COMPLEX ANALYSIS IN DATA SCIENCE

Project dissertation submitted to the Bangalore University for the award of the degree

MASTER OF SCIENCE IN MATHEMATICS

Submitted by

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CERTIFICATE

This is to certify that the Project dissertation entitled **"COMPLEX ANALYSIS IN DATA SCIENCE"** is a bonafide record of project work carried out by **Manu C. S.** under my guidance and suprivision in fulfillment of the requirements for the degree of **Master of Science in Mathematics**. Bangalore University, Bangalore and that no part of this dissertation has been submitted earlier for the award of any degree, diploma, fellowship or any other similar title.

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Place : Bangalore

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DECLARATION

I here by declare that the dissertation entitled **"COMPLEX ANALYSIS IN DATA SCIENCE"** is a genuine record of the project work carried out by me under the guidnce of **Dr Harina P. Waghamore**, Department of Mathematics, Bangalore University, Bangalore, for the award of the degree of **Master of Science in Mathematics**. Also, work of this project, either partially or fully, has not been submitted to any other University or Institution for the award of any degree.

Date :

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ABSTRACT

Complex analysis, a branch of mathematics that investigates functions of complex numbers, plays a pivotal role in various domains of data science. This paper provides an overview of advanced techniques in complex analysis and their applications in the field of data science. presents the application of complex analysis in signal processing and image analysis. Techniques such as the Fourier and Laplace transforms, derived from complex analysis, are employed to analyze and process signals and images efficiently. Extends the discussion to statistical modeling and machine learning. Complex-valued functions and their derivatives find applications in modeling multivariate datasets with inherent phase information. The paper illustrates how complex analysis aids in enhancing the performance of classification and regression algorithms, especially when dealing with non-linear relationships. This paper underscores the pivotal role of complex analysis in advancing data science methodologies. By harnessing the power of complex numbers and functions, researchers and practitioners can develop more robust and efficient algorithms for a wide range of applications, from signal processing to quantum computing. This synthesis of complex analysis and data science opens up new avenues for innovation and paves the way for the next generation of data-driven solutions.

Contents

1 Introduction

1.1 Complex analysis

Complex analysis is a branch of mathematics that deals with complex numbers, their functions, and their calculus. In simple terms, complex analysis is an extension of the calculus of real numbers to the complex domain. We will extend the notions of continuity, derivatives, and integrals, familiar from calculus to the case of complex functions of a complex variable. In doing so we will come across analytic functions, which form the centerpiece of this introduction. In fact, to a large extent complex analysis is the study of analytic functions.

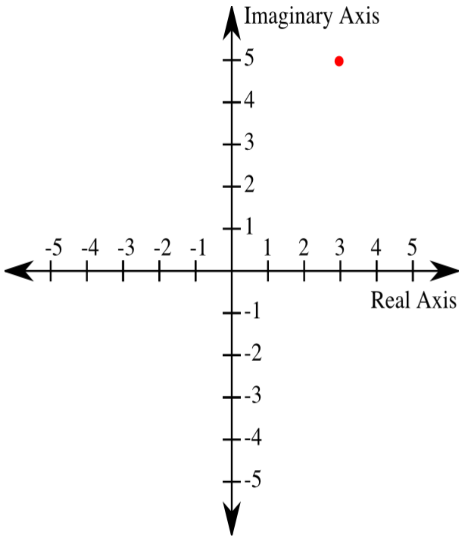
The basic ingredient of complex analysis is an analytic function, or that we know so well in calculus as a differentiable function. Any complex number z can be thought of as a point in a plane (x, y) so $z = x+iy$, where i=$\sqrt{-1}$. In a similar fashion, any complex function of a complex variable z can be separated into two functions, as in, $f(z) = u(z) + iv(z), or, f(x, y) = u(x, y) + iv(x, y)$. Clearly, such functions depend on two independent variables and have two separable functions, so plotting the function would need a four-dimensional space, which is difficult to imagine. Of course, the first starting point of the calculus of complex functions is to start with continuity of the function and then slowly move into the differentiability in the complex domain. use of complex analysis also has applications in engineering fields such as nuclear, aerospace, mechanical and electrical engineering. As a differentiable function of a complex variable is equal to its Taylor series (that is, it is analytic), complex analysis is particularly concerned with analytic functions of a complex variable (that is Holomorphic function)

1.2 Data Science

Data science combines math and statistics, specialized programming, advanced analytics, artificial intelligence (AI), and machine learning with specific subject matter expertise to uncover actionable insights hidden in an organization's data. These insights can be used to guide decision making and strategic planning. The accelerating volume of data sources, and subsequently data, has made data science is one of the fastest growing field across every industry. As a result, it is no surprise that the role of the data scientist was dubbed the "sexiest job of the 21st century" by Harvard Business Review. Organizations are increasingly reliant on them to interpret data and provide actionable recommendations to improve business outcomes. The data science lifecycle involves various roles, tools, and processes, which enables analysts to glean actionable insights

1.3 Geometrical Representation of Complex Analysis

As we know that every complex number ($z = a+ib$) is represented by a unique point p(a, b) on the complex plane and every point on the complex plane represents a unique complex number. To represent any complex number $z = (a+ib)$ on the complex plane follow these conventions: Real part of $z(Re(z) = a)$ becomes the X-coordinate of the point p Imaginary part of $z(Im(z) = b)$ becomes the Y-coordinate of the point p And finally $z(a+ib) \rightarrow p(a, b)$ which is a point on the complex plane. The plane on which the complex numbers are uniquely represented is called the Complex plane or Argand plane or Gaussian plane. The Complex plane has two axes: X-axis or Real Axis and Y-axis or Imaginary Axis.



X-axis or Real Axis

All the purely real complex numbers are uniquely represented by a point on it. Real part Re(z) of all complex numbers are plotted with respect to it.That's why X-axis is also called Real axis.

Y-axis or Imaginary Axis

All the purely imaginary complex numbers are uniquely represented by a point on it. Imaginary part Im(z) of all complex numbers are plotted with respect to it. That's why Y-axis is also called Imaginary axis.

1.4 Fundamental of Complex Analysis

Complex analysis is a branch of mathematics that deals with functions of complex numbers. It plays a fundamental role in various areas of data science, including signal processing, image processing, control systems, quantum mechanics, and more. Here are some of the key fundamentals of complex analysis relevant to data science:

- **Complex Numbers**

  A complex number is a number of the form $a + bi$, where a and b are real numbers, and i is the imaginary unit. Example: 3+4i.

- **Complex Functions**

  In data science, complex functions are used to model and analyze phenomena that can be naturally represented using complex numbers. A complex function $f(z)$ maps complex numbers to complex numbers and can be expressed as $f(z) = u(x, y)+iv(x, y)$, where u and v are real-valued functions of real variables x and y.

- **Complex Differentiation**

  Complex differentiation involves the concept of a derivative of a complex function. A complex function $f(z) = u(x, y) + iv(x, y)$ is said to be differentiable at a point $z_0 = x_0 + iy_0$ if the partial derivatives u_x, u_y, v_x, and $v_y$exist and satisfy the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$.

- **Analytic Functions**

  A complex function $f(z)$ is said to be analytic in a region if it is differentiable at every point within that region. Analytic functions play a crucial role in various applications of complex analysis.

- **Complex Integration**

  Complex integration involves integrating complex-valued functions over curves or regions in the complex plane. The Cauchy's Integral Theorem is a fundamental result in complex integration stating that if $f(z)$ is analytic in a simply connected domain D and C is a closed curve in D, then:

  $$\oint_c f(z)dz = 0.$$

  Provided that C is continuously differentiable.

- **Residue Theory**

  Residue theory is a powerful tool in complex analysis used for calculating certain types of integrals. The residue of a function at a particular point is a complex number that encodes information about the behavior of the function in the vicinity of that point.

$$Res(f, z_0) = \frac{1}{2\pi i} \oint_c f(z)dz.$$

- **Cauchy's Residue Theorem**

  This theorem extends Cauchy's Integral Theorem to include integrals over closed curves that enclose singularities. It states that:

$$\oint_c f(z)dz = 2\pi i \sum Res(f, z_k),$$

  where the sum is taken over all singularities enclosed by the curve C.

- **Euler's Formula**

  Euler's formula relates complex exponentials to trigonometric functions:

$$E = cos(v) + isin(v).$$

  These fundamentals of complex analysis provide the mathematical tools needed to analyze and understand complex-valued functions and their behaviour. In data science, these concepts are particularly important in applications involving signals, images, control systems, and quantum mechanics.

1.5 Importance of Complex Analysis in Data science

Complex analysis plays a crucial role in data science, especially in fields like signal processing, image processing, and control systems. Here are some key reasons why complex analysis is important in data science, along with mathematical expressions:

- **Frequency Domain Analysis**

  Complex numbers are used to represent the frequency domain of signals. The Fourier Transform, which is a fundamental tool in signal processing, allows us to analyze signals in terms of their frequency components. The complex exponential $e^{-i\omega t}$ is used to represent sinusoidal signals, where ω is the angular frequency.

  $$x(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt.$$

- **Convolution Theorem**

  The convolution theorem states that the convolution of two functions in the time domain is equivalent to the pointwise multiplication of their Fourier transforms. This is expressed as:

  $$x(t) * h(t) \Leftrightarrow X(\omega)H(\omega).$$

- **Laplace Transforms**

  In control systems and differential equations, Laplace transforms are used to analyze the behavior of linear time-invariant systems. The Laplace transform of a function $f(t)$ is defined as:

  $$F(s) = \int_{0}^{\infty} e^{-st}f(t)dt.$$

Here, s is a complex number

- **Complex Matrices**

  In data science, especially in areas like quantum computing, complex matrices are used to represent quantum states and operations. These matrices often involve complex eigenvalues and eigenvectors.

