Integration of 1-forms on Graphs

Introduction

Given a connected graph $G = (V, E)$ with V vertices, and E edges and a 1-form $v : E \to \mathbb{R}^D$. We're looking for the 0-form $x : V \to \mathbb{R}^D$ minimizing the error:

$$E(x) = \sum_{(i,j)\in E} ||dx_{ij} - v_{ij}||^2$$

where $dx : E \to \mathbb{R}^D$ is the differential of x.

Obs. 1: It is sufficient to consider the case $D = 1$, because the problem can be reduced to solving D independent 1-D problems on the same graph for each coordinate.

Obs. 2 (Abuses of notation): We use V to denote both the set of vertices and the total number of vertices of the graph. And we denote v to the 1-form and v_i is the $i-th$ vertex.

Obs. 3: Vertices and edges can be enumerated in many different ways. We will enumerate vertices and edges according to a traversal order of a spanning tree. This process determine a consistent orientation of the edges: an edge will be oriented from a lower vertex to a greater vertex:

$$e_{ij} : v_i \to v_j \ (for\ i < j)$$

Integration process

The proposed process for integrating v is as follows:

1. Construct a spanning tree T for G
2. Choose an order of traversal for T and orient the edges from parent to child
3. Enumerate the vertices and tree edges according to the order of traversal (See figure 1)
4. We will refer to the remaining graph edges (those which do not belong to the tree) as "loop-edges". Orient the loop-edges from parent to child.
5. Construct the oriented incidence matrix D: this is an $E \times V$ sparse matrix with one row per graph edge and one column per vertex. For each edge e_{ij} $(i < j)$ we set the $i-th$ column entry to -1 and the $j-th$ column to 1 (See figure 2). Rows representing tree-edges are ordered such that the $N-1 \times N-1$ upper-right submatrix is lower triangular.

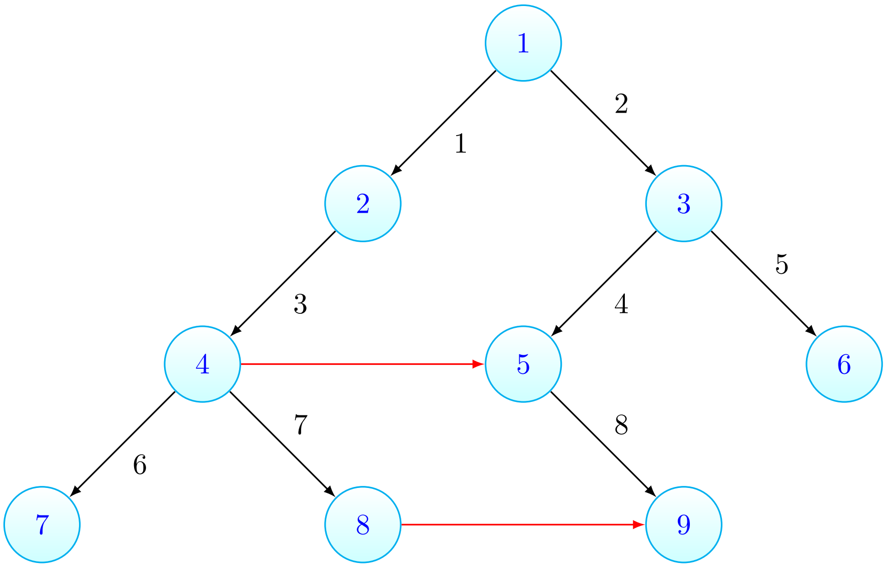

Figure 1: BFS traversal of a spanning tree. Nodes and edges are enumerated according to tree traversal. Edges are oriented from parent to child. Red edges correspond to $loop - edges$.

$$
\left(\begin{array}{c|cccccccc}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{array}\right)
$$

Figure 2: Incidence matrix correspondig to graph in figure 1. The first column corresponds to the tree root, it can be dropped. Green and blue submatrixes correspond to tree-edges and loop-edges, respectively. The tree-edge submatrix is lower triangular.

Properties of the oriented incidence matrix D and other facts

Dropping the root column of D

We can rewrite the minimization problem in matrix form as follows:

$$E(x) = \sum_{(i,j)\in E} ||dx_{ij} - v_{ij}||^2 = ||Dx - v||^2$$

Note that as we are dealing with an integration problem, the solutions are equivalent *modulo a translation*. The consequence of this fact is that we can fix the value of the tree root (ie. $x_1 = 0$). Thus we can drop the first column of D. (See figure 2)

Relation to the *Laplacian* matrix and rank of D

The directed incidence matrix has the following property:

$$L = D^t D$$

where L is the *Laplacian* matrix of G. The rank of L is: $n - c$ (where c is the number of connected components of G). In our case, as G is connected, the rank of L (and consequently the rank of D) is $n - 1$.

The problem is equivalent to solve a linear system

To solve the minimizatiom problem is equivalent to find where the gradient of $E(x)$ vanishes:

$$\nabla E = [\frac{\partial E}{\partial x_1}, \ldots, \frac{\partial E}{\partial x_n}] = D^t Dx - D^t v = 0$$

It is equivalent to solve the linear system:

$$D^t Dx = D^t v$$

As the rank of D is $n - 1$, the linear system may not have a solution. But solving the problem by iterative methods will converge to the closest "possible" guess.

The full rank matrix M based on D

A full rank $E \times E$ matrix can be constructed based on D. We drop the first column of D (as pointed before) to obtain D' (a $E \times (V-1)$ matrix). Suppose that we rewrite D' as follows:

$$D' = \begin{bmatrix} T \\ L \end{bmatrix} \tag{1}$$

Where the rows of T correspond to tree edges and the rows of L to loop-edges. We can append $E - V + 1$ linearly independent columns spannning the orthogonal complement of D':

$$D' = \begin{bmatrix} TB \\ LI \end{bmatrix} \tag{2}$$

Where I is the indentity matrix and the following orthogonality condition is satisfied:

$$0 = \begin{bmatrix} T \\ L \end{bmatrix}^t \begin{bmatrix} B \\ I \end{bmatrix} \tag{3}$$

This implies that

$$B = -(L)^{-t} A^t$$

TODO: Tengo que entender la matriz B del apunte de Gabriel.

Otras partes:

- Descripción del problema de Poisson y Laplace - Mostrar ejemplo simple de que la elección del árbol generador es fundamental - Problema combinatorio de elegir mejor árbol generador. Es NP? Se puede atacar con métodos exctos y heurísticos?