Probabilistic Robotics Course

Gaussian Distribution

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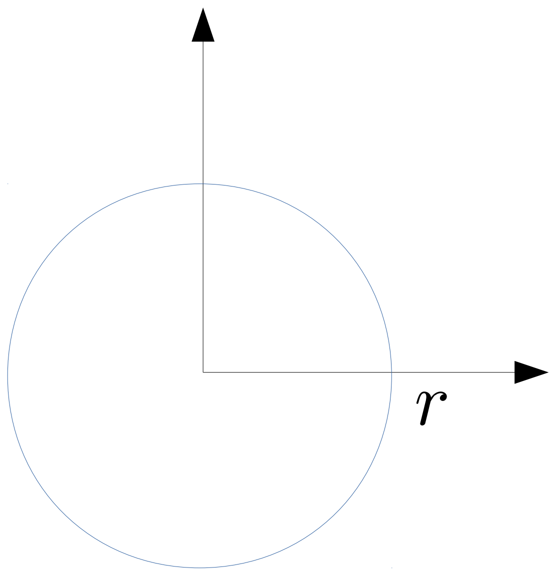
Outline

- Drawing Ellipses
- Parametrizations
- Drawing Gaussians
- Classical Parametrization
  - Marginalization
  - Conditioning
  - Chain Rule
  - Affine Functions
  - Quasi-Affine Functions

Circles

A circle looks like that

$$x^2 + y^2 = r^2$$

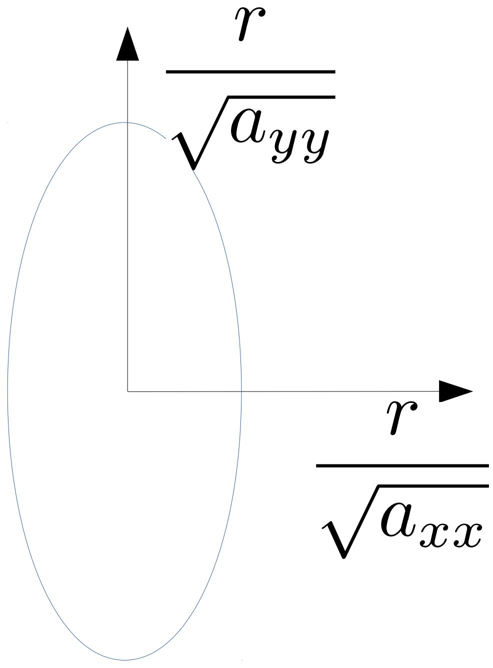


it is a slice of a paraboloid

Scaled Circles

A scaled circle like that

$$a_{xx}x^2 + a_{yy}y^2 = r^2$$



Slanted Circles

A slanted scaled circle like that

$$a_{xx}x^2 + a_{xy}xy + a_{yy}y^2 = r^2$$

that can be rewritten as:

$$\begin{pmatrix} x & y \end{pmatrix} \underbrace{\begin{pmatrix} a_{xx} & \frac{a_{xy}}{2} \\ \frac{a_{xy}}{2} & a_{yy} \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x \\ y \end{pmatrix} = r^2$$

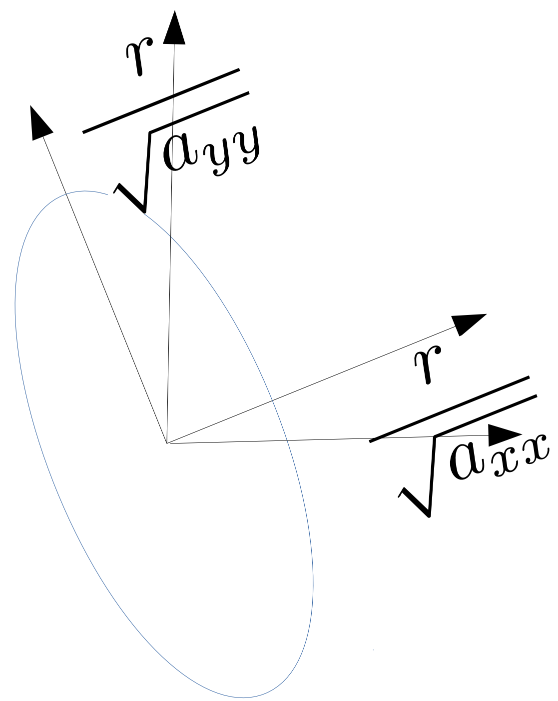
The matrix **A** admits an eigenvalue decomposition

$$\mathbf{A} = \mathbf{R}^T \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \mathbf{R}$$

Slanted Circles (cont)

