 1 ... |V|
  - Edges are **`(from, to, weight)`**
- We enumerate nodes by length of their shortest paths
  - In the first loop, we pick x and update distances (A) to all adjacent nodes
  - When we pick a node k, we already have computed its distance to x in A
  - We adapt the current best distances to all neighbors of k we haven't picked yet
- Once we picked all nodes, we are done

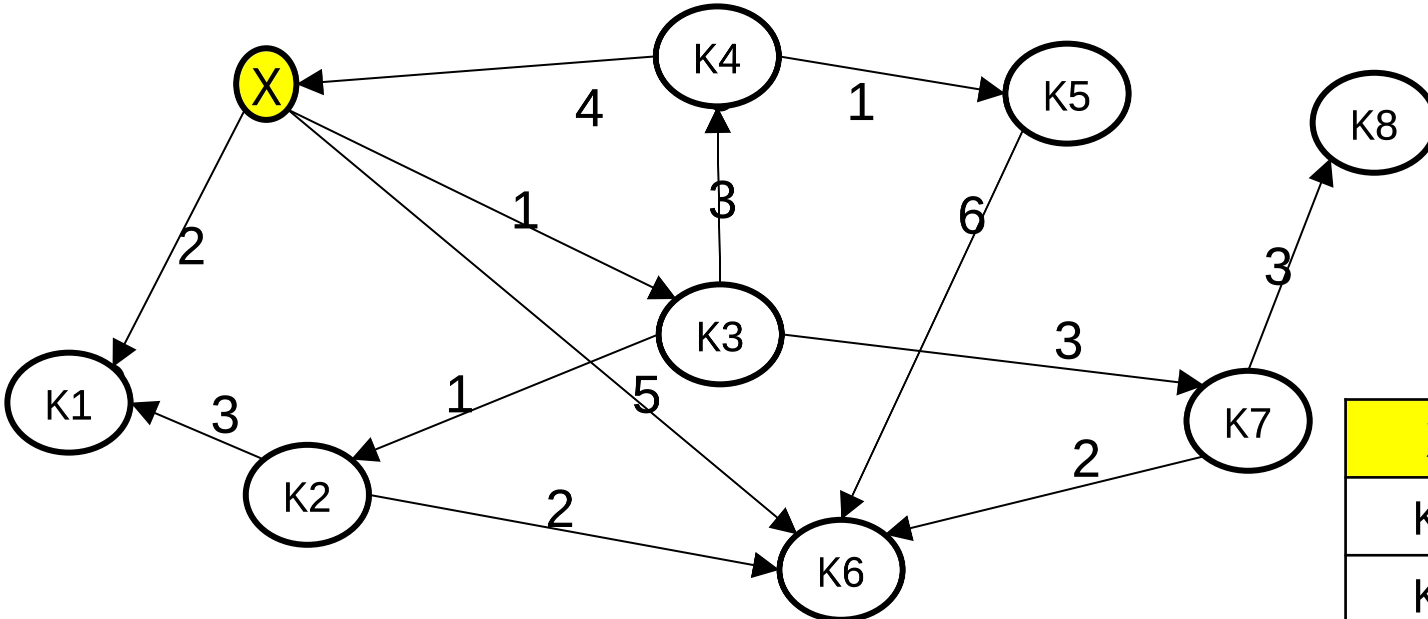
Example for Algorithm



X	0
K1	
K2	
K3	
K4	
K5	
K6	
K7	
K8	

- Pick x

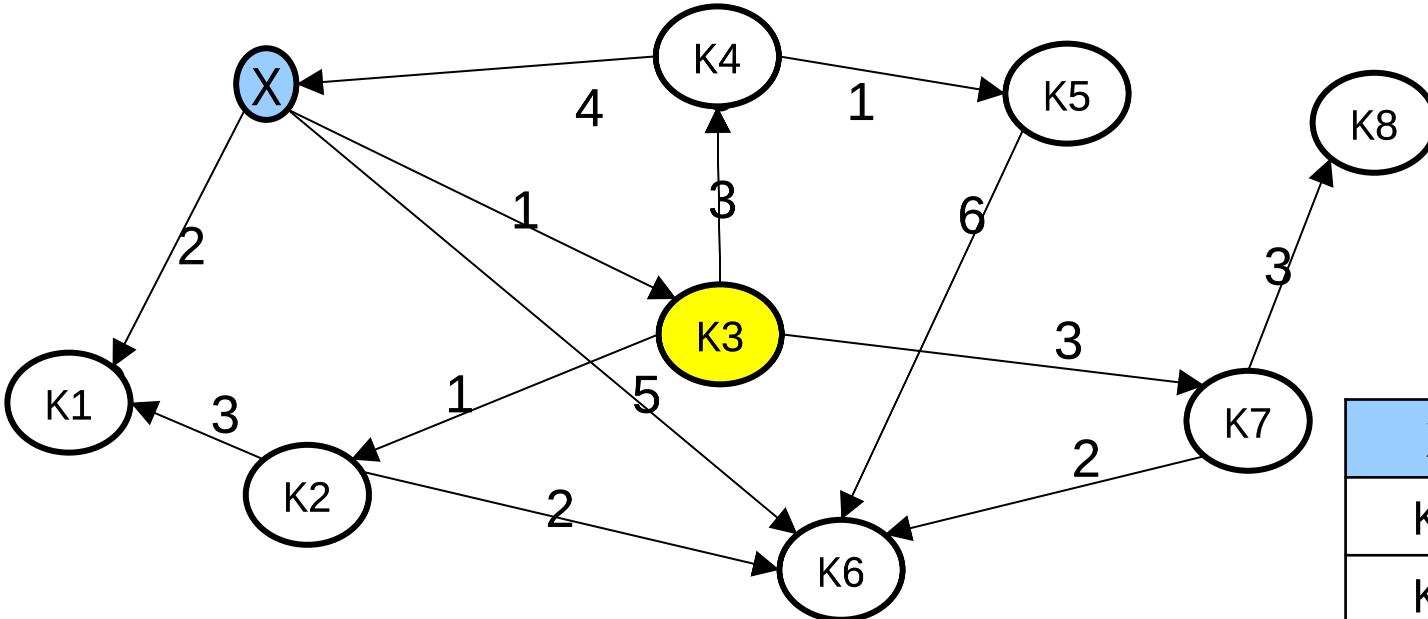
Example for Algorithm



X	0
K1	2
K2	
K3	1
K4	
K5	
K6	5
K7	
K8	

- Pick x
- Adapt distances to all neighbors

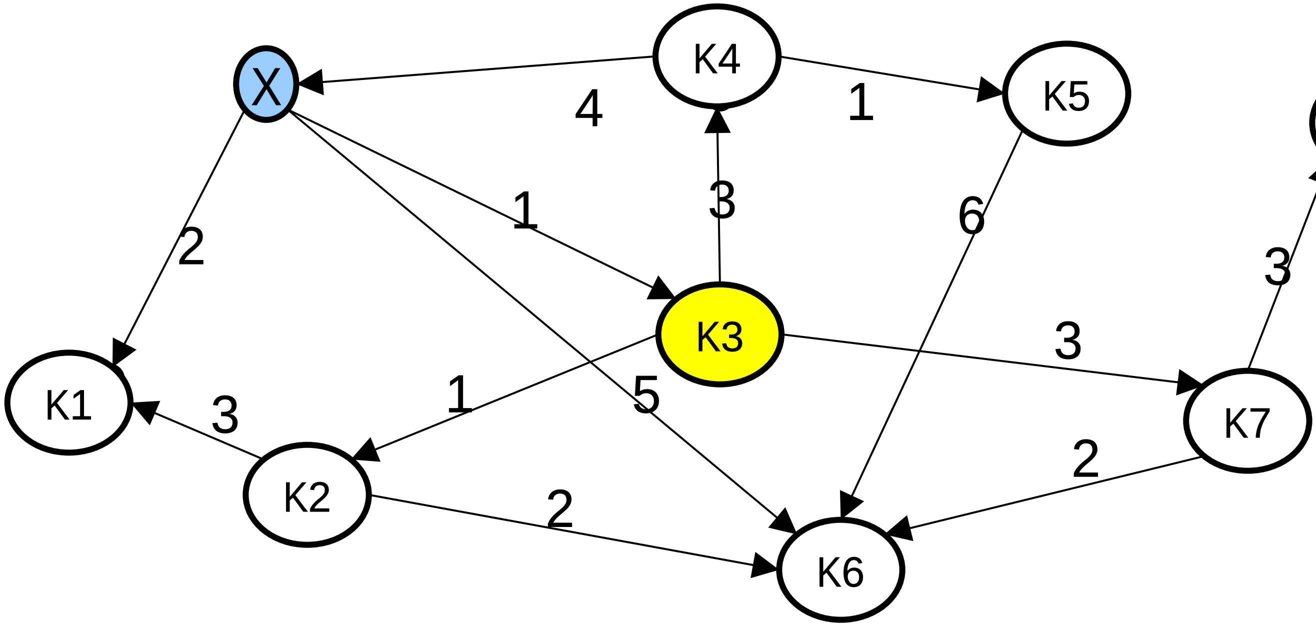
Example for Algorithm



X	0
K1	2
K2	
K3	1
K4	
K5	
K6	5
K7	
K8	

- Pick K3 (closest to x)