- **Image Processing**

  Complex numbers are used to represent the magnitude and phase of pixels in images. The Discrete Fourier Transform (DFT) is a key tool in image processing, and it involves complex exponentials.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{i2\pi kn}{N}}.$$

- **Analytical Solutions**

  Complex analysis provides techniques for finding analytical solutions to differential equations involving complex functions. For example, Cauchy's Integral Theorem and Residue Theorem allow us to evaluate complex integrals.

$$\oint_c f(z)dz = 2\pi i \sum_{nk=1} Res(f, z_k).$$

- **Control Systems**

  Complex analysis is crucial in the study of stability and control of dynamic systems. The complex plane and Nyquist stability criterion are central concepts in control theory $G(s)H(s) = -1$ at s=critical point

- **Probability and Statistics**

  Complex random variables are used in probability theory, especially in fields like quantum mechanics and electrical engineering.

$$E[X] = \int xp(x)dx.$$

These mathematical expressions and concepts demonstrate the importance of complex analysis in data science. It provides powerful tools for analyzing and manipulating data in various domains, making it an essential branch of mathematics for data scientists working on advanced applications.

1.6 Complex Analysis in Machine learning

Machine learning is a branch of artificial intelligence (AI) and computer science which focuses on the use of data and algorithms to imitate the way that humans learn, gradually improving its accuracy. Overview of how complex numbers can be used in certain machine learning algorithms give algorithm and mathematical expression Complex numbers can be used in certain machine learning algorithms, particularly in areas where dealing with phase information or complex-valued data is crucial. Here are two examples of machine learning algorithms that utilize complex numbers:

Convolutional Neural Networks (CNNs) for Image Processing

- **Algorithm Overview**

  CNNs are a type of deep learning model widely used for image processing tasks. They consist of multiple layers of convolutional filters that scan through the input image to extract features.

- **Complex Numbers Usage**

  In CNNs, complex numbers can be employed to represent images that have both amplitude and phase information. This is particularly useful in tasks where phase information plays a significant role, such as in medical imaging (e.g., MRI) or in some computer vision applications.

- **Mathematical Expression**

  Let I be the complex-valued input image, where $I(x, y) = A(x, y)e^{i\phi(x,y)}$. represents the amplitude and phase at each pixel. The convolution operation in the complex domain can be defined as: where K is the complex-valued convolution kernel.

Complex-Valued Neural Networks (CVNNs)

- **Algorithm Overview**

  CVNNs are neural networks that deal with complex-valued inputs, weights, and activations. They extend traditional neural networks to handle complex numbers directly.

- **Complex Numbers Usage**

  CVNNs are particularly useful in tasks where the data is inherently complex, such as in signal processing or in tasks involving oscillatory phenomena. They can also be used to model interactions between real and imaginary components. Mathematical Expression: Consider a complex-valued input $z = x + iy$, where x and y are the real and imaginary components, respectively. The activation function and weight updates are defined using complex arithmetic. For instance, the activation function might be $f(z) = \tanh(z)$, where *tanh* is computed using complex arithmetic

2 Cauchy-Riemann equation and Conformal mapping

2.1 Cauchy-Riemann equation

In this Section we consider two important features of complex functions. The Cauchy-Riemann equations provide a necessary and sufficient condition for a function $f(z)$ to be analytic in some region of the complex plane. A mapping between the z-plane and the w-plane is said to be conformal if the angle between two intersecting curves in the z-plane is equal to the angle between their mappings in the w-plane. Such a mapping has widespread uses in solving problems in fluid flow and electromagnetics, for example, where the given problem geometry is somewhat complicated.

The Cauchy-Riemann equations

Remembering that $z = x+iy$ and $w = u+iv$, we note that there is a very useful test to determine whether a function $w = f(z)$ is analytic at a point. This is provided by the Cauchy-Riemann equations. These state that $w = f(z)$ is differentiable at a point $z = z_0$ if, and only if,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

at that point When these equations hold then it can be shown that the complex derivative may be determined by using either

$$\frac{df}{dz} = \frac{\partial f}{\partial x} \qquad \text{or} \qquad \frac{df}{dz} = -\frac{\partial f}{\partial y}$$

(The use of 'if, and only if,' means that if the equations are valid, then the function is differentiable and vice versa.)

If we consider $f(z) = z^2 = x^2 - y^2 + 2ixy$ then $u = x^2 - y^2$ and $v = 2xy$ so that

It should be clear that, for this example, the Cauchy-Riemann equations are always satisfied; therefore, the function is analytic everywhere. We find that

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y, \frac{\partial v}{\partial x} = 2y, \frac{\partial v}{\partial y} = 2x$$

It should be clear that, for this example, the Cauchy-Riemann equations are always satisfied; therefore, the function is analytic everywhere. We find that

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = 2x + 2iy = 2z \quad \text{or , equivalently,} \ \frac{df}{dz} = -i$$

$$\frac{\partial f}{\partial y} = -i(-2y + 2ix) = 2z$$

This is the result we would expect to get by simply differentiating $f(z)$ as if it were a real function. For analytic functions this will always be the case i.e. for an analytic function $f_0(z)$ can be found using the rules for differentiating real functions.

2.2 Conformal mapping

Conformal maps are a class of complex functions that preserve angles locally. More formally, a function f is considered conformal in a region D if it is holomorphic (analytic) and its derivative $f'(z)$ is never zero in D.

we saw that the real and imaginary parts of an analytic function each satisfies Laplace's equation. We shall show now that the curves u(x, y) = constant and v(x, y) = constant intersect each other at right angles (i.e. are orthogonal). To see this we note that along the curve u(x, y) = constant we have du = 0. Hence

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

Thus, on these curves the gradient at a general point is given by

$$\frac{dy}{dx} = -\frac{(\frac{\partial u}{\partial x})}{(\frac{\partial u}{\partial y})}$$

Similarly along the curve v(x, y) = constant, we have

$$\frac{dy}{dx} = -\frac{(\frac{\partial v}{\partial x})}{(\frac{\partial v}{\partial y})}$$

The product of these gradients is

$$\frac{(\frac{\partial u}{\partial x})(\frac{\partial v}{\partial x})}{(\frac{\partial u}{\partial y})(\frac{\partial v}{\partial y})} = -\frac{(\frac{\partial u}{\partial x})}{(\frac{\partial u}{\partial y})}\frac{(\frac{\partial u}{\partial y})}{(\frac{\partial u}{\partial x})} = -1$$

where we have made use of the Cauchy-Riemann equations. We deduce that the curves are orthogonal. As an example of the practical application of this work consider two-dimensional electrostatics. If u = constant gives the equipotential curves then the curves v = constant are the electric lines of force. Figure shows some curves from each set in the case of oppositely-charged particles near to each other; the dashed curves are the lines of force and the solid curves are the equipotentials In ideal fluid flow the curves v = constant are the streamlines of the flow. In these situations the function w = u + iv is the complex potential of the field

2.2.1 Properties of Conformal mapping

- **Angle Preservation**

  Conformal maps preserve angles between curves. Specifically, if y_1 and y_2 are smooth curves that intersect at a point z in their respective images under f, then the angle between the curves at z is the same as the angle between their images at $f(z)$.

- **Angle Preservation** A conformal map is a local biholomorphism, meaning it is bijective (both one-to-one and onto) and holomorphic in a neighborhood around each point in the domain.
- **Smoothness and Continuity**

  Conformal maps are infinitely differentiable in their domain (except at singularities, if any). They also preserve continuity.
- **Invariance of Length Ratios**

  If f is conformal in a domain D, and y is a smooth curve in D, then the ratio of the lengths of $f(y)$ and y is constant. This is a consequence of the preservation of angles.
- **Preservation of Circles and Lines**

  If a conformal map f is defined in a domain containing a circle or a straight line, it maps the circle or line to another circle or line (or possibly a point or a straight line in certain cases).
- **Conformal Equivalence**

  Two regions in the complex plane are conformally equivalent if there exists a conformal map between them. Conformal equivalence is an equivalence relation; it is reflexive, symmetric, and transitive.
- **Möbius Transformations**

  A special class of conformal maps are the Möbius transformations, which are transformations of the form $f(z) = \frac{cz+d}{az+b}$, where a,b,c,d are complex numbers with $ad - bc \neq 0$ Möbius transformations include translations, dilations, rotations, and inversions.
- **Applications**

  Conformal maps have wide-ranging applications in physics and engineering, particularly in areas such as fluid dynamics, electrostatics, heat conduction, and more. They are used to model and analyze various physical phenomena. Overall, conformal maps are powerful tools in complex analysis with applications in a wide range of scientific and engineering disciplines. They allow us to study complex systems in a way that simplifies their geometry while preserving important analytic properties.

2.2.2 Conformal mapping can be applied in the visualization and analysis

Conformal mapping can be a powerful tool in data visualization and analysis, particularly in cases where preserving local angles and relationships is important. Here are some examples of how conformal mapping can be applied in this context:

- **Geographic Data Visualization**

  Conformal maps can be used to project geographic data onto a flat surface while preserving angles. This is crucial in cartography for creating accurate maps. For instance, the Mercator projection is a well-known conformal map used for navigation.

- **Image Processing and Computer Vision**

  Conformal maps can be applied in image registration and warping. For instance, when aligning images, it's important to preserve the spatial relationships between features. Conformal mapping can help achieve this by locally preserving angles.