All this because of a_{xy}

The off diagonal components "rotate" the ellipsoid

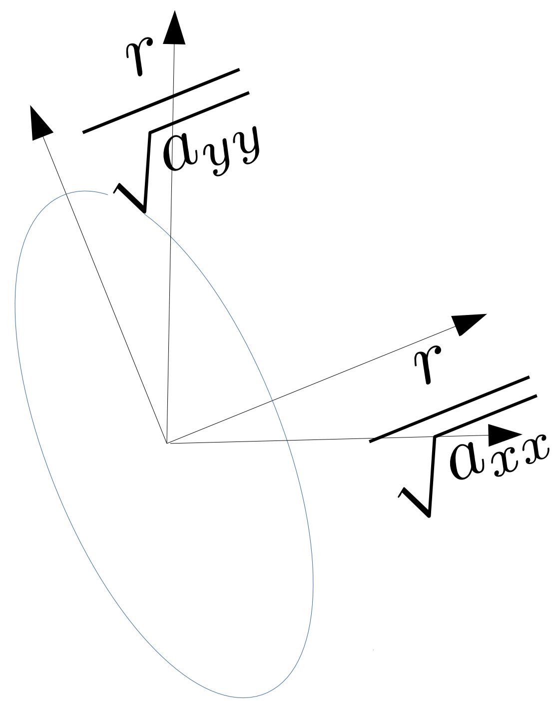


Who is guilty for the rotation?

Breaking News

Ellipses can also be translated

$$\mathbf{A} = \left[\mathbf{R}\begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix}\right]^T \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \left[\mathbf{R}\begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix}\right] = r^2$$



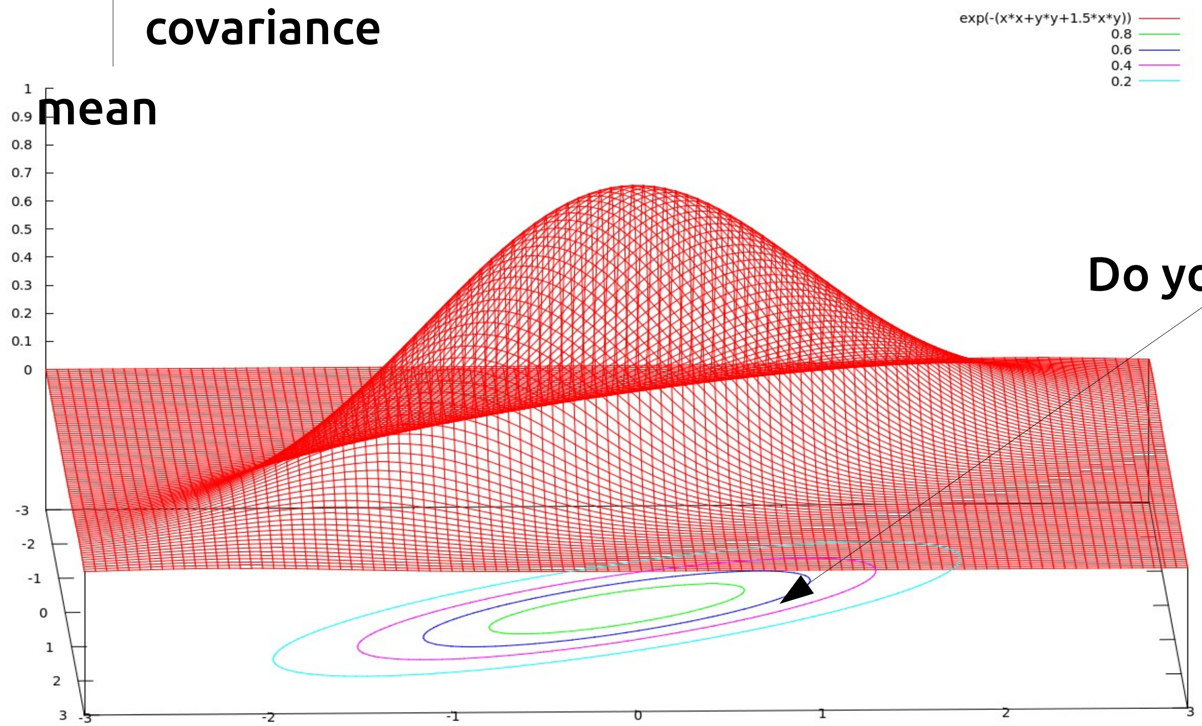
Gaussian

The pdf of a Gaussian distribution has the following form

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mu, \mathbf{\Sigma}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \mathbf{\Sigma}}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right)$$

covariance

mean



Do you see the ellipse???