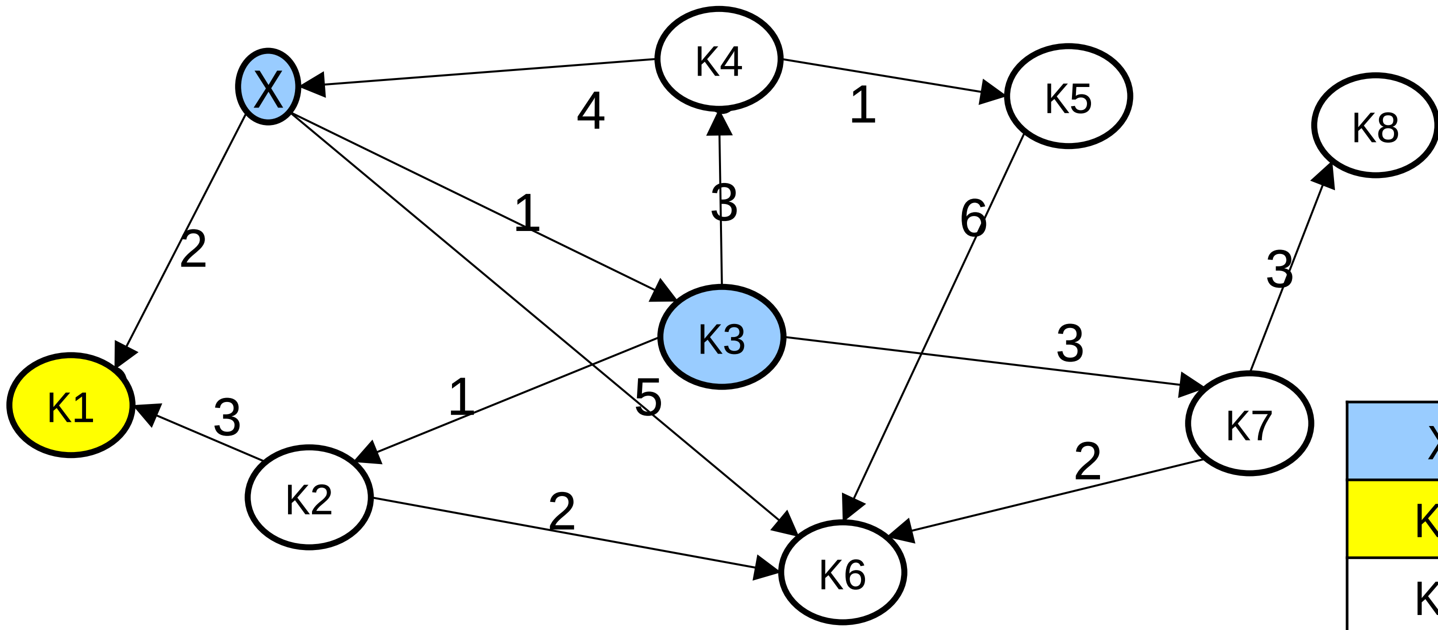
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	
K6	5
K7	4
K8	

- Pick K3
- Adapt distances (from x) to all neighbors (of K3)

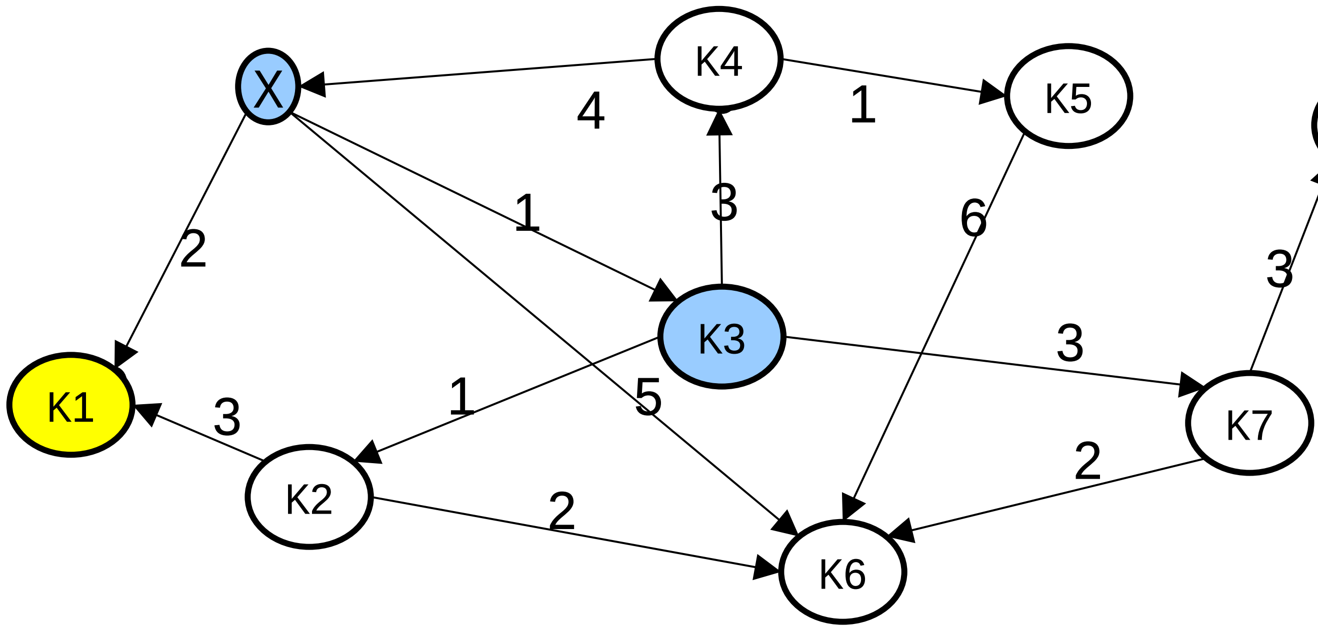
Example for Algorithm



- Pick K1 (or K2)

X	0
K1	2
K2	2
K3	1
K4	4
K5	
K6	5
K7	4
K8	

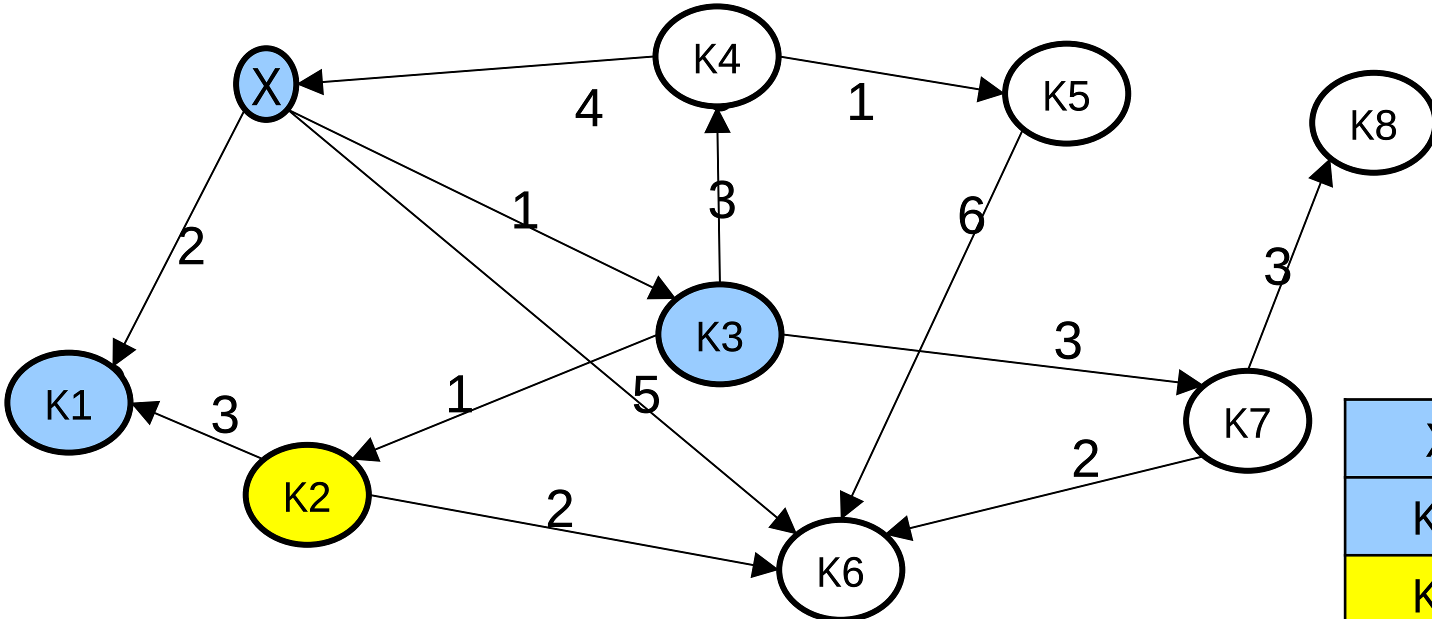
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	
K6	5
K7	4
K8	