- **Mesh Generation and Finite Element Analysis**

  In computational engineering, conformal maps can be used to generate structured meshes for simulations. They can ensure that the mesh elements preserve angles and accurately represent the geometry of the physical system.

- **Analyzing Networks and Graphs**

  Conformal mapping can be used to visualize and analyze complex networks, such as social networks, transportation networks, or electrical circuits. It can help in understanding the global and local properties of the network while preserving the relative distances and connections.

- **Data Clustering and Dimensionality Reduction**

  Conformal mapping can be applied to represent high-dimensional data in lower-dimensional spaces, while preserving local distances and relationships. This can be useful in tasks like clustering and visualization of data clusters.

- **Electromagnetic Field Analysis**

  In electromagnetism, conformal mapping can be used to analyze and visualize the behavior of electric and magnetic fields around complex geometries. This is crucial in designing antennas and other electromagnetic devices. Overall, conformal mapping provides a powerful mathematical framework for visualizing and analyzing complex data sets in a way that preserves important geometric properties. It finds applications in various scientific and engineering disciplines where accurate representation of spatial relationships is crucia.

3 Fourier Transform and Complex Analysis

There are some naturally produced signals such as nonperiodic or aperiodic, which we cannot represent using Fourier series. To overcome this shortcoming, Fourier developed a mathematical model to transform signals between time (or spatial) domain to frequency domain and vice versa, which is called 'Fourier transform'

3.1 Relation between Complex Analysis and Fourier Transform

Complex analysis and Fourier transforms are intimately connected through the use of complex exponential functions. This relationship is fundamental in many areas of mathematics, engineering, and physics. Here's an overview of how complex analysis and Fourier transforms are related:

- **Complex Exponential Functions**

  In complex analysis, the complex exponential function is defined as:

$$e^{ix} = \cos(x) + i\sin(x).$$

  This function plays a central role in complex analysis and is used to represent oscillatory behavior in many physical phenomena.

- **Fourier Series**

  The Fourier series is a representation of a periodic function as a sum of complex exponential functions:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

  Here, cn are complex coefficients that are obtained through integration and depend on the original function $f(x)$. The Fourier series allows us to analyze functions in terms of their frequency components.

- **Complex Fourier Transform**

  The Fourier transform is a generalization of the Fourier series for non-periodic functions. The complex Fourier transform of a function $f(x)$ is defined as:

$$F(k) = \int_{-\infty}^{\infty} e^{-ikx} dx.$$

  The relationship between the Fourier transform and complex analysis is evident in the presence of the complex exponential function e^{-ikx}, which is characteristic of Fourier transforms.

- **Analyticity and Fourier Transforms**

  Functions that are analytic (holomorphic) in a certain domain of the complex plane have a Fourier transform that is well-defined and convergent for a certain range of frequencies. This is a powerful connection between the two fields.

- **Convolution and Multiplication Theorems**

  Complex analysis provides tools like the convolution theorem and multiplication theorem which state that the Fourier transform of the convolution of two functions is the pointwise product of their Fourier transforms, and vice versa. These theorems have important applications in signal processing, communication systems, and more.

- **Filtering and Spectral Analysis**

  Complex analysis is used to analyze and design filters, where the behavior of the filter is understood in terms of its frequency response. The Fourier transform is a key tool in spectral analysis, helping to identify the frequency components in a signal.

- **Control Systems and Signal Processing**

  Complex analysis and Fourier transforms are extensively used in control theory and signal processing. They play a crucial role in the analysis and design of systems that involve feedback, filtering, and modulation. In summary, complex analysis and Fourier transforms are intertwined through the use of complex exponential functions, which provide a powerful framework for analyzing functions in terms of their frequency content. This relationship is essential in various fields of mathematics, physics, engineering, and signal processing.

Applications

3.2 Fourier Transform and signal Processing

- **Function Properties**

  We begin by stating the properties of the functions that we will investigate and by provid- ing appropriate definitions. The functions we are considering are piecewise-continuous complex-valued functions of real variables; that is, functions mapping values in R or sub- sets thereof into the complex plane. A function is piecewise continuous if it contains no infinite discontinuities, and each finite subinterval of its domain contains a finite number of discontinuities. We require piecewise continuity for its favorable properties with respect to integration, as will become clear. We will also investigate periodic functions. These are functions that have the property that $f(x) = f(x + Tn)$ for each x in the domain of the function and for each integer n. The positive real value T is such that it is the smallest such value to satisfy this property; we say the function has a period of T , or that the function is T-periodic. Using this definition, each function defined on some interval I with length T , such as [0, T] or $\left[-\frac{T}{2}, \frac{T}{2}\right]$, the domain of the function can be extended to all of $\mathbb{R}$ and made T -periodic. If the function is continuous on I, its periodic extension is continuous on R if its values at the left and right endpoints of I are equal. Otherwise, the function will be piecewise-continuous on $\mathbb{R}$. Lastly, the set E is is defined by to be the set of piecewise-continuous complex-valued 1-periodic functions on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. We will consider this set and appropriate subsets as vector spaces of functions with respect to different inner products.

- **Vector Space of Real Periodic Function**

  we first consider the subset of E consisting of real-valued functions f: $\left[\frac{-1}{2}, \frac{1}{2}\right] \to$ R, and define an inner product for this subspace as

$$\langle f, g\rangle = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\overline{g(x)}dx.$$

where f, g$\in$ E are real-valued, and $\overline{g(x)}$ denotes the complex conjugate of g(x). Note that,for any real-valued function f , we have $\overline{f(x)}$= f (x). This subspace of real-valued functions is spanned by the set of functions is spanned by the set of function.

$$\left\{\frac{1}{\sqrt{2}}, \cos(2\pi x), \sin(2\pi x), \cos(4\pi x), \sin(4\pi x), \ldots\right\}$$
$$= \left\{\frac{1}{\sqrt{2}}, \cos(2\pi nx), \sin(2\pi nx)\right\}.$$

The first of these functions accounts for a vertical shift, while each cosine and sine de- scribe the even and odd portions of a particular frequency. Together, the pair $\cos(2\pi nx)$ and $\sin(2\pi nx)$ can be thought of as describing a single sinusoid of period $\frac{1}{n}$ that may be horizontally translated (or phase shifted, as this is referred to in many physical applications). This can be easily verified through the use of the angle sum formulas for cosine or sine. Note that each of these functions is orthogonal to each other function, since the inner product between any two distinct functions in the set is equal to 0. Each function is also a unit vector since the inner product of each function with itself is equal to 1. Thus, this set forms an orthonormal basis for the subspace of real-valued functions on E. Each function in this subspace can be represented by a linear combination of these basis vectors as follows:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(2\pi nx) + b_n \sin(2\pi nx)],$$

where the coefficients

$$a_n = \langle f(x), \cos 2\pi nx \rangle = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\cos(2\pi nx)dx, \qquad for \quad n = 0, 1, 2, \ldots$$

$$b_n = \langle f(x), \cos 2\pi nx \rangle = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\sin(2\pi nx)dx, \qquad for \quad n = 0, 1, 2, \ldots$$

This representation of the function is the real Fourier series of f . Note that equality does not necessarily hold without considering the convergence of this infinite summation. Imposing the restriction that f has finite-valued one-sided derivatives at all points x∈$(-\frac{1}{2}, \frac{1}{2})$, including the left-derivative at the right endpoint and vice-versa, is sufficient to provide pointwise convergence for the series on $[-\frac{1}{2}, \frac{1}{2}]$. An equivalent condition is the requirement that f' be piecewise continuous, thus providing $f' \in$ E. By Dirichlet's Theorem, for each function $f \in$ E with $f' \in$ E, its real Fourier series will converge to the average of the one-sided limits at each point. For points in the domain where f is continuous, the one-sided limits are necessarily equal, and the series converges to the value of the function at that point. At the endpoints, the series will converge to the average of the function's one-sided limit at each endpoint. As brief aside, consider the case of the vector space of "arrows" in $\mathbb{R}_{\ltimes}$; the inner product of a vector with a particular basis vector informally represents how much that vector "points" in the same direction as the basis vector. In our vector space of functions, each real coefficient a_n and b_n analagously represents how much the function f "corresponds with" the cosine or sine with period $\frac{1}{n}$(or, with frequency n). Another interpretation is that the coefficient describes how much each particular frequency is present in the function.

- **Vector Space of E**

  Turning our attention to the entire function space E, we will need a set of complex-valued basis vectors. Similar to how real-valued sinusoidal functions can be expressed using a (real) linear combination of a cosine and sine with equal period, we can use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to express complex-valued periodic functions. We define an inner product for the vector space E as

$$\langle f, g \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\overline{g(x)}dx.$$

  One orthonormal basis for E is the set of complex exponentials

$$\left\{e^{i2\pi nx}\right\}_{n=-\infty}^{\infty} = \{\ldots, e^{-i4\pi x}, e^{-i2\pi x}, 1, e^{i2\pi x}, e^{i4\pi x}, \ldots\}.$$

For a particular complex exponential, varying x corresponds to a rotation around the unit circle in the complex plane. With this interpretation, the functions containing positive integer values for n produce counterclockwise rotation, and those with negative n produce clockwise rotation. The complex Fourier series of a function f ∈ E with f ′ ∈ E is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx}.$$

Again, each coefficient cn is the inner product of f with the appropriate complex exponential:

$$c_n = \langle f(x), e^{i2\pi nx} \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\overline{e^{i2\pi nx}}dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)e^{i2\pi nx}dx. \tag{1}$$

As with the real Fourier series, these complex coefficients c_n can be thought of as describing the frequency content of the function f . Taking cues from the field of complex analysis, the modulus of c_n conveys information about the amplitude of that sinusoid, and its argument conveys information about the phase. This effectively encodes all of the necessary information about the periodic function.