Why Gaussians are Cool

Gaussian distributions are closed under

- sum
- affine transformation (Ax+b)
- conditioning
- marginalization

This means that in order to implement the above operations, one only needs to compute the **parameters** of the result, from the parameters of the input

Moment Parametrization

The one seen in the previous slide is known as moment parameterization

The parameters can be calculated from a (large) set of samples as

$$\mu = \frac{1}{N} \sum \mathbf{x}^{(i)}$$

$$\boldsymbol{\Sigma} = \frac{1}{N} \sum (\mathbf{x}^{(i)} - \mu)(\mathbf{x}^{(i)} - \mu)^T$$

Moment Parametrization

The parameters are defined as the 1st and 2nd order moments of the distribution

$$\mu = \int_{\Omega} \mathbf{x}\, p(\mathbf{x}) d\mathbf{x} = \mathbb{E}[\mathbf{x}]$$

$$\boldsymbol{\Sigma} = \int_{\Omega} (\mathbf{x} - \mu)(\mathbf{x} - \mu)^T\, p(\mathbf{x}) d\mathbf{x} = \mathbb{E}[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^T]$$

Canonical Parametrization

Another parametrization is the so called canonical, useful for conditioning

$$\begin{aligned} \nu &= \mathbf{\Sigma}^{-1}\mu \quad \text{information vector} \\ \mathbf{\Omega} &= \mathbf{\Sigma}^{-1} \quad \text{information matrix} \end{aligned}$$

$$\mathcal{N}^{-1}(\mathbf{x};\nu,\mathbf{\Omega}) = \frac{\exp\left(\frac{1}{2}\nu^T\mathbf{\Omega}^{-1}\nu\right)\sqrt{\det\mathbf{\Omega}}}{(2\pi)^{n/2}}\exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{\Omega}\mathbf{x}+\mathbf{x}^T\nu\right)$$

Partitioned Gaussian Densities

The space can be split in two subspaces

The density is over a joint distribution

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix} \quad \mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix} \quad \nu = \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix}$$

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{aa} & \mathbf{\Sigma}_{ab} \\ \mathbf{\Sigma}_{ba} & \mathbf{\Sigma}_{bb} \end{pmatrix} \quad \mathbf{\Omega} = \begin{pmatrix} \mathbf{\Omega}_{aa} & \mathbf{\Omega}_{ab} \\ \mathbf{\Omega}_{ba} & \mathbf{\Omega}_{bb} \end{pmatrix}$$