- Pick K1
- Adapt distances to all neighbors
  - There are none

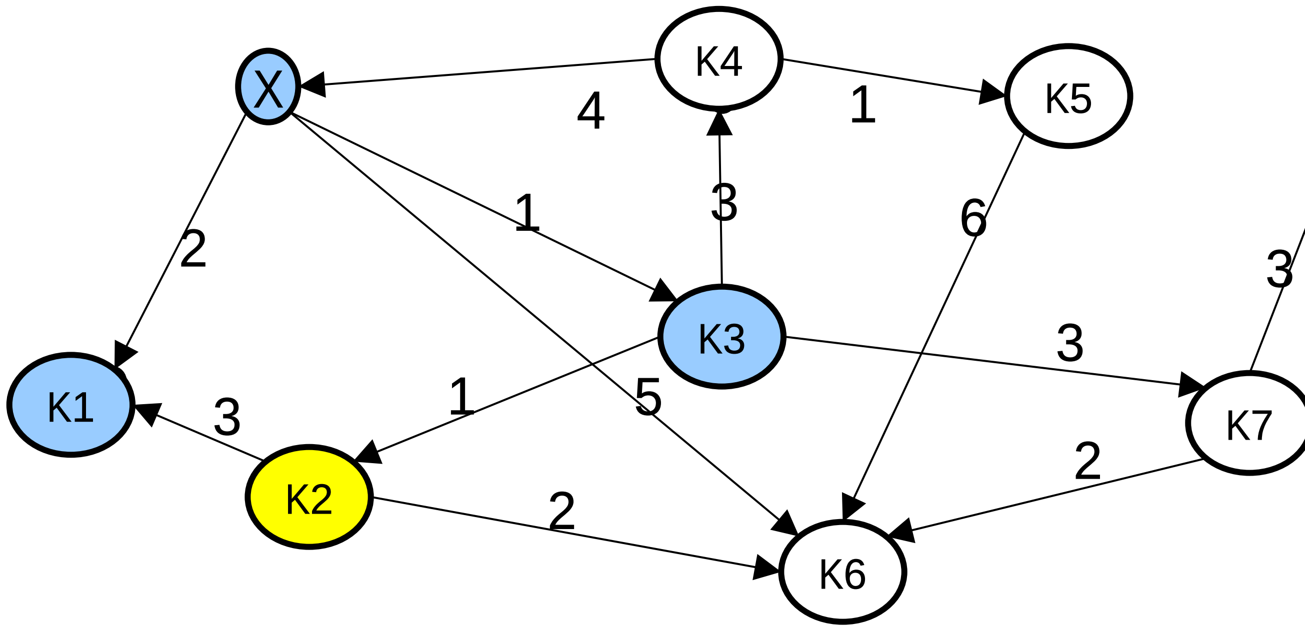
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	
K6	5
K7	4
K8	

- Pick K2

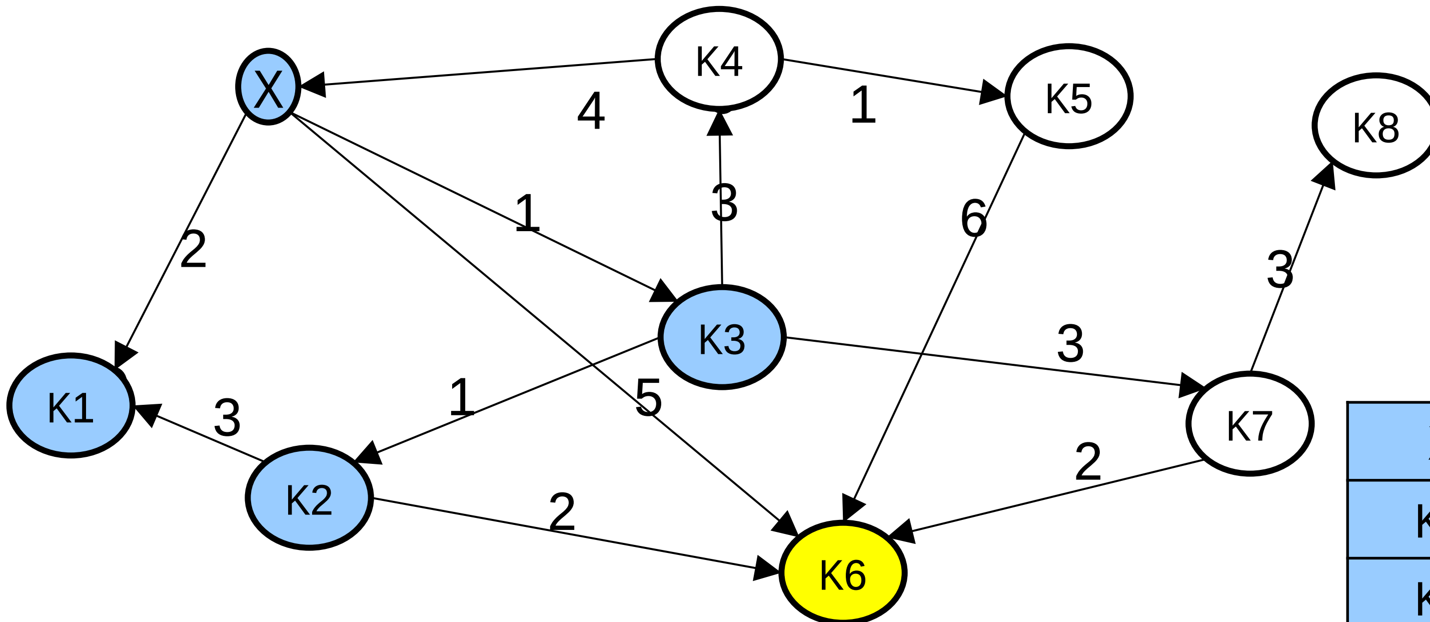
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	
K6	4
K7	4
K8	

- Pick K2
- Adapt distances to all neighbors
  - K1 was picked already – ignore
  - We found a shorter path to K6

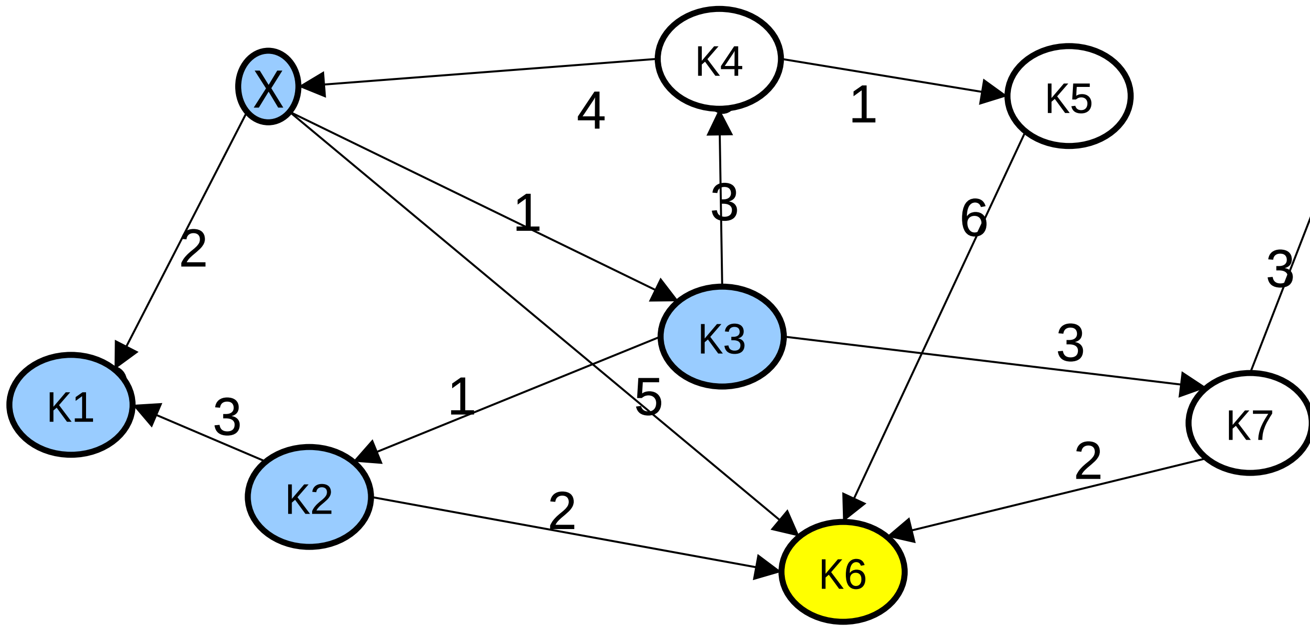
Example for Algorithm



- Pick K6 (or K4 or K7)