- **The Fourier Transform**

To motivate the formulation of the Fourier transform, we pose the question: Given a real- valued function f, can we create a new function to describe its frequency content? Fortunately, the coefficients of the complex Fourier series lead us towards a solution. Changing our perspective, given some f $\in$ E, the complex inner product (1) for E can be thought of as a mapping from an integer n, representing a particular frequency, to a complex value c_n that contains amplitude and phase information of the complex sinusoid that corresponds to that frequency present in f . However, since this integral is only taken over $[-\frac{1}{2}, \frac{1}{2}]$, it is only able to capture information about f on that interval. If we wish to consider functions f that are defined on all of $\mathbb{R}$, we could change the inner product to an improper integral from $-\infty$ to ∞. This allows the integral to capture information over the entire real line; however, in doing so, we must revisit the convergence of the integral (1). Obtaining finite values for each c_n is desirable, so we desire the integrand f(x) $e^{-i2\pi nx}$ to be absolutely integrable. Since $|e^{-i2\pi nx}| = 1$, that f is absolutely integrable on $\mathbb{R}$ is a sufficient condition to provide

$$\int_{-\infty}^{\infty} |f(x)e^{-i2\pi nx}|dx \leq \infty.$$

Lastly, to describe all frequencies present in the function, rather than only those with integer value, we can change from the discrete variable n $\in \digamma$ to a continuous variable $\xi \in \mathbb{R}$ With all of the pieces in place, the Fourier transform of a function f (x) is defined by as

$$F(\xi) = \mathcal{F}[f(x)] = \int_{-\infty]}^{\infty} f(x)e^{-i2\pi x\xi}dx. \tag{2}$$

This is sometimes referred to as the forward Fourier transform, and we will refer to it as such. Similarly, the inverse Fourier transform of F(ξ) is defined to be

$$f(x) = \mathcal{F}^{-1}[F(\xi)] = \int_{-\infty}^{\infty} f(\xi)e^{i2\pi x\xi}d\xi. \qquad (3)$$

The forward transform takes in a function $f : \mathbb{R} \to \mathbb{C}$ and produces a function $F : \mathbb{R} \to \mathbb{C}$. Naturally, its inverse produces f given F. To make better sense of the relationship between these variables x and ξ, we turn back to the physical interpretation of these two functions. We can model a spatial wave-function (such as the peaks and troughs along the cross section of a body of water) with a function f that relates the position x to the wave's height f(x). This gives x units of length or distance. For the exponential function to have a dimensionless argument, we give ξ units of reciprocal distance, which describes cycles per unit distance. This describes the spatial frequency of the wave. The transformed function F(ξ) takes a particular spatial frequency as input and returns a complex number that describes the amplitude and phase of the spatial wave with that frequency that best corresponds with f. Another interpretation of domains for the functions f and F are time and temporal frequency, respectively. For example, an audio signal can be modeled as a function of time, f(t), and its transform F(ν) describes the frequency content of the audio. If t is given in seconds, then ν has units of Hertz. There are many other pairs of variables that can be related in this manner. In the field of quantum physics, the Fourier transform relates the position of a particle to its momentum. Pairs of variables that are related in this way are sometimes referred to as conjugate variables.

- **Properties of the Fourier Transform**

  In our studies of the transform, it was particularly interesting to see the relationship between two actions on a function through the Fourier transform. Perhaps the most straightforward is the linearity of the transform, such that

$$\mathcal{F}[af + bg](\xi) = aF(\xi) + bG(\xi).$$

  This follows from the linearity of the integral used in the definition. The transform of purely real- or imaginary-valued functions f also displays interesting symmetries. For instance, if f is real-valued, then F (-ξ) = $\overline{F(\xi)}$. Similarly, the parity of the function f reveals additional symmetries. Exploiting these symmetries can reduce the amount of computation required to obtain F . More interestingly, given c $\in \mathbb{R}$ the two relationships $\mathcal{F}[e^{i2\pi nx} \cdot f(x)](\xi) = F(\xi - c)$ and $\mathcal{F}[f(x - c)](\xi) = e^{-i2\pi c\xi} \cdot F(\xi)$ indicate that a horizontal translation in one domain corresponds to a complex rotation in the other domain. Again, since the complex values of F describe the amplitude and phase, this interpretation provides that a horizontal translation (phase shift) corresponds to a change in the phase angle of the sinusoids describing the function. Perhaps the most powerful property of the Fourier transform for signal processing appli- cations is given in the Convolution Theorem . Simply put, it states that the transform (forward or inverse) of the convolution of two functions is equivalent to the product of their transforms. This is equivalent to the following two statements:

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g] = F \cdot G. \tag{4}$$

$$\mathcal{F}^{-1}[F * G] = \mathcal{F}^{-1}[F] \cdot \mathcal{F}^{-1}[G] = f \cdot g. \tag{5}$$

where

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y).g(x-y)dy. \tag{6}$$

denotes the convolution of any two complex-valued functions f and g of real variables. Though the convolution of two functions is demanding and often has few symmetries aiding in computation, the Convolution Theorem allows the convolution of two functions to be computed through transforming two functions, performing pointwise multiplication, and taking the inverse transform of the resulting product.

- **Multidimensional Fourier Transform**

  To consider functions of multiple variables, we first consider our variables x and ξ as elements of $\mathbb{R}^n$, such that

$$x = (x_1, x_2, x_3, \ldots x_n),$$

  and

$$\xi = (\xi_1, \xi_2, \xi_3, \ldots \xi_n),$$

  Utilizing the dot product on $\mathbb{R}^n$, we find

$$x.\xi = x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n \in \mathbb{R}. \tag{7}$$

  The multidimensional Fourier transform and its inverse transform relate the functions f (x) and F (ξ) as follows:

$$F(\xi) = \mathcal{F}[f(x)] = \int_{\mathbb{R}^n} f(x)e^{-i2\pi x.\xi}dx. \tag{8}$$

$$f(x) = \mathcal{F}^{-1}[F(\xi)] = \int_{\mathbb{R}^n} F(x)e^{-i2\pi x.\xi}d\xi. \tag{9}$$

To better understand how this is computed, we expand the dot product in the exponent using (7) to obtain

$$F(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i2\pi x\xi}dx.$$

$$= \int_{\mathbb{R}^n} f(x)e^{i2\pi(x_1\xi_1+x_2\xi_2+\ldots+x_n\xi_n)}dx.$$

$$= \int_{\mathbb{R}^n} f(x_1, x_2, \ldots, x_n)e^{-i2\pi x_1\xi_1}e^{-i2\pi x_2\xi_2}\ldots e^{-i2\pi x_n\xi_n}dx_1dx_2\cdots dx_n.$$

$$= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x_1, x_2, x_3, \ldots x_n)e^{-i2\pi x_1\xi_1dx_1}\right]e^{-i2\pi x_2\xi_2}dx_2\cdots e^{-i2\pi x_n\xi_n}dx_n.$$

The innermost integral, written here with respect to x1, is simply the one-dimensional Fourier transform with respect to x_1; the resulting function is a function of $\xi_1, x_2, \ldots, x_n$ Continuing in this manner, we find that the multidimensional transform can be computed with n independent Fourier transforms, one along each dimension.

3.3 Image Processing

As we have touched on previously, the Fourier transform has extensive applications to signal processing. A number of signal processing techniques, such as filtering, are modeled using the convolution of two functions. Since the convolution is very computationally intensive, especially so in higher dimensions, it is common practice to take advantage of the Convolution Theorem. If we first compute the transform of each function, multiply the transformed functions, and finally compute the inverse transform of their product, the result is exactly the convolution of the two initial functions. The greatest advantage comes when considering the symmetries of the Fourier trans- form, allowing us to perform fewer computations for the forward and inverse transforms. Some of these symmetries, along with memory-management techniques, allow a far more computationally efficient algorithm for computing the Fourier transform on discrete data sets . In the case of image processing, we will consider two-dimensional Fourier transforms. We can model a grayscale image as a function $f(x)$, which assigns an intensity value to each coordinate $x = (x_1, x_2)$ in the plane. (A full-color image can be treated as a triplet of such functions, one for each of the red, green, and blue channels in the image.) As is often the case when relating our equations to physical phenomena, this will be a real-valued function with a range [0, 1]; a value of 0 represents black pixels and 1 represents a white pixel. The transformed function F (ξ) assigns a value to each spatial frequency pair ξ= (ξ_1, ξ_2). Here, interpreting these elements as vectors introduces the notion of a direction for the planar wave described by the corresponding sinusoid, as seen in Figure 1. An illustrative example is beneficial to understanding how the Convolution Theorem applies to image processing. Let $f(x)$ again be a grayscale image. We consider some function $h(x)$ that acts as a filter function which produces a processed image $g(x) = (f * h)(x)$. With

the Convolution Theorem, we find

$$g(x) = (f * h)(x).$$

$$= \mathcal{F}^{-1}[F.H]. \tag{10}$$

(a) (b) (c)

Figure 1: Left to Right: Low,intermediate and high frequency planar waves.The red arrows drawn in image denotes a vectors representation of ξ=(ξ_1, ξ_2)

In practice, we care less about h(x) than we do its transform H(ξ); instead we can choose a function H to modify the frequency content of f however we desire [6]. For instance, if we wish to blur the image, we can boost its low frequency content relative to its high frequencies. Similarly, sharpening the image corresponds to increasing its high frequency content relative to its low frequency content. It is worth noting that the low frequency vectors closest to $(\xi_1, \xi_2) = (0,\ 0)$ correspond to the overall brightness of the image. This can be seen in Figure 2. We now demonstrate the application of (10) . We can define a filter function

$$H(\xi_1, \xi_2) = |\xi_1 \xi_2| + 0.3, \tag{11}$$

constructed as a sharpening filter that simultaneously brightens the image. This function H and the resulting function g are shown in Figure 3. showcase various other filtering functions and their corresponding output.