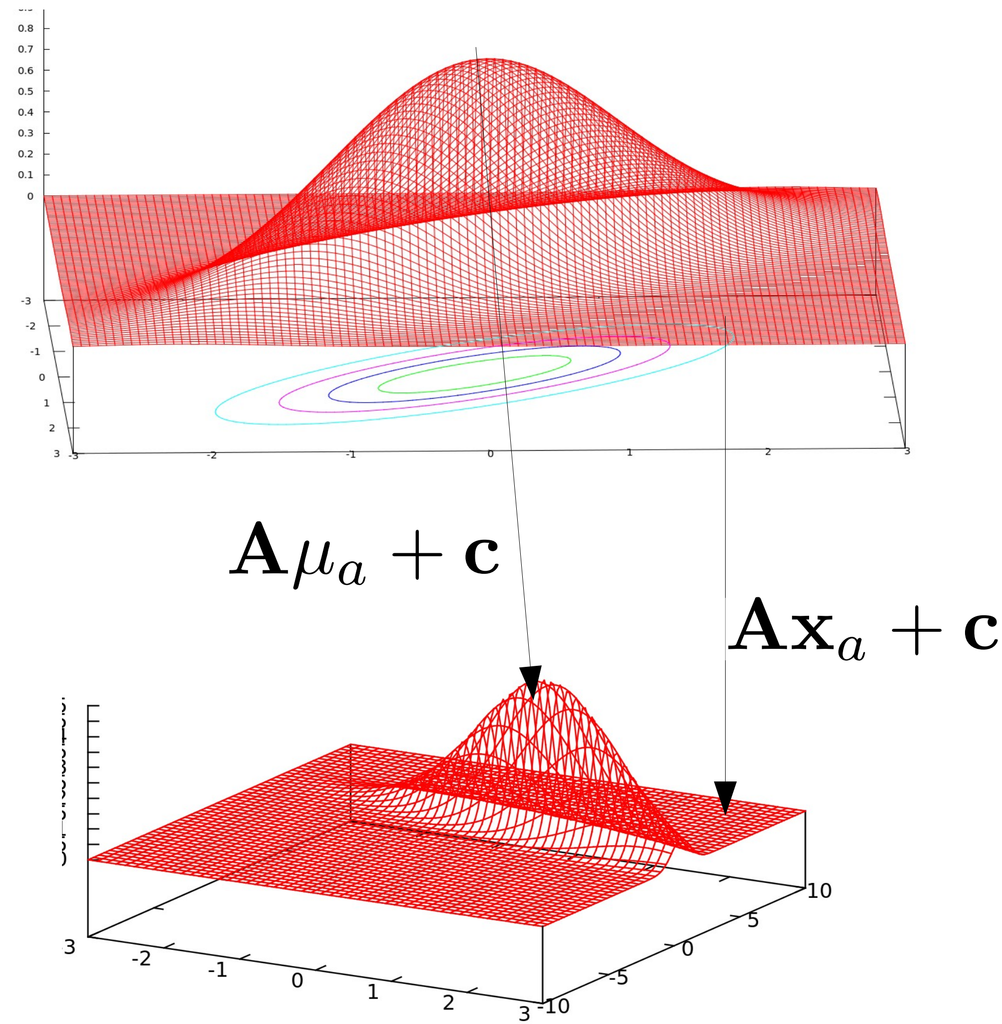
Affine Transformation

Let $\mathbf{x}_a$ be a Gaussian random variable such that

$$\mathbf{x}_a \sim \mathcal{N}(\mathbf{x}_a, \mu_a, \mathbf{\Sigma}_a)$$

Let $\mathbf{x}_b = \mathbf{f}(\mathbf{x}_a) = \mathbf{A}\mathbf{x}_a + \mathbf{c}$

an affine transformation of $\mathbf{x}_a$



$\mathbf{x}_b$ is Gaussian:

$$p(\mathbf{x}_b) = \mathcal{N}(\mathbf{x}_b; \mu_b, \mathbf{\Sigma}_b)$$

The parameters are

$$\mu_b = \mathbf{A}\mu + \mathbf{c} \qquad \mathbf{\Sigma}_b = \mathbf{A}\mathbf{\Sigma}\mathbf{A}^T$$

Taylor Expansion

For non-linear transformations, we can approximate the function around a linearization point.

$$\begin{aligned}\mathbf{f}(\mathbf{x}) &\simeq \mathbf{f}(\mathbf{x}_0) + \underbrace{\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}}}_{\mathbf{A}}(\mathbf{x} - \mathbf{x}_0) \\ &= \mathbf{A}\mathbf{x} + \underbrace{\mathbf{f}(\mathbf{x}_0) - \mathbf{A}\mathbf{x}_0}_{\mathbf{b}}\end{aligned}$$

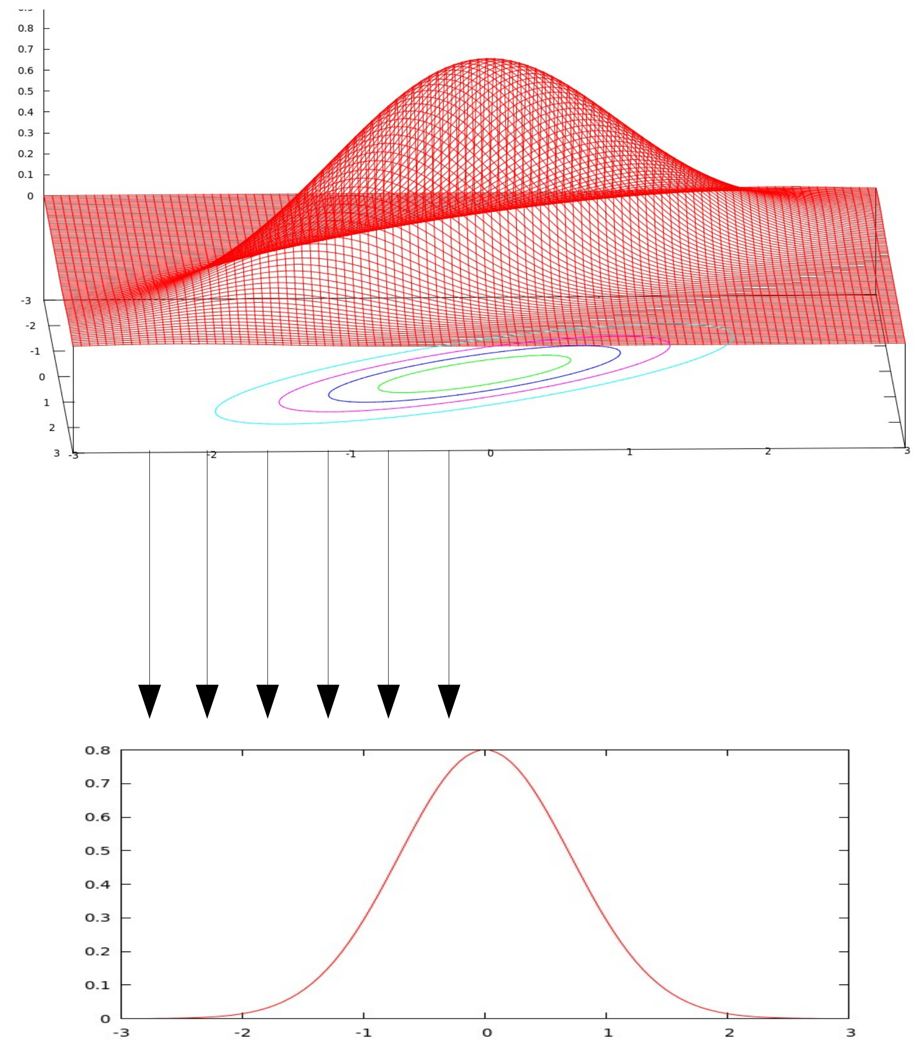
- This reduces the transformation to an affine transform