X	0
K1	2
K2	2
K3	1
K4	4
K5	
K6	4
K7	4
K8	

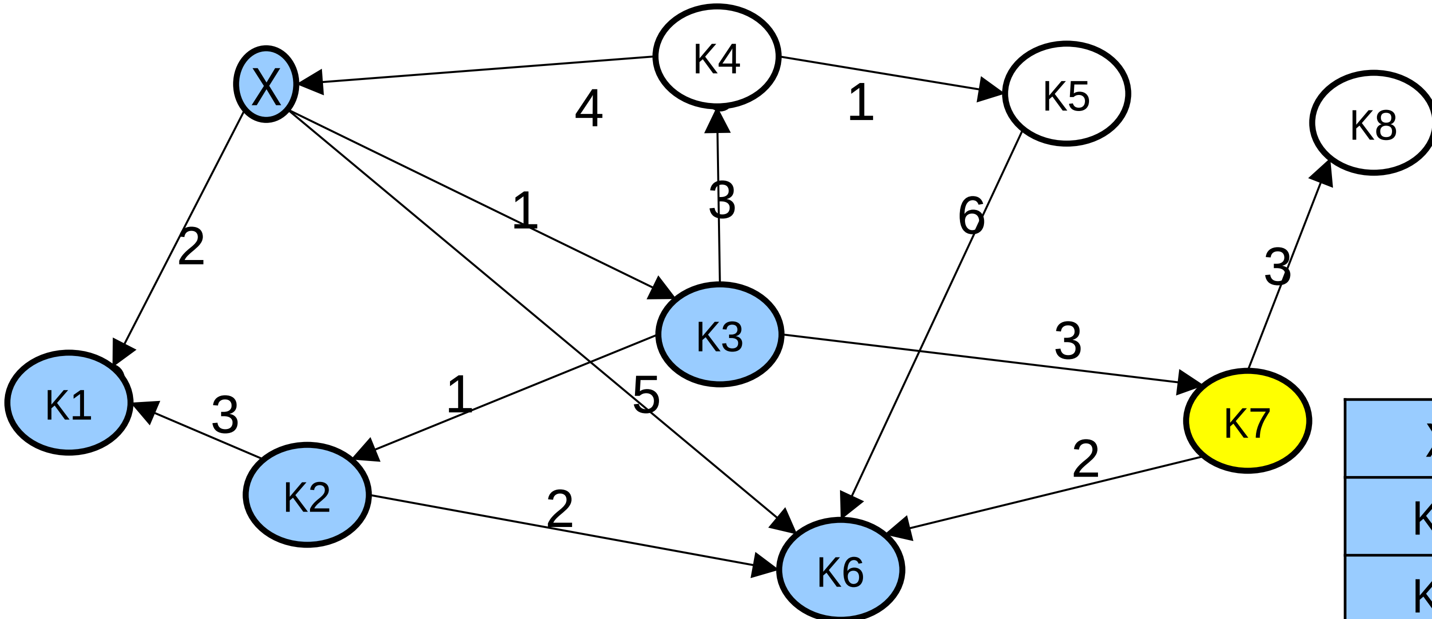
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	
K6	4
K7	4
K8	