(a) (b)

Figure 2: Left: An example input image,representing f(x). Right:The magnitude of the transformed function F(ξ); the values have been log scaled to improve visual contract of the values close to zero

(a) (b)

Figure 3: processing image (left) produced by filter H(ξ)=$|\xi_1\xi_2| + 0.3$(right)

Beyond Image Processing

Image processing problems similar to those posed in eqn. (10) are solved by designing a filter function H such that the resulting image $g(x)$ has certain desirable properties. A far more formidable set of problems are inverse problems . Better stated, given a function g(x), we wish to undo the effects of convolution by some unknown filtering operation to find the original image $f(x)$. There are many physical processes which can be modeled with a convolution that we may wish undo through solving such an inverse problem. A simple one-dimensional case is the reconstruction of an audio signal that has been corrupted by noise or other artifacts. In imaging, we may wish to correct a photograph that was taken out-of-focus. Many portions of medical imaging, especially 3-dimensional imaging techniques such as computed tomography (CT) or magnetic resonance imaging (MRI), are dependent on undoing a con- volution introduced by the physical imaging process. The resulting function obtained is a three-dimensional reconstruction of the imaging field. The solutions to many inverse problems can sometimes be ill-posed since artifacts and noise introduced by audio recording or imaging can include non-linear terms. These inverse problems are then solved as optimization problems, where we wish to find a recon- structed function with the understanding that, in most cases, it will not be identical to the original function. For any hope of a useful solution, it is essential to have a thorough understanding of the physical processes being modeled.

• 3.4 Sound waves and Fourier Analysis

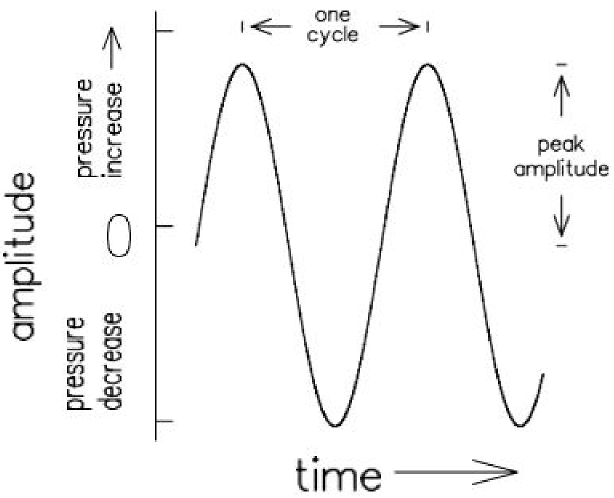
Wave forms

Auditory information tells us about small movements, which generate the sounds that our ears detect. Small movements produce sound by causing fluctuations of the air's pressure in their vicinity. These fluctuations follow the source's movements so that there are alternating increases and decreases in the air's pressure. This generates a sound wave as the fluctuations spread through the air. Sound informs us about the nature of the movement that produced it because the pressure fluctuations follow the source's movement. Thus, a graph of pressure against time, as a sound wave passes, mirrors the distant movements of the sound source. This graph is called a waveform.

Sine waves

A simple kind of movement is like the repeating pattern a pendulum's motion. Rapid movements of this kind produce sounds called pure tones, known also as sine waves, sine tones, sine-wave tones or sinusoids. Sounds of this precise type are not particularly common in our everyday environment, but they can be heard when a tuning fork is sounded, or when we whistle. The waveform of such a sound is shown below

Each repetition of the wave's pattern is called a cycle. The rate at which



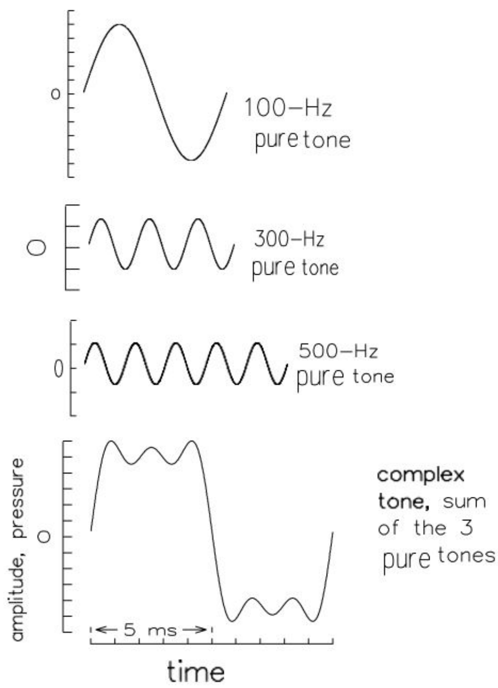
(a)

cycles arrive is the rate that they were produced at the source. The number of cycles arriving in one second is known as the frequency of the sound, measured in Hertz and abbreviated Hz. 1000 Hz is a kilohertz, abbreviated kHz (see also appendix 1). Thus, slow vibrations give low frequency sounds and faster vibrations give high frequency sounds. Frequencies of pure tones that the ear can detect range from about 20 Hz up to nearly 20 kHz. The height of the wave's crest is called its peak amplitude. The power of a pure tone increases with its peak amplitude and is measured in decibels, abbreviated dB . This aspect of the waveform indicates the strength of the source's vibration. Phase is a term used to describe position within the wave's cycle, where there are 360 degrees (or 2π radians) in a full cycle. Thus, 180 degrees (or π radians) is half way through a cycle, 90 degrees (or 2πradians) is one quarter of a cycle, and so on.

Complex tones

Typical everyday sounds are said to be complex because they contain a range of frequency components. This is true of individual speech sounds and individual notes in music as well as the clatters, clangs, bangs, bumps, rasps, rumbles, buzzes, beeps, fillips, fizzes, crunches, crackles, thumps, thuds, etc., that populate our auditory world. When components at higher frequencies are removed from such sounds they can seem muffled, or if lower frequencies are missing they can sound tinny. Individually, a single frequency component from a complex sound is just like a pure-tone with a particular frequency and dB. Therefore, different complex sounds can be obtained by adding together pure tones with different frequencies and dB. One complex sound is shown on the left, along with its 3 components. Adding together the pressures of each of the pure-tone components at each point in time forms the waveform of the complex sound.

A complex sound can be analysed to find its frequency components. This

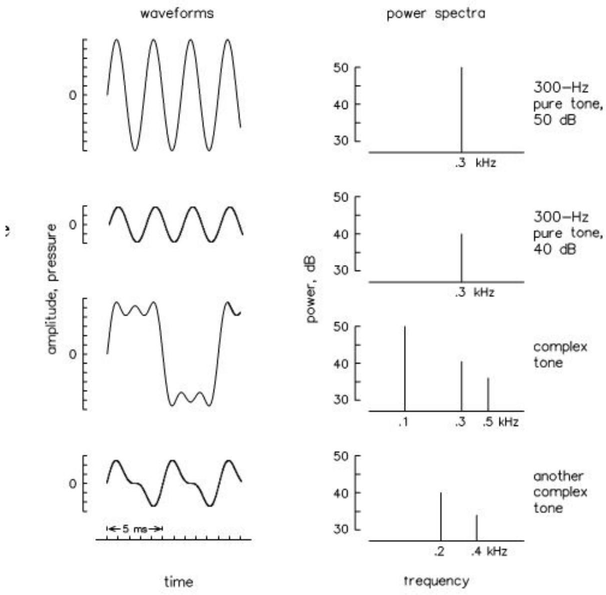


(b)

is called Fourier analysis or frequency analysis. Ohm (1843) suggested that the ear performs a Fourier analysis in his 'acoustical law', and in broad terms he was correct. It is therefore common to represent sounds in terms of their frequency components, and such a representation is called the power spectrum

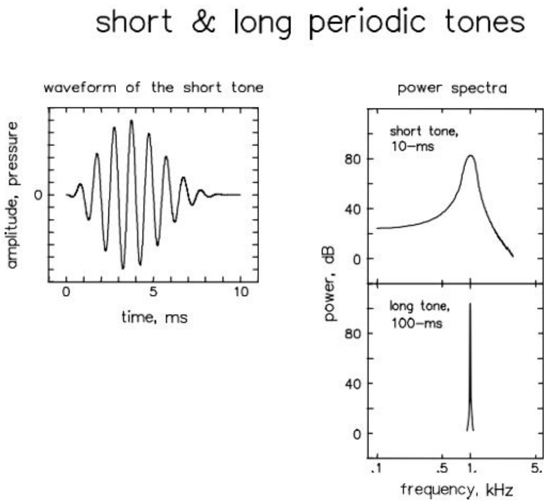
- **Periodic sound**

  Power spectra of simple and complex sounds are shown on the right along with their waveforms. These are called periodic sounds because their waveforms repeat. Their spectra are vertical lines whose left-right position represents the frequency of a component, while the line's height represents the power of a component in dB. When a periodic sound is turned on or off, extra frequency components are added. The power spectra of such sounds contain numerous closely spaced frequency components, so in these cases the dB values are connected together by a continuous line. Spectra of a short and a longer tone are



(c)