- The approximation holds only around a linearization point.
- The farther f is from being linear, the worse the approximation

Marginalization

Let $\mathbf{x}^T = (\mathbf{x}_a^T \ \mathbf{x}_b^T)$ be a Gaussian random variable such that

$$\mathbf{x} \sim \mathcal{N}(\mathbf{x}, \mu, \mathbf{\Sigma})$$



The marginal

$$p(\mathbf{x}_a) = \int_{\mathbf{x}_b} p(\mathbf{x}_a, \mathbf{x}_b) d\mathbf{x}_b$$

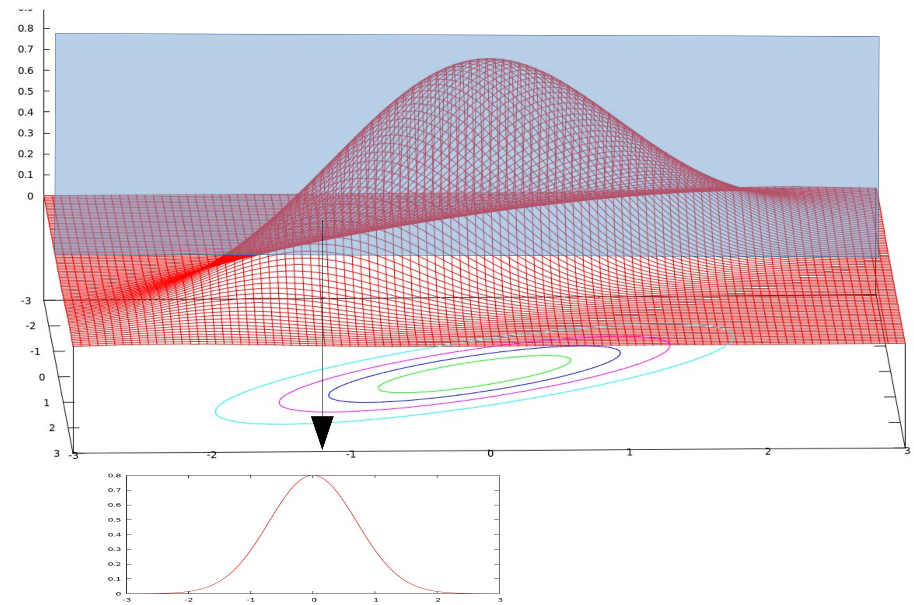
is Gaussian with parameters

$$p(\mathbf{x}_a) = \mathcal{N}(\mathbf{x}_a; \mu_a, \mathbf{\Sigma}_a)$$

Conditioning

Let $\mathbf{x}^T = (\mathbf{x}_a^T \ \mathbf{x}_b^T)$ be a Gaussian random variable such that

$$\mathbf{x} \sim \mathcal{N}(\mathbf{x}, \mu, \boldsymbol{\Sigma})$$



The conditional

$$p(\mathbf{x}_a \mid \mathbf{x}_b) = \frac{p(\mathbf{x}_a, \mathbf{x}_b)}{\int_{\mathbf{x}_a} p(\mathbf{x}_a, \mathbf{x}_b) d\mathbf{x}_a}$$

is Gaussian with parameters

$$p(\mathbf{x}_a) = \mathcal{N}(\mathbf{x}_a; \mu_{a|b}, \boldsymbol{\Sigma}_{a|b})$$