- Pick K6
- Adapt distances to all neighbors
  - There are none

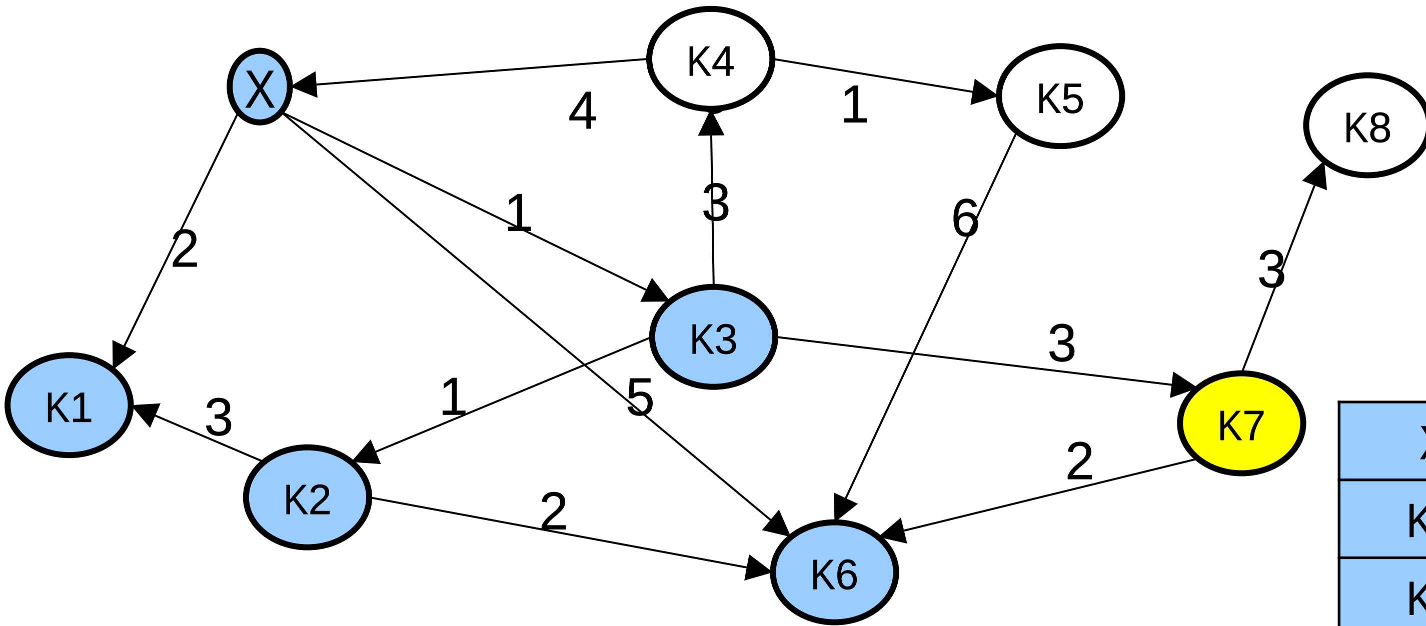
Example for Algorithm



- Pick K7

X	0
K1	2
K2	2
K3	1
K4	4
K5	
K6	4
K7	4
K8	

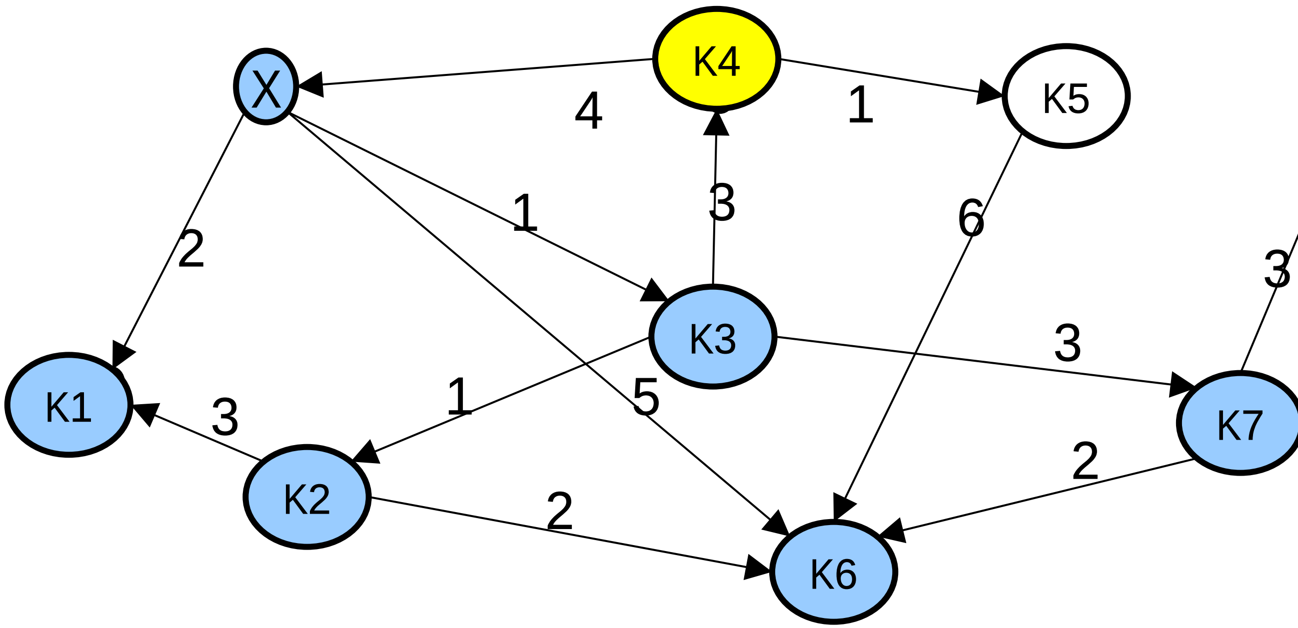
Example for Algorithm



- Pick K7
- Adapt distances to all neighbors
  - K6 was visited already

X	0
K1	2
K2	2
K3	1
K4	4
K5	
K6	4
K7	4
K8	7

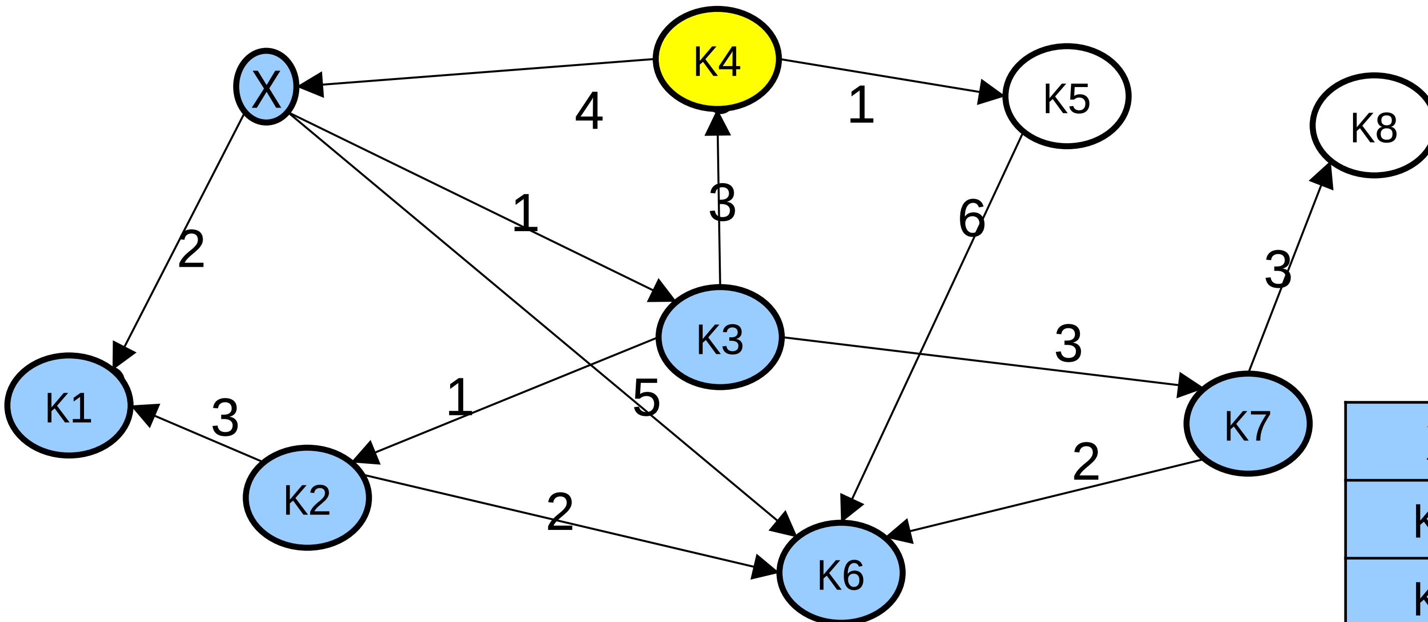
Example for Algorithm



- Pick K4

X	0
K1	2
K2	2
K3	1
K4	4
K5	
K6	4
K7	4
K8	7

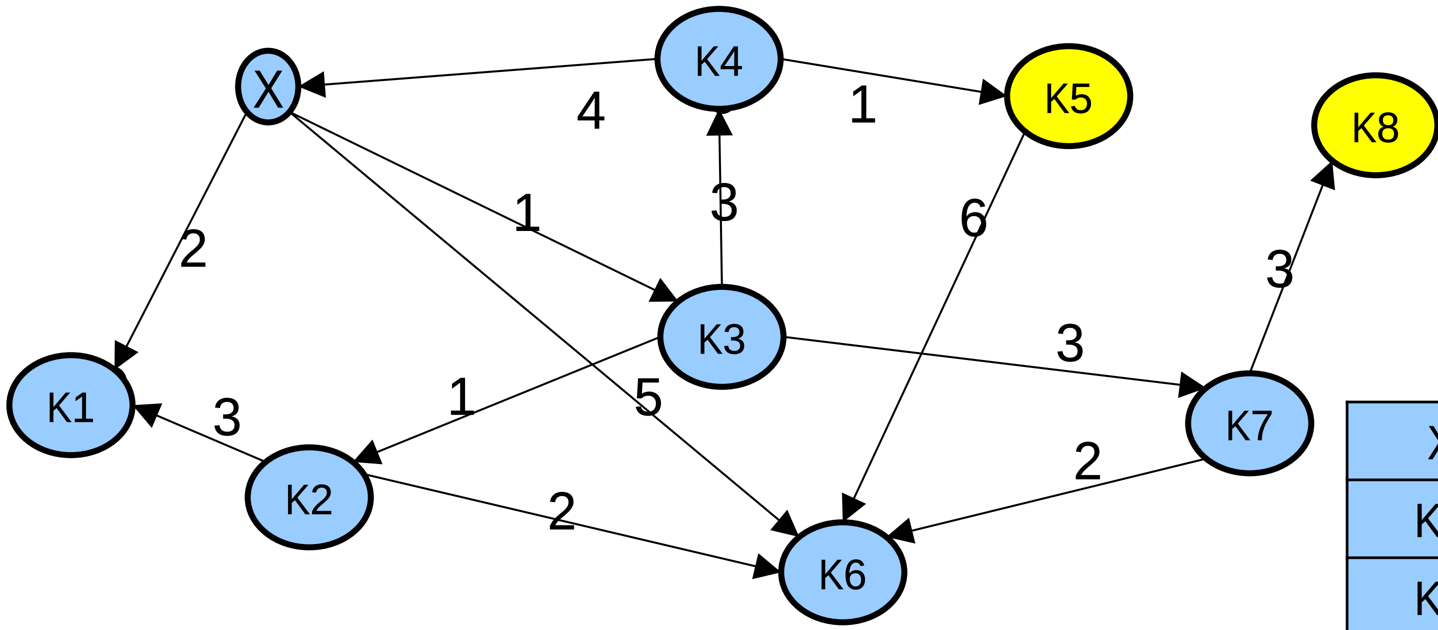
Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	5
K6	4
K7	4
K8	7

- Pick K4
- Adapt distances to all neighbors
  - X was visited already

Example for Algorithm



X	0
K1	2
K2	2
K3	1
K4	4
K5	5
K6	4
K7	4
K8	7

- Pick K5 ... Pick K8

A Closer Look

```
1.  G = (V, E);
2.  x : start_node;     # x∈V
3.  A : array_of_distances;
4.  ∀i: A[i]:= ∞;
5.  L := V;
6.  A[x] := 0;
7.  while L≠∅
8.    k := L.get_closest_node();
9.    L := L \ k;
10.   forall (k,f,w)∈E do
11.     if f∈L then
12.       new_dist := A[k]+w;
13.       if new_dist < A[f] then
14.         A[f] := new_dist;
15.       end if;
16.     end if;
17.   end for;
18. end while;
```

- Algorithm seems to work
  - Proof and analysis will follow later
  - Hint: 8 is passed-by |V| times and 12 at most |E| times
- Central: **`get_closest_node()`**
  - Needs to find the node k in L for which A[k] is the smallest
  - A[k] is changed a lot during a run
- Searching A? Resorting A?
- Better: Priority queue
  - List of tuples (o, v) (object,value)
  - Central operation: Return tuple where v is smallest

Content of this Lecture

- Priority Queues
- Using Heaps
- Using Fibonacci Heaps

Priority Queues

- A priority queue (PQ) is an ADT with 3 essential operations
  - **`add(o,v)`**: Add element o with value (priority) v
    - May be also bulk insert – convert a list in a priority queue
  - **`getMin()`**: Retrieve element with highest priority
  - **`removeMin()`**: Remove element with smallest value
- Typical additional operations
  - **`merge(p1, p2)`**: Merge two PQs into one (properly sorted)
  - **`delete(o)`**: Delete o from PQ
  - **`changeValue(o,v)`**: Change value of o to v

Applications

- Games (e.g. chess)
  - The machine explores next movements but cannot look at all of them; give each move an assumed benefit and explore moves with highest benefit first (also called A* algorithm)
- Event simulators
  - While events are handled, new events are generated for the future; manage all events in a PQ sorted by event time and always pull the next event
- Quality of Service in a network
  - When bandwidth is limited, sort all transmission requests in a PQ and transmit by highest priority
- …

Naive Implementations (with |Q|=n)

- Using a linked list
  - **`add`** requires O(1)
  - **`getMin`** requires O(n) [bad]
  - **`deleteMin`** requires O(1) (if we keep the pointer after a getMin)
  - **`merge`** requires O(1)
- Using a linked list sorted by priority
  - **`add`** requires O(n) [bad]
  - **`getMin`** requires O(1)
  - **`deleteMin`** requires O(1)
  - **`merge`** requires O(n+m)

Maybe Arrays?