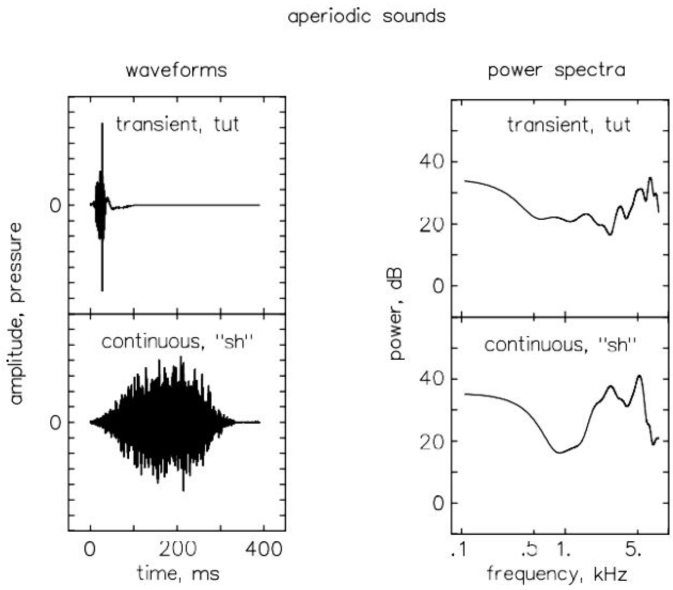
shown above. Notice that the longer tone's continuous spectrum approaches the 'ideal' of a line spectrum. This is increasingly the case as the tone gets longer.



(d)

Aperiodic sounds

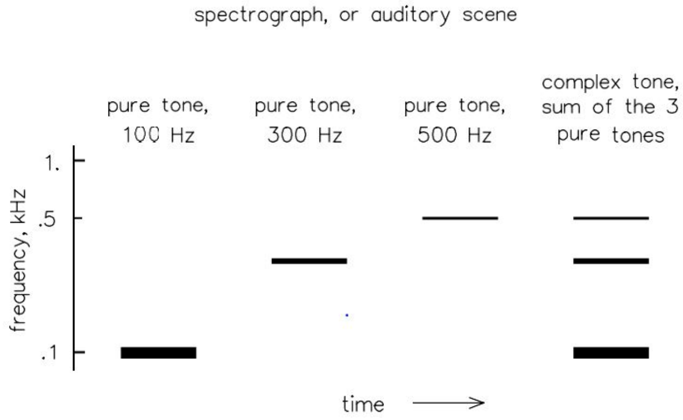
Sounds with waveforms that do not repeat are called aperiodic sounds. Their power spectra are also continuous because they also contain numerous, closely spaced frequency components. Transients are sounds of this type, such as the tut (as in tut- tutting) whose waveform and power spectrum are shown on the right. The other aperiodic sounds are continuous and have amplitudes that fluctuate haphazardly from moment to moment. These sounds have a noise-like quality, such as the spoken "sh" that is also shown here.



(e)

Auditory scenes

The spectra of many everyday sounds tend to change during the sound. To represent this, graphs called 'auditory scenes' or 'spectrographs' are plotted. These show the frequencies of the sound's components on the vertical axis, and time on the horizontal axis. The darkness of the plotted points tries to represent the dB of components. This sort of display is shown below. The spectra of these particular sounds do not actually change with time. However,



(f)

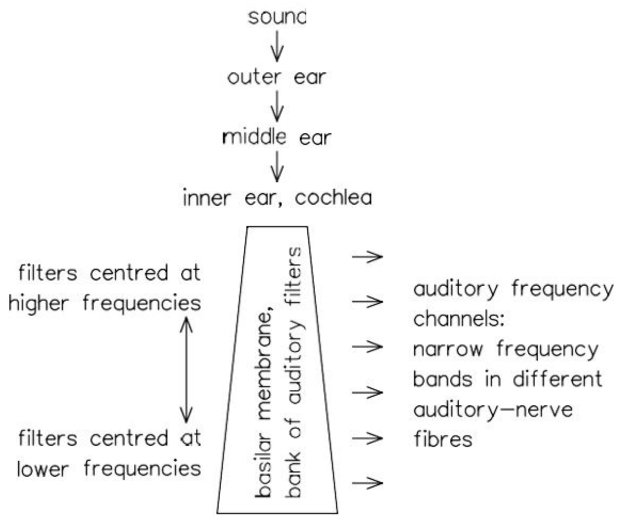
you can compare this type of display with the waveforms and power spectra of these same sounds that have been shown above.

Basilar membrane

The ear gives us a 'display' that resembles an auditory scene, broadly as shown below. The ear provides a Fourier analysis of the sound at each moment of time. The basilar membrane acts like a bank of 'auditory filters'. Each filter picks out a different, narrow band of frequencies from the sound, and sends this information along one of the 25,000 fibres of the auditory nerve. This allocation of components to different fibres is a possible 'place code' for their frequencies. The fibres are like 'frequency channels' that are closely spaced and distributed over the entire range of audible frequencies. There is substantial overlap between the frequency ranges of neighbouring channels.

Auditory-nerve firing

The movement at a particular point on the basilar membrane follows the waveform of a frequency component, or a narrow range of frequency components, in the sound. This causes movement of hair cells which 'fire' if they are displaced by a sufficient amount. The firing of a hair cell causes an impulse to travel down the corresponding auditory-nerve fibre. The firing only happens



(g)

at a particular point in the wave's cycle, as long as the component's frequency is lower than about 5kHz. This 'phase locking' (Rose et al., 1968) gives rise to time intervals between neural impulses that are either the duration of the wave's cycle, or this duration multiplied by an integer (whole number), as shown below. These intervals are therefore closely related to the component's frequency, so there is here a possible 'time code' for frequency.

Individually, the fibres shown do not fire on every cycle of the wave, but, taking their firings together as a group, there is an impulse from at least one of them on each cycle. This is called the 'volley principle' as it allows a group of fibres to fire more rapidly than any one individual fibre.

Frequency resolution

The precision of frequency coding by the ear is called its frequency selectivity, frequency resolving power or frequency tuning. It can be measured in essentially two ways. One method tries to assess activity within a single channel in response to components with different frequencies. This gives a measurement known as the shape of the auditory filter. Narrow filter-shapes indicate good frequency resolution. The other method tries to assess activity across different frequency channels in response to a sound with a single, fixed frequency component. This gives what is called an 'excitation pattern'; whose spread across channels is also narrow if frequency resolution is good. Both of these types of measurement can be made in living breathing people, albeit indirectly. The methods rely on the perceptual phenomenon of masking, which seems largely to be determined by characteristics of the early stages of auditory processing that have been described here.

Analysis and synthesis

Although sounds often have several frequency components, we do not generally experience the components as separate entities when the sound is played. This indicates that the ear's frequency analysis is accompanied by a synthesis of the products of analysis. Auditory phenomena that come about through this synthesis will also be considered in subsequent lectures. In graphs representing sounds it is usual to plot frequency on an axis that has a log (logarithmic) scale. This gives equal frequency ratios an equal spatial separation. Thus, the spacing between 1kHz and .5kHz, a ratio of 2 to 1, is the same as the spacing between 10kHz and 5kHz. Similarly, the 10 to 1 ratio of 1kHz and .1kHz gives these two frequencies the same spacing as that between 10kHz and 1kHz. It is said that a pair of notes in music is separated by a musical interval, and it is the ratio of the frequencies of the two notes that defines the interval. Thus, the ratio 2 to 1 is called an octave. This is divided into 12 equal intervals called semitones. Eight of these 12 steps are the notes of the diatonic scale (the white notes on a piano; doh, re, me, etc.). Four semitones are one third of an octave, which is an interval with a frequency ratio of 1.25 to 1. Decibels are not plotted on a log scale on graphs representing sounds. However, the dB is found by taking the logarithm of a ratio of two sound-pressures. It is thus a logarithmic unit. In determining the dB of a sound, the first step is to find what is called the 'root mean square pressure', p(rms), from the sound's waveform. This is found by considering amplitudes (pressures) at points on the waveform that are closely spaced in time. The square of each value is found, and then the mean (average) of these squared values is taken over a reasonably long interval of time. The square root of this mean is p(rms). The next step is to divide p(rms) by a 'reference pressure'. Different dB scales have different reference pressures so they are relative scales like those for temperature. Thus, different numbers from different dB scales can describe the same sound level. For sensation level, dB(SL), the reference pressure is the p(rms) of a sine wave when it is at threshold for a human listener. In the scientific measurement of sound the reference pressure is a standardised value, 0.00002 Newtons per square metre. The scale based on this universal standard is called sound pressure level, dB(SPL). Finally, p(rms) of the sound is divided by the reference pressure, the log (base 10) of this ratio is taken, and this logarithm is multiplied by 20 to give the dB value

3.5 Audio signal processing for Machine Learning

The fourier transform using complex number well the idea here is the whenever we apply a fourier transform end up with for each frequency for each like puritan is a pair of parameter os one is magnitude and phase and the idea is that we can use this magnitude and phase as polar co ordinates of a complex number is other words can encode both of co efficient magnitude and phase as polar co ordinate of a complex number

- **Complex Fourier Transform co-efficients**

$$\varphi_f = armax_{\varphi\in[0,1)} \left\{ \int s(t).sin(2\pi.(ft-\varphi)dt \right\},$$

$$d_f = max_{\varphi\in[0,1)} \left\{ \int s(t).sin(2\pi.(ft-\varphi)dt \right\},$$

$$c = |c| \cdot e^{i\gamma}.$$

here we have φ is the phase and d_f is magnitude we have the definition of complex number

$$c = |c| \cdot e^{i\gamma}.$$

in polar co-ordinates. We get a fourier transform co-efficient this is the formula basically we can encode with the magnitude as well a phase in a single complex number

$$c_f = \frac{d_f}{\sqrt{2}}.e^{-i2\pi\varphi_f}.$$

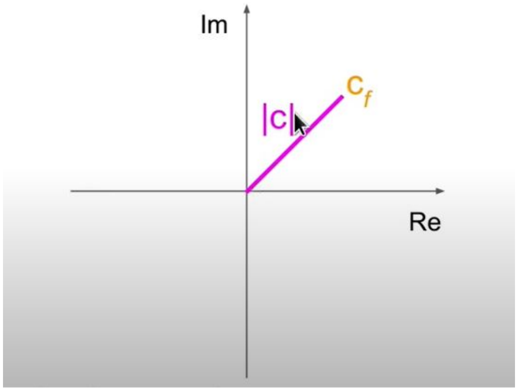
this is called complex fourier transform co-efficient each of the frequency we decompose our original single inside.

$$c_f = \frac{d_f}{\sqrt{2}}.e^{-i2\pi\varphi_f}.$$

we can just like map this a complex co-efficient to this complex definition of a well polar represent of a complex number so the absolute value of c gets mapped to the magnitude divided by square roots 2 why we use square root of 2 because of narmalization and divided by constant γ is the angle over here and γ is equal to 2 $\pi\varphi_f$.