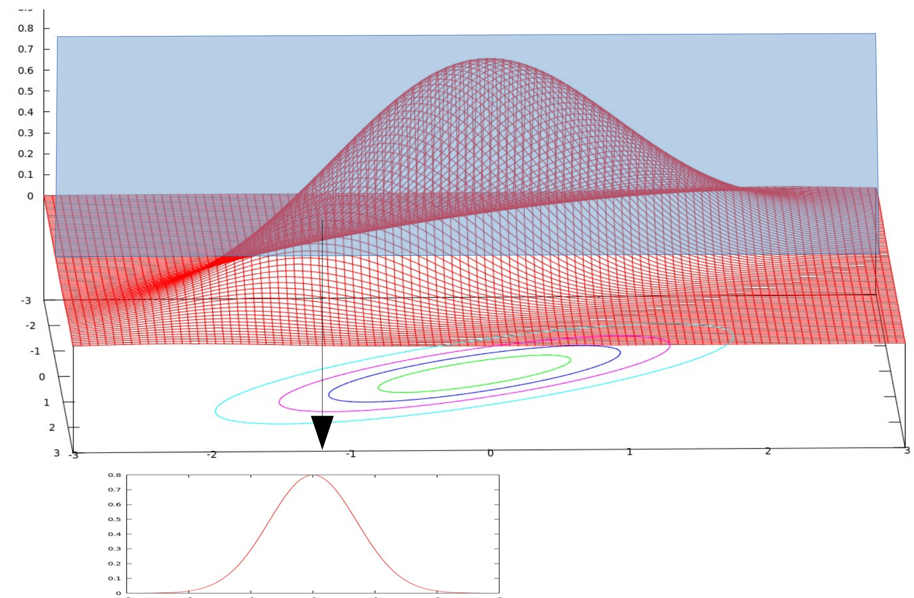
$$\mu_{a|b} = \mu_a + \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}(\mathbf{x}_b - \mu_b)$$

$$\boldsymbol{\Sigma}_{a|b} = \boldsymbol{\Sigma}_{aa} - \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}\boldsymbol{\Sigma}_{ba}$$

Conditioning (2)

Let $\mathbf{x}^T = (\mathbf{x}_a^T \ \mathbf{x}_b^T)$ be a Gaussian random variable such that

$$\mathbf{x} \sim \mathcal{N}^{-1}(\mathbf{x}, \nu, \mathbf{\Omega})$$



The conditional

$$p(\mathbf{x}_a \mid \mathbf{x}_b)$$

is Gaussian with parameters

$$\nu_{a|b} = \nu_a - \mathbf{\Omega}_{ab}\mathbf{x}_b$$

$$\mathbf{\Omega}_{a|b} = \mathbf{\Omega}_{aa}$$

$$p(\mathbf{x}_a) = \mathcal{N}^{-1}(\mathbf{x}_a; \nu_{a|b}, \mathbf{\Omega}_{a|b})$$

Chain Rule

We know

$$p(\mathbf{x}_a) = \mathcal{N}(\mathbf{x}_a; \mu_a, \mathbf{\Sigma}_a).$$

$$p(\mathbf{x}_b|\mathbf{x}_a) = \mathcal{N}(\mathbf{x}_b; \underbrace{\mathbf{A}\mathbf{x}_a + \mathbf{c}}_{\mu_{b|a}}, \mathbf{\Sigma}_{b|a})$$