- Using a sorted array
  - `add` requires O(n) [We find the position in log(n), but then have to free a cell by moving all elements after this cell]
  - `getMin` requires O(1)
  - `deleteMin` requires O(n)
- PQs are typically used in applications where elements are inserted and removed all the time
- We need a DS that can change its size dynamically at very low cost
- We want constant or at most log-time for all operations

Content of this Lecture

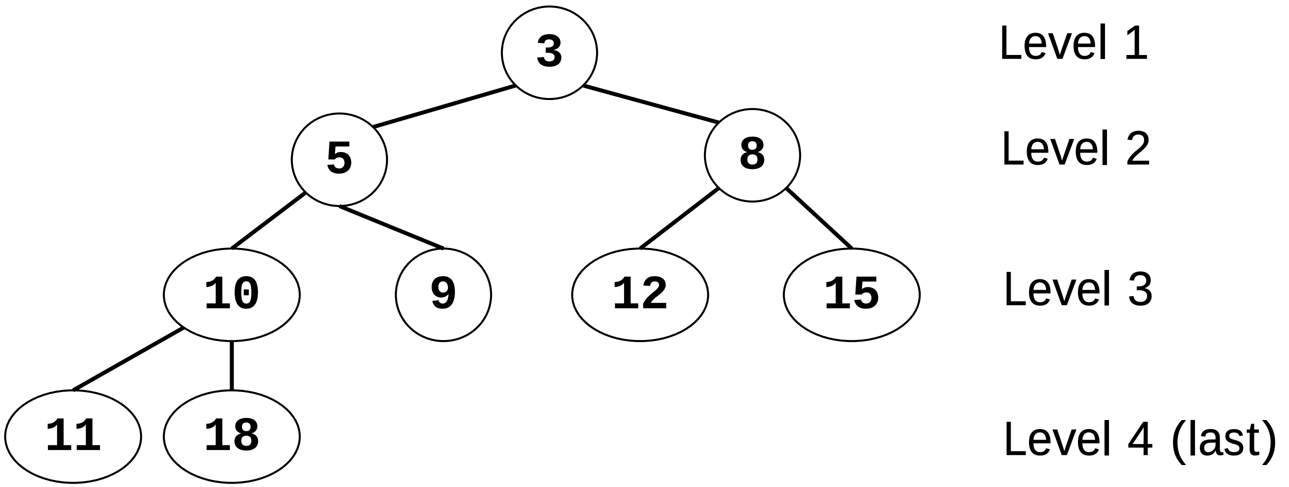
- Priority Queues
- Using Heaps
  - Heaps
  - Operations on Heaps
  - Heap Sort
- Using Fibonacci Heaps

Heap-based PQ

- Unsorted lists require O(n) for **`getMin()`**
  - We don't know where the smallest element is
- Sorted lists require O(n) for **`add()`**
  - We don't know where to put the new element
- Can we find a way to keep the list "a little sorted"?
  - Actually, we only want the smallest element at a fixed position
  - All other elements can be at arbitrary places
  - **`add() / deleteMin()`** should be faster than O(n), because they don't need to keep the entire list sorted
- One such structure is called a heap

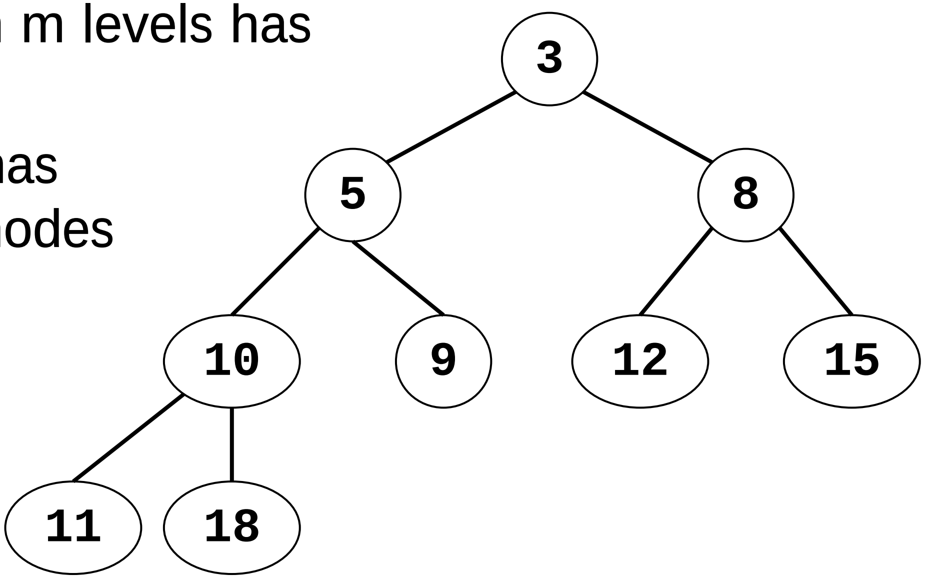
Heaps

- Definition
  A heap is a labeled binary tree for which the following holds
  - *Form-constraint (FC): The tree is complete except the last level*
    - *I.e.: Every node has exactly two children*
  - *Heap-constraint (HC): The value of any node is smaller than that of its children*



Properties

- Order
  - A heap is "a little" sorted: We know the smallest element (root)
  - We know the order for some pairs of elements (parent-successors), but for many pairs we don't know which is bigger (e.g. nodes in the same level)
- Size
  - A complete binary tree with m levels has 2^m-1 nodes
  - A heap with m levels thus has between $2^{m-1}+1$ and 2^m-1 nodes
  - A heap with n nodes has ceil(log(n+1)) levels