- **Visualization of fourier transform co-efficient**

  c_f are fourier transform co-efficient for frequency f is equal to complex



plane.Distance from complex number from the origin is given the absolute value of co-efficient itself. γ is the angle between the positive real axis and line connect the r coefficient complex number with the origin and γ is $2\pi\varphi_f$ multiply by the phase is between 0 and 1 it means trace the whole circle ẇhen we increase the phase what happend is that we are rotating clockwise so if we increase the phase

- **Continuous Audio signal**

  Continuous $g(t)$ this is function $g : R \to R$ get the real number input and output another real number x axis we have time for each time we get back an amplitude or intensity value this function G is continous audio signal get a real number time as input and it provide us it output a another real number actual sound pressure intensity or amplitude

$$g(f) = c_f.$$

This is stark contrast with thw actual complex fourier transform so that complex fourier transform we can indicate with the g of f and output of this our fourier transform coefficient.

$$g = \mathbb{R} \to \mathbb{C}$$

This function take real number which is frequency and it express in hertz but then it doesnot output a real number but rather it output a complex number and complex number that it output is the fourier transform coefficient.

Complex Fourier transform

Formula for both magnitude and phase come out from the fourier transform

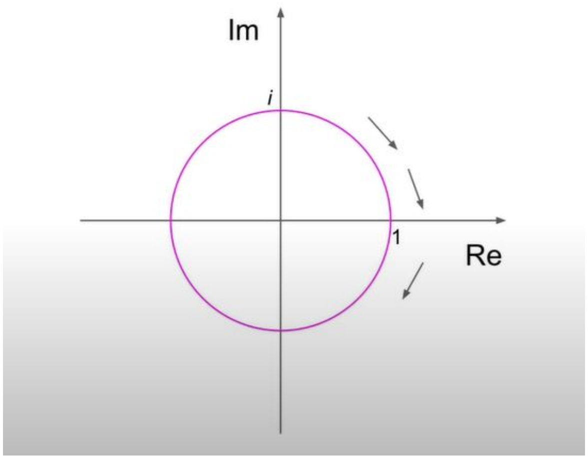
$$\varphi_f = armax_{\varphi\in[0,1)} \left\{ \int s(t).sin(2\pi.(ft-\varphi)dt \right\},$$

$$d_f = max_{\varphi\in[0,1)} \left\{ \int s(t).sin(2\pi.(ft-\varphi)dt \right\},$$

here we have the definition of complex fourier transform

$$\widehat{g}(f) = \int g(t).e^{-2\pi ft}dt.$$

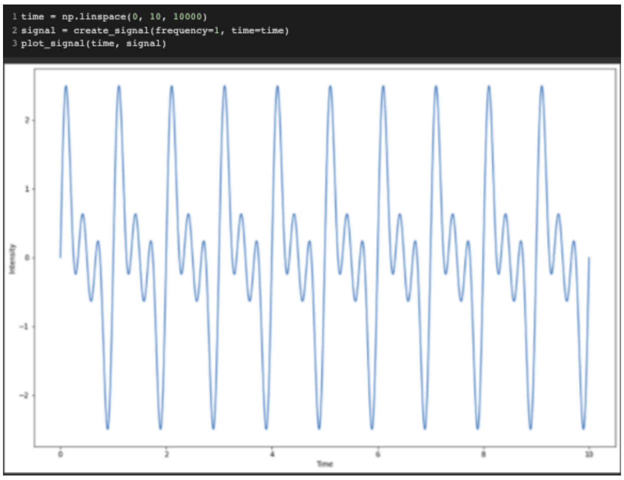
we look into difference parts of above formula $e^{-2\pi ft}$ basically traces the



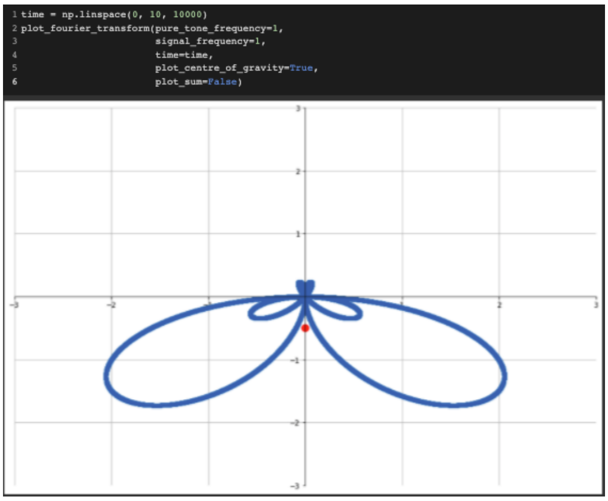
unit circle in the complex plane ft increases trace the whole unit circle but now tracing unit circle going clockwise speed at which we can complete a circle depends on this f value which is the frequency is equal to 1 hertz then it take us to go through to just complete one full circle now frequency is equal to 2 it will take us half of a second that beacause of the periodic represent the sinusoidal a sine wave using complex number

$$\widehat{g}(f) = \int g(t).e^{-2\pi ft}dt.$$

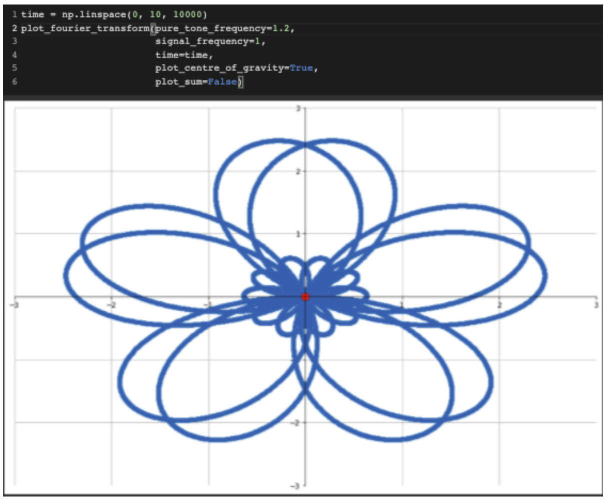
$e^{-2\pi ft}$ decomposing the original complex signal g of f complex fourier transform formula is the signal g of f now we created a custom signal complex fourier transform is the multiplication between our original pure



tone frequency 1 Hz and sine wave and pass the fourier transform



if we change the pure tone frequency 1.2 Hz we getting signals like Eulers



formula $e^{iy} = cos(\gamma) + isin(\gamma)$.

$$\widehat{g}(f) = \int g(t).e^{-2\pi ft}dt.$$

Rewritten the initial complex fourier transform

$$\widehat{g}(f) = \int g(t).e^{-i2\pi ft}dt = \int g(t) - cos(-2\pi ft)dt + i \int g(t).sin(-2\pi ft)dt.$$

$|\widehat{g}(f)|$ magnitude of the fourier transform complex fourier transform co-efficient

$$c_f = \frac{d_f}{\sqrt{2}}.$$

$$d_f = \sqrt{2}.|g(f)|.$$

co-efficient magnitude d_f its basically equal to absolute value of fourier transforms of f multiply square of 2 now interms of the phase we can define by taking γ and angle

$$\varphi = -\frac{\gamma f}{2\pi}.$$

- **Inverse fourier transform**

  The inverse Fourier Transform is also used in image processing, where it is used to convert images from the frequency domain to the spatial domain. This allows for image enhancement and restoration, as well as the removal of noise and other unwanted artifacts.