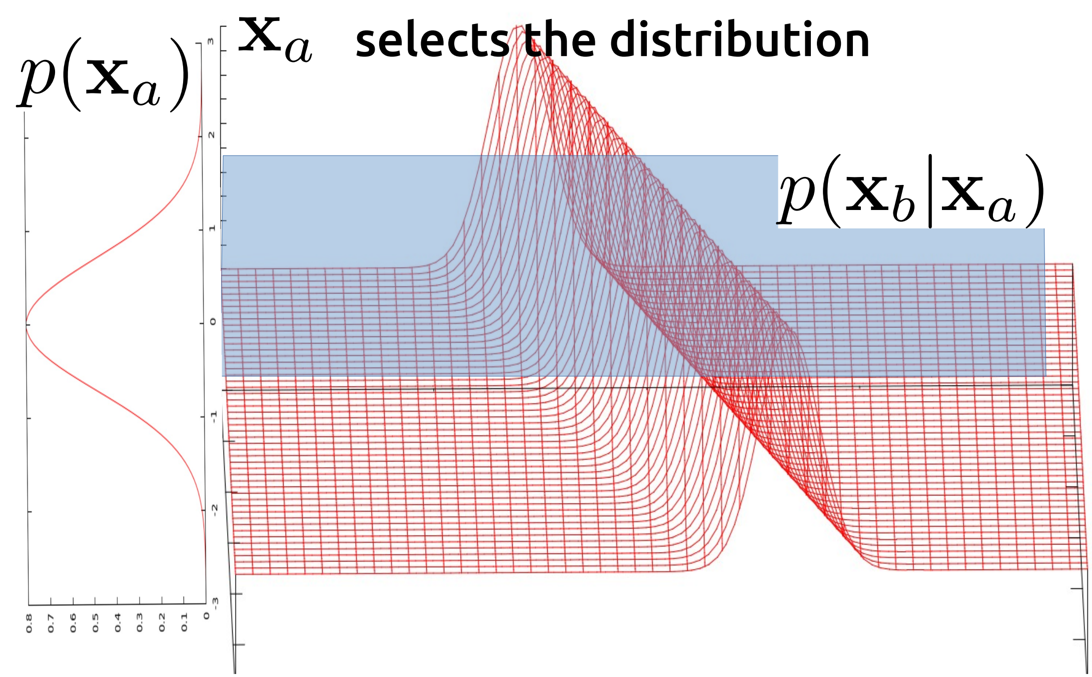
We want to compute

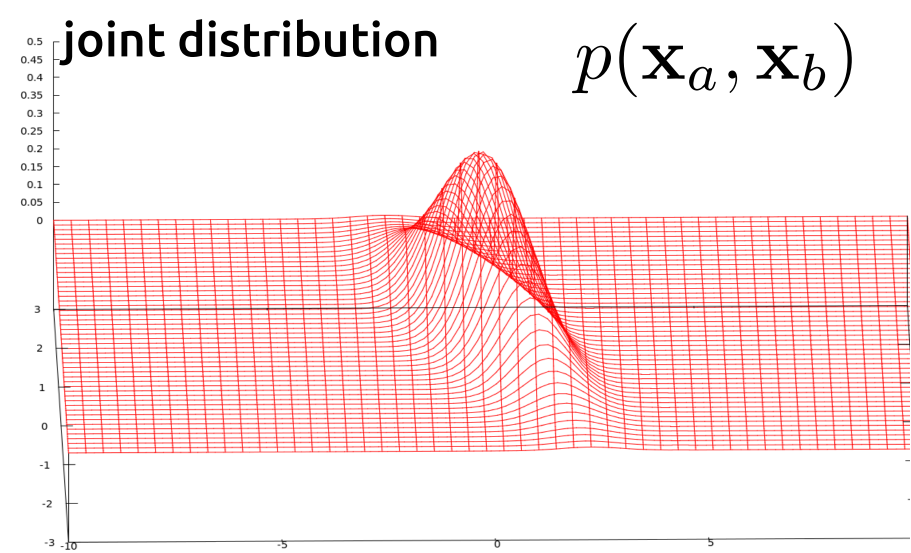
$$p(\mathbf{x}_a, \mathbf{x}_b) = \mathcal{N}\left(\mathbf{x}_{a,b}; \mu_{a,b}, \mathbf{\Sigma}_{a,b}\right)$$

The parameters are

$$\mu_{a,b} = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix} = \begin{pmatrix} \mu_a \\ \mathbf{A}\mu_a + \mathbf{c} \end{pmatrix}$$

$$\mathbf{\Sigma}_{a,b} = \begin{pmatrix} \mathbf{\Sigma}_a & \mathbf{\Sigma}_a \mathbf{A}^T \\ \mathbf{A}\mathbf{\Sigma}_a & \mathbf{\Sigma}_{b|a} + \mathbf{A}\mathbf{\Sigma}_a\mathbf{A}^T \end{pmatrix}$$