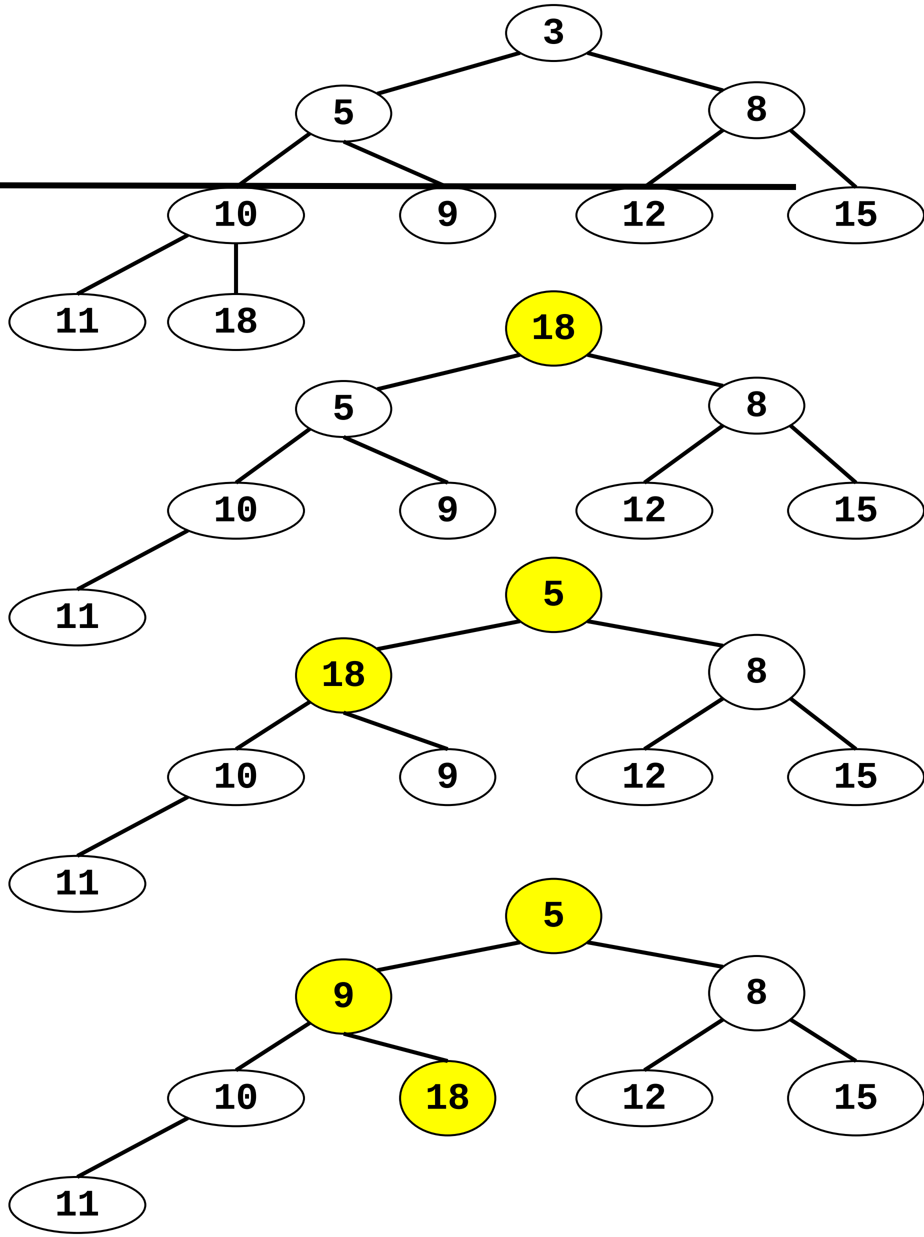


Operations

- Assume we store our PQ as a heap
- Clearly, **`getMin()`** is possible in O(1)
  - Keep a pointer to the root
- But ...
  - How can we perform **`deleteMin()`** – such that the new structure again is a heap?
  - How can we add an element to a heap – such that the new structure again is a heap?
  - How can we turn a list into a heap?

DeleteMin()

- We first remove the root
  - Creates two heaps
  - We must connect them again
- We take the „last“ node, place it in root, and sift it down the tree
  - Last node: right-most in the last level (actually, we can take any from the last level)
  - Sifting down: Exchange with smaller of both children as long as at least one child is smaller than the node itself



Analysis - Correctness

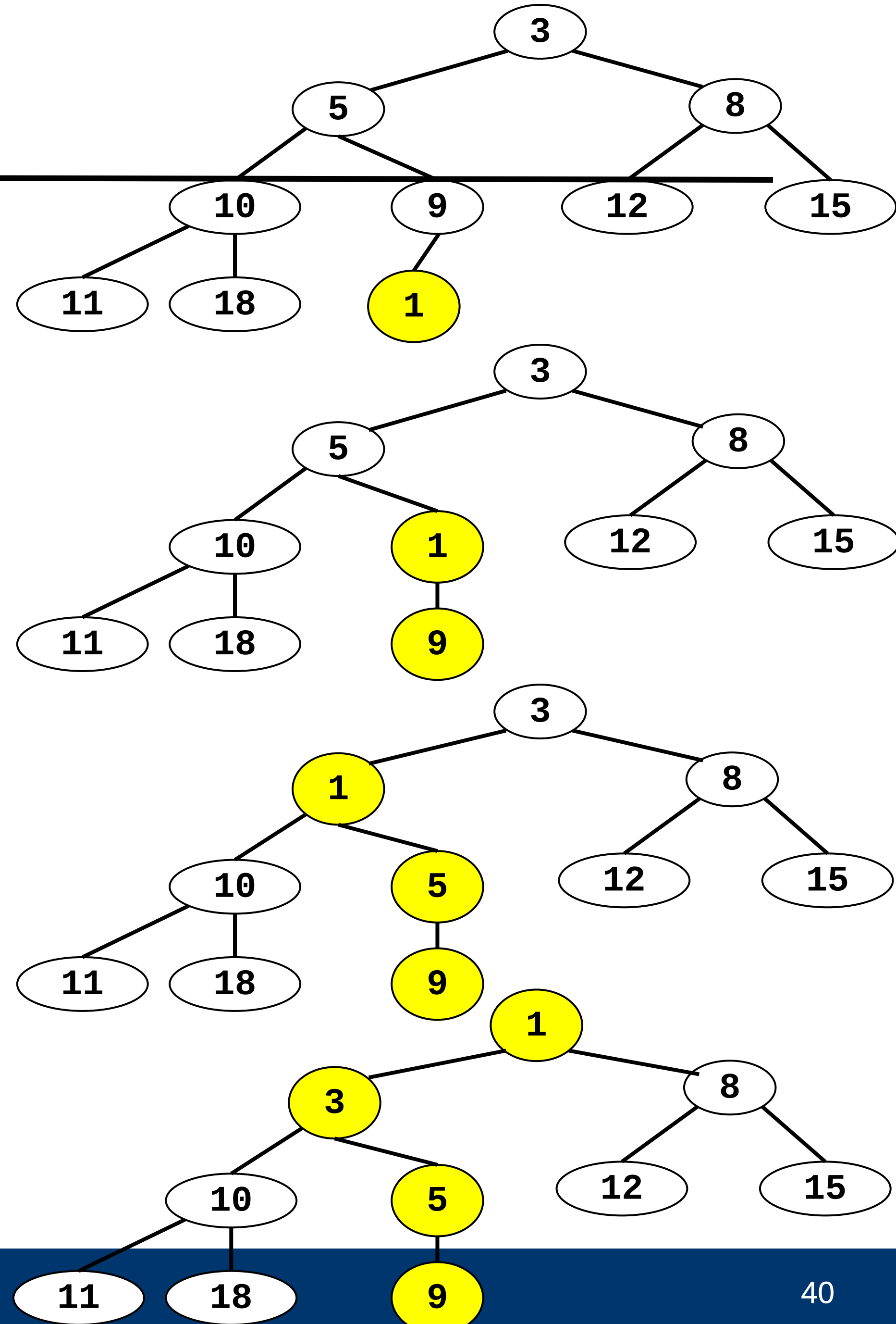
- We need to show that FC and HC still hold
- HC: Look at the tree after we moved a node k. k may
  - ... be smaller than its children. Then HC holds and we are done
  - ... be larger than at least one child k2. Assume that k2 is the smaller of the two children (k1, k2) of k. We next swap k and k2. The new parent (k2) now is smaller than its children (k1, k), so the HC holds
  - After the last swap, k has no children – HC holds
- FC: We remove one node, then we sift down
  - Removing last node doesn't affect FC as we remove in the last level
  - Sifting does not change the topology of the tree (we only swap)

Analysis - Complexity

- Recall that a heap with n nodes has ceil(log(n+1)) levels
- During sifting, we perform at most one comparison and one swap in every level
- Thus: O(ceil(log(n+1))) = O(log(n))

Add() on a Heap

- Cannot simply add on top
- Idea: We add new element somewhere in last level and sift up
  - We might need a new level
  - Sifting up: Compare to parent and swap if parent is larger



Analysis

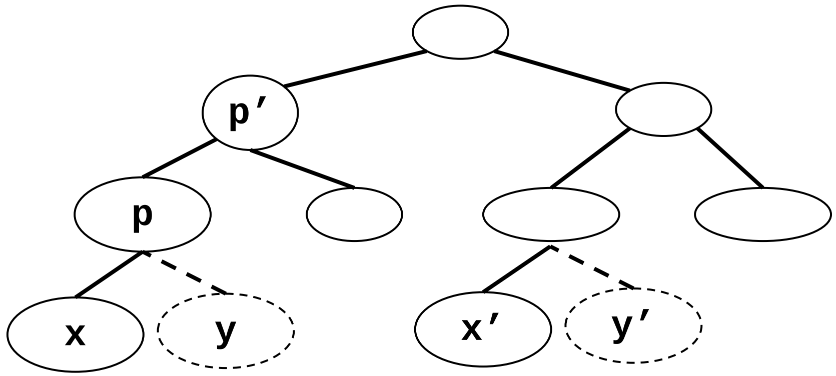
- Correctness
  - HC
    - If parent has only one child, HC holds after each swap
    - Assume a parent k has children k1 and k2, k2 was swapped there in the last move, and k2<k. Since HC held before, k<k1, thus k2<k<k1. We swap k2 and k, and thus the new parent is smaller than its children. On the other hand, if k2≥k, HC holds immediately (and we don’t swap).
  - FC: See **`deleteMin()`**
- Complexity: O(log(n))
  - See **`deleteMin()`**