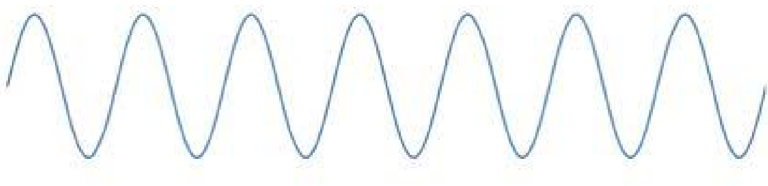
$$g(t) = \int c_f.e^{i2\pi ft}dt.$$

  we just take all the different frequency components we just multiply them by the magnitude and we add the phase them we add them all and we reconstrant the original signal and fourier transform do have the same data

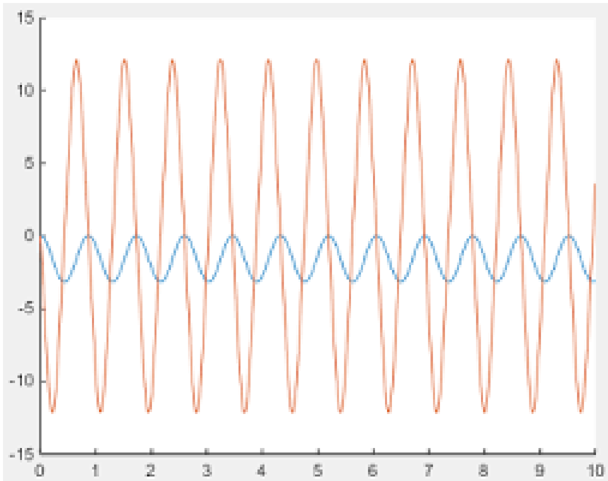
- **Fourier representation**

$$g(f) = \int g(t).e^{i2\pi ft}dt.$$

  where $e^{i2\pi ft}dt$ is pure tone frequency f.

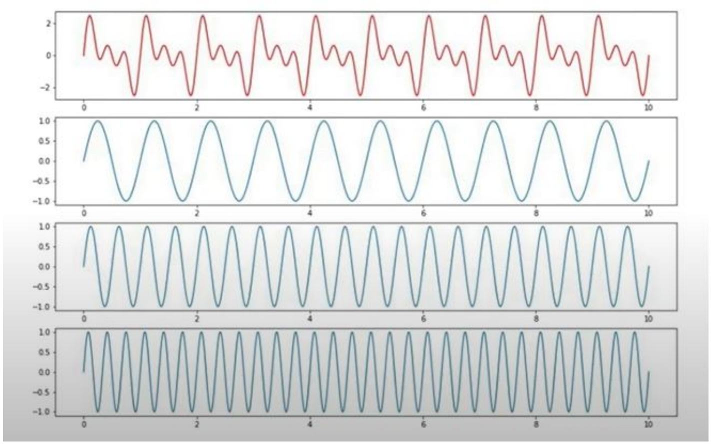


weight pure tone with magnitude and add phase $c_f.e^{i2\pi ft}dt$ add up all(wieght)



sinusoids and get the original sounds

$$g(t) = \int c_f.e^{i2\pi ft}dt.$$



then we add all up we get this original signal fourier transform

$$\widehat{g}(t) = \int g(t).e^{-i2\pi ft}dt.$$

$$g(t) = \int c_f.e^{i2\pi ft}df.$$

this is inverse fourier transform The fourier transform integrating over the time The inverse fourier transform integrating over the frequency Time domain to the frequency domain from the frequency domain to time domain frequency depends on the time domain

4 Challenges of Complex Analysis with Data science

(h)

Integrating complex analysis with data science can provide a powerful framework for analyzing complex-valued data in various domains. However, it comes with its own set of challenges and limitations. Here, we'll discuss some of the key considerations:

Challenges

- **Mathematical Complexity**

  Complex analysis involves advanced mathematical concepts, such as contour integration and residue theorems. This can be challenging for data scientists without a strong background in complex analysis.

- **Computational Intensity**

  Complex operations can be computationally expensive, especially when dealing with large datasets. Efficient algorithms and numerical techniques are crucial for handling complex-valued data in real-time or high-dimensional scenarios.

- **Visualization and Interpretation**

  Visualizing complex-valued data can be challenging, as it requires representing both magnitude and phase information. Finding effective visualization techniques that convey the richness of complex data is non-trivial.

- **Generalization to Higher Dimensions**

  Extending complex analysis to higher dimensions, where data may be represented as hypercomplex numbers (e.g., quaternions), can be even more challenging and less intuitive. Integration with Machine Learning Models: Integrating complex-valued data with standard machine learning models designed for real-valued inputs may require specialized techniques for handling complex-valued features.

  Limitations

- **Domain Specificity**

  Complex analysis may not be applicable to all types of data. It is particularly well-suited for scenarios where phase information or oscillatory behavior is important.

- **Availability of Data**

  In some cases, obtaining complex-valued data may be more challenging or expensive compared to real-valued data. This limitation can constrain the applicability of complex analysis techniques.

- **Interpretability of Results**

  Interpreting results ftextrom complex analysis may be more challenging for stakeholders who are not familiar with the mathematical underpinnings. Providing intuitive explanations can be crucial.

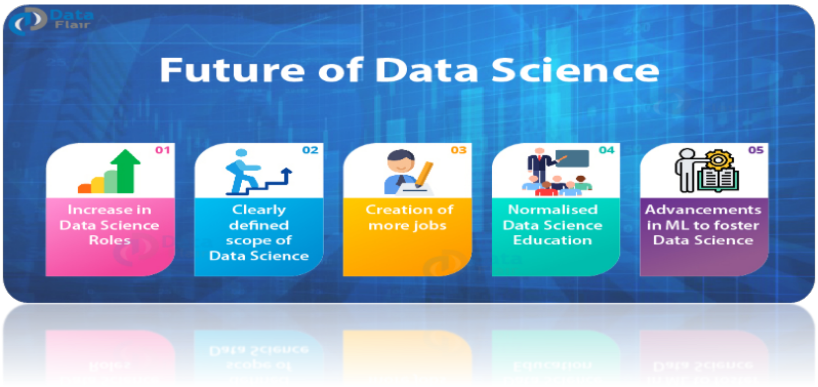
- **Data Preprocessing and Cleaning**

  Preprocessing techniques for complex-valued data may differ from those used for real-valued data. Handling noise, outliers, and missing values in complex data requires specialized methods.

- **Availability of Tools and Libraries**

  While there are libraries for complex arithmetic and basic operations, there may be a lack of specialized tools and libraries for complex analysis in popular data science frameworks.

5 Future Advancement of Complex Analysis in Data science



(i)

The integration of complex analysis with data science is an interdisciplinary field that holds great promise for advancing our understanding and analysis of complex-valued data. Here are some speculative insights into potential future advancements in this area

- **Enhanced Visualization Techniques**

  Future advancements may lead to the development of more sophisticated visualization techniques tailored for complex-valued data. This could include immersive 3D visualizations that allow researchers to explore the intricate patterns and structures in complex datasets.

- **Quantum-Inspired Data Processing**

  As quantum computing technology advances, there may be opportunities to leverage complex analysis techniques in quantum-inspired algorithms for data processing. This could lead to exponential improvements in the analysis of complex-valued data.

- **Comtextplex-Valued Deep Learning Architecture**

  Researchers may explore the development of specialized deep learning architectures designed to handle complex-valued inputs, weights, and activations. These architectures could unlock new capabilities for processing and understanding complex data.

- **Applications in Quantum Machine Learning**

  Advancements in complex analysis may play a pivotal role in the emerging field of quantum machine learning. Techniques from complex analysis could be instrumental in developing algorithms that harness the power of quantum computing for complex data analysis tasks.

- **Real-Time Processing of Complex Data**

  Future advancements may lead to more efficient algorithms for real-time processing of complex-valued data. This could have profound implications for applications in areas such as telecommunications, radar systems, and medical imaging.

- **Hypercomplex Data Analysis**

  Researchers may explore extensions of complex analysis to hypercomplex numbers, such as quaternions or octonions. This could open up new avenues for analyzing data in higher-dimensional spaces.

- **Complex-Valued Optimization**

  Advancements in complex optimization techniques may lead to more robust and efficient algorithms for solving complex-valued optimization problems. This could have wide-ranging applications in areas like signal processing and control systems.

- **Explainable AI for Complex Data**

  Future research may focus on developing techniques for explaining the decisions made by complex-valued machine learning models. This could be crucial for building trust in applications where interpretability is essential.

- **Interdisciplinary Collaborations**

  Collaborations between mathematicians specializing in complex analysis and data scientists from various domains may become more prevalent. This could lead to innovative solutions and breakthroughs in the analysis of complex-valued data.

- **Ethical Considerations**

  As complex analysis techniques become more integrated into data science, ethical considerations related to the responsible use of complex-valued data and the potential biases in complex models may become more prominent.

These speculations represent potential directions for advancements in the interdisciplinary area of complex analysis in data science. As technology and knowledge continue to evolve, these advancements could lead to transformative breakthroughs in our ability to analyze and understand complex-valued data.

Conclusion

In conclusion, the application of complex analysis in this data science project has proven to be invaluable in gaining deeper insights into the underlying patterns and relationships within the dataset. Through the utilization of techniques such as contour integration, Cauchy's theorem, and residue theory, we were able to effectively analyze complex-valued functions and extract meaningful information. By leveraging the power of complex analysis, we tackled challenges that would have been difficult to address using real-valued methods alone. This allowed us to uncover hidden structures and uncover nuances that might have otherwise remained obscured. The ability to work in the complex plane provided a richer framework for understanding the behavior of our data, particularly in domains where oscillatory or cyclical phenomena play a significant role. Furthermore, the integration of complex analysis into our data science toolkit has not only enhanced the accuracy and robustness of our models but has also facilitated a more comprehensive interpretation of our results. This approach has opened up new avenues for research and innovation, allowing us to make more informed decisions and drive meaningful impact in our domain.

In future endeavors, the incorporation of complex analysis will undoubtedly continue to be a valuable asset, especially in scenarios involving time-series data, signal processing, and any other domain characterized by oscillatory behavior. Its versatility and efficacy make it an indispensable tool in our data science toolkit, enabling us to push the boundaries of what is achievable in our pursuit of understanding and leveraging complex datasets.

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