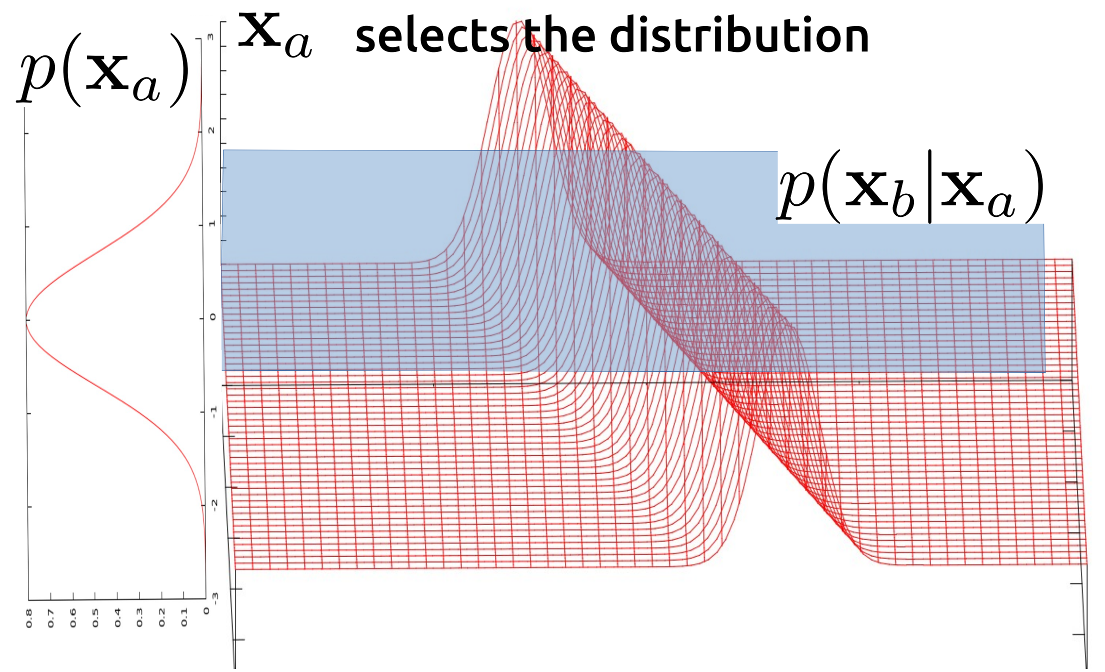


Chain Rule (2)

We know

$$p(\mathbf{x}_a) = \mathcal{N}(\mathbf{x}_a; \mu_a, \mathbf{\Sigma}_a).$$

$$p(\mathbf{x}_b|\mathbf{x}_a) = \mathcal{N}(\mathbf{x}_b; \mathbf{A}\mathbf{x}_a + \mathbf{c}, \mathbf{\Sigma}_{b|a})$$



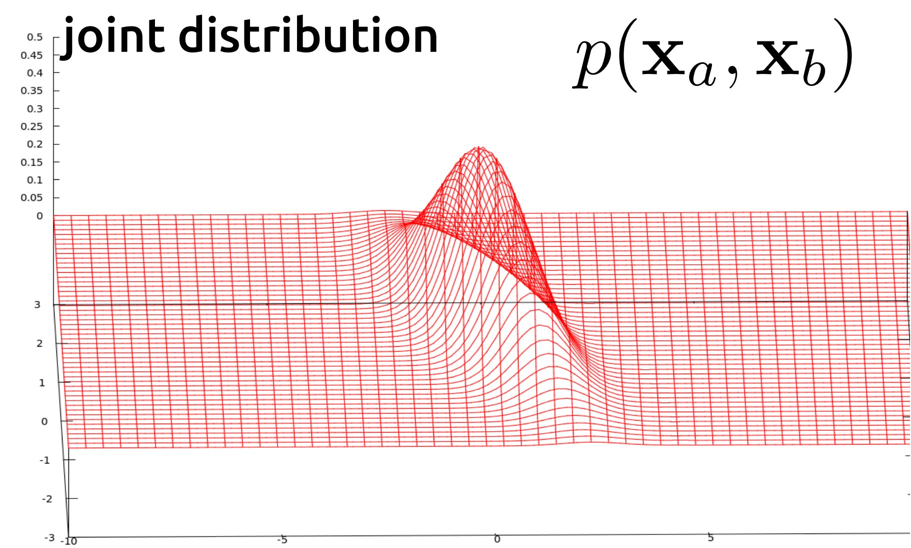
We want to compute

$$p(\mathbf{x}_a, \mathbf{x}_b) = \mathcal{N}\left(\mathbf{x}_{a,b}; \mu_{a,b}, \mathbf{\Sigma}_{a,b}\right)$$

The parameters are

$$\mu_{a,b} = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix} = \begin{pmatrix} \mu_a \\ \mathbf{A}\mu_a + \mathbf{c} \end{pmatrix}$$

$$\mathbf{\Omega}_{a,b} = \begin{pmatrix} \mathbf{A}^T\mathbf{\Omega}_{b|a}\mathbf{A} + \mathbf{\Omega}_a & -\mathbf{A}^T\mathbf{\Omega}_{b|a} \\ -\mathbf{\Omega}_{b|a}\mathbf{A}^T & \mathbf{\Omega}_{b|a} \end{pmatrix}$$



References

Further detailsare here (warmly recommended)

- *Thomas Schoen*, On Manipulating the Multivariate Gaussian Density