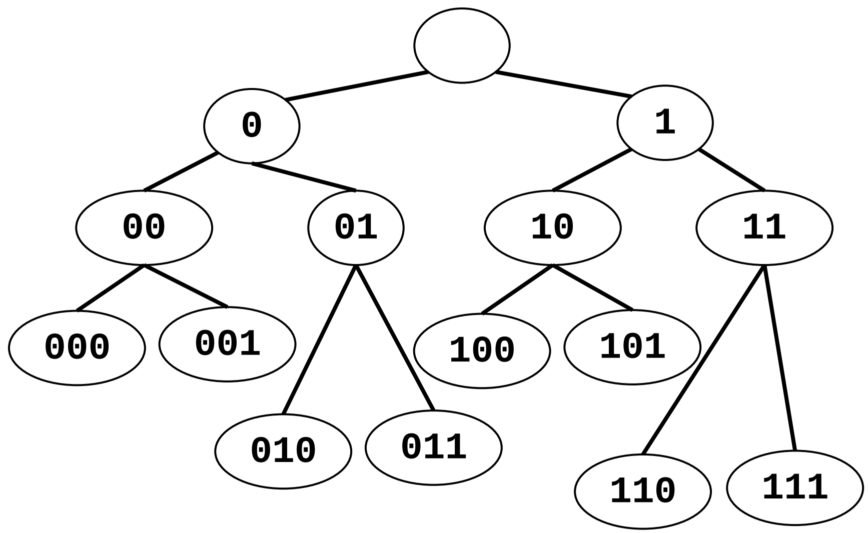
How to Find the Next Free / Last Occupied Node

- What do we need to find?
  - For **`deleteMin`**, we use the right-most leaf on the last level
  - For **`add`**, we add after the leaf right from the last leaf
- We actually need the parent k
  - From n, we can compute in O(1) the position p of the last leaf in the last level: p = n – 2^(floor(log(n)))
    - Or log(n+1) for **`add`**
  - The parent k of the node at p has index floor(p/2)'th in level d-1
  - The parent k' of k has index floor(p/4)'th in level d-2
  - ...
  - Now, in each node, we can decide whether to go left or right
  - Fast trick: Use the binary representation of p

Illustration

- For **`deleteMin`**, we need x (or x'); for **`add`**, we need y (or y')
  - p(x)=0, p(y)=1, p(x')=4, p(y')=5
  - Binary: 000, 001, 100, 101
- Go through bitstring from left-to-right
- Next bit=0: Go left
- Next bit=1: Go right
- Allows finding k in O(log(n))





Summary

	Linked list	Sorted linked list	Heap
getMin()	O(n)	O(1)	O(1)
deleteMin()	O(1)	O(1)	O(log(n))
add()	O(1)	O(n)	O(log(n))
merge()	O(1)	O(n1+n2)	O(log(n1)*log(n2))
Space	n add. pointer	n add. pointer	n add. pointer

Heaps can also be kept efficiently in an array – no extra space, but limit to heap size

But merge() requires O(n1+n2) or O(n1*log(n2+n1)) when using an array

Creating a Heap

- We start with an unsorted list with n elements
- Naïve algorithm: Start with empty heap and perform n additions
  - Obviously requires O(n*log(n))
- Better: Bottom-Up-Sift-Down
  - Build a tree from the n elements fulfilling the FC (but not HC)
    - Simple fill a tree level-by-level – this is in O(n)
  - Sift-down all nodes on the second-last level
  - Sift-down all nodes on the third-last level
  - ...
  - Sift down root

Analysis

- Correctness
  - After finishing one level, all subtrees starting in this level are heaps because sifting-down ensures FC and HC (see **`deleteMin()`**)
  - Thus, when we are done with the first level (root), we have a heap
- Analysis
  - We look at the cost per level h (1 ... log(n)=d)
  - For every node at level h, we need at most d-h operations
  - At level h≠d, there are 2^{h-1} nodes
    - For nodes at level d, we don't do anything
  - Over all levels, this yields

$$T(n)=\sum_{h=1}^{d-1} 2^{h-1}*(d-h)=\sum_{h=1}^{d-1} h*2^{d-h-1}=2^{d-1}\sum_{h=1}^{d-1}\frac{h}{2^h}\le n*\sum_{h=1}^{\infty}\frac{h}{2^h}=n*2=O(n)$$

Heap Sort

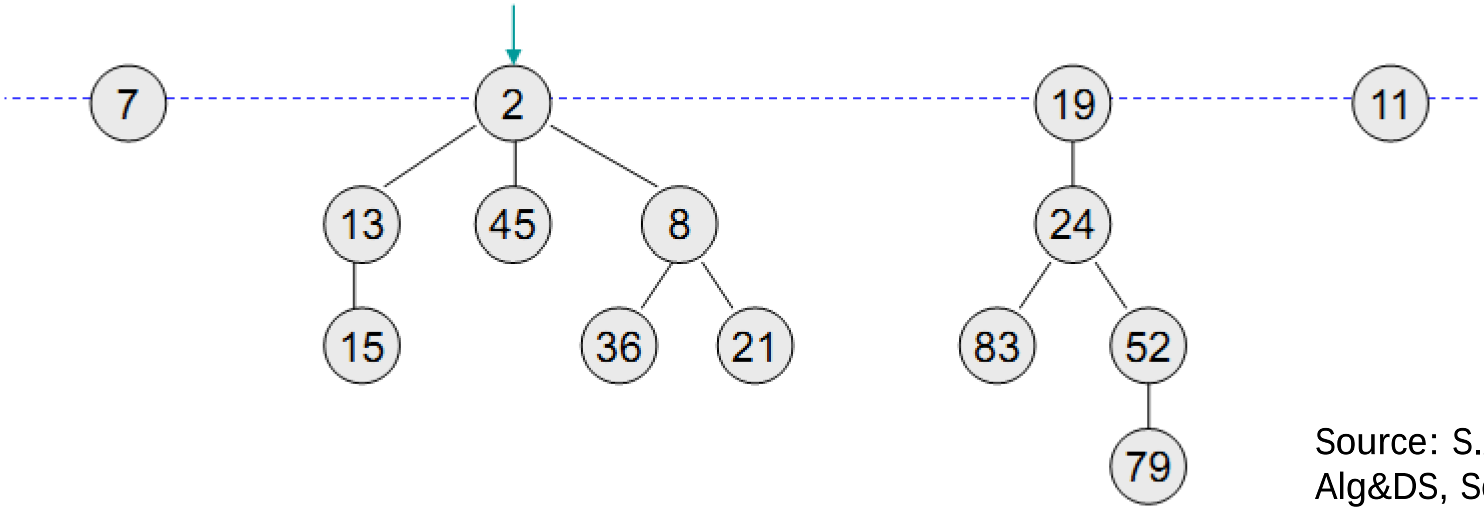
- Heaps also are a suitable data structure for sorting
- Heap-Sort (a classical one)
  - Given an unsorted list, first create a heap in O(n)
  - Repeat
    - Take the smallest element and store in array in O(1)
    - Re-build heap in O(log(n))
      - Call **`deleteMin( root)`**
  - Until heap is empty – after n iterations
- Thus: O(n*log(n))
  - Average-case only slightly better
- Can be implemented in-place when heap is stored in array
  - See [OW93] for details

Content of this Lecture

- Priority Queues
- Using Heaps
- Using Fibonacci Heaps

Fibonacci-Heaps (very rough sketch)

- A Fibonacci Heap (FH) is a forest of (non-binary) heaps with disjoint values
  - All roots are maintained in a double-linked list
  - Special pointer (**`min`**) to the smallest root
  - Accessing this value (**`getMin()`**) obviously is O(1)



Source: S.Albers, Alg&DS, SoSe 2010

Maintainance of a FH

- FHs are maintained in a lazy fashion
  - **`add(v)`**: We create a new heap with a single element node with value v. Add this heap to the list of heaps; adapt min-pointer, if v is smaller than previous min
    - Clearly O(1)
  - merge(): Simple link the two root-lists and determine new min (as min of two mins)
    - Clearly O(1)
- Deleting an element (**`deleteMin()`**) needs more work
  - Until now, we just added single-element heaps
  - Thus, our structure after n add() is an unsorted list of n elements
  - Finding the next min element after **`deleteMin()`** in a naïve manner would require O(n)

deleteMin() on FH

- Method is not complicated
  - We first remove the min element
  - We then go through the root-list and merge heaps with the same rank (=# of children) until all heaps in the list have different ranks
  - Merging two heaps in O(1): (1) Find the heap with the smaller root value; (2) Add it as child to the root of the other heap
- But analysis is fairly complicated
  - The above method is O(n) in worst case
    - But after every clean-up, the root-list is much smaller than before
    - Subsequent clean-ups need much less time
  - Amortized analysis shows: Average-case complexity is O(log(n))
  - Analysis depends on the growth of the trees during merge – these grow as the Fibonacci numbers

Disadvantage

- Though faster on average, Fibonacci Heaps have unpredictable delays
- No log(n) upper bound for every operation
- Not suitable for real-time applications etc.

Summary

	Linked list	**Sorted linked list**	**Heap**	**Fibonacci Heap**
getMin()	O(n)	O(1)	O(1)	O(1)
deleteMin()	O(1)	O(n)	O(log(n))	O(log(n))*
add()	O(1)	O(n)	O(log(n))	O(1)
merge()	O(1)	O(n1+n2)	O(log(n))	O(1)