Robotics 1

Trajectory planning

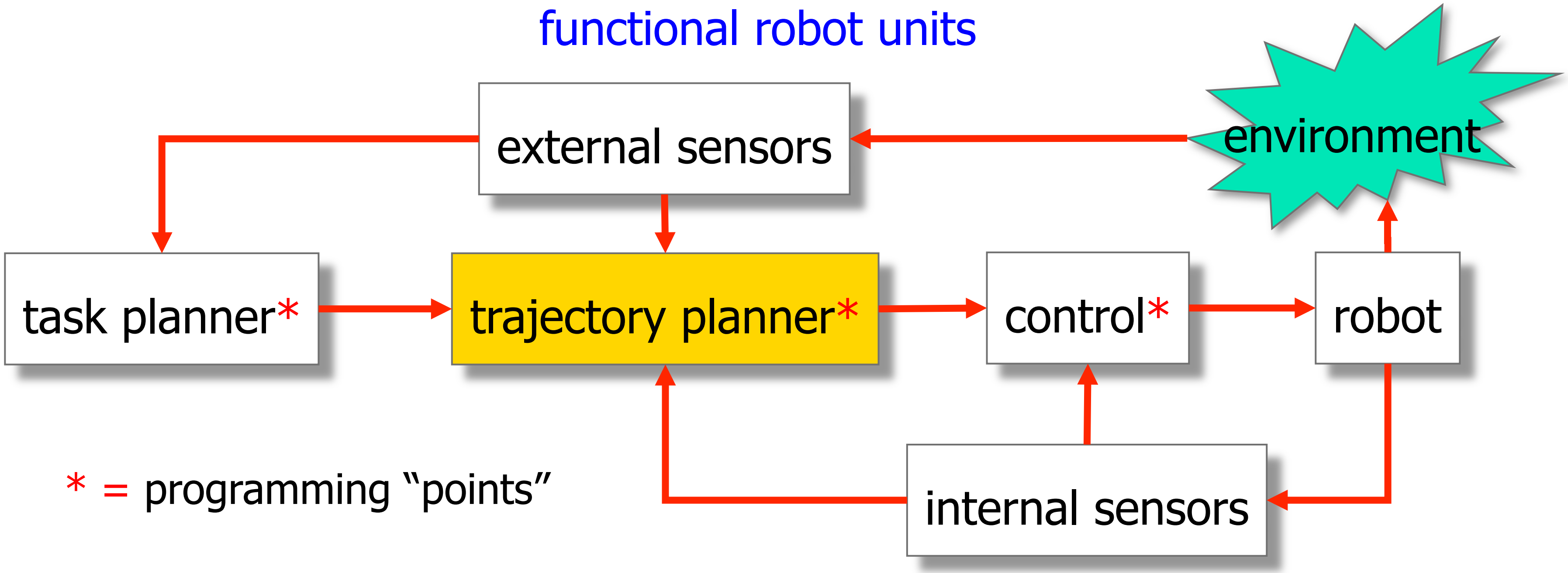
Prof. Alessandro De Luca

Dipartimento di Ingegneria Informatica Automatica e Gestionale Antonio Ruberti

Sapienza Università di Roma

Trajectory planner interfaces

functional robot units



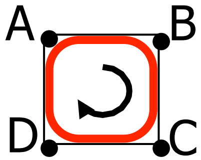
robot action described as a sequence of poses or configurations (with possible exchange of contact forces) → **TRAJECTORY PLANNER** → reference profile/values (continuous or discrete) for the robot controller

Trajectory definition
a standard procedure for industrial robots

1. define Cartesian pose points (position+orientation) using the teach-box
2. program an (average) velocity between these points, as a 0-100% of a maximum system value (different for Cartesian- and joint-space motion)
3. linear interpolation in the joint space between points sampled from the built trajectory

examples of additional features

a) over-fly



b) sensor-driven STOP

c) circular path through 3 points

main drawbacks

- semi-manual programming (as in "first generation" robot languages)
- limited visualization of motion

➡ a mathematical formalization of trajectories is useful/needed

From task to trajectory

TRAJECTORY — of motion $p_d(t)$; of interaction $F_d(t)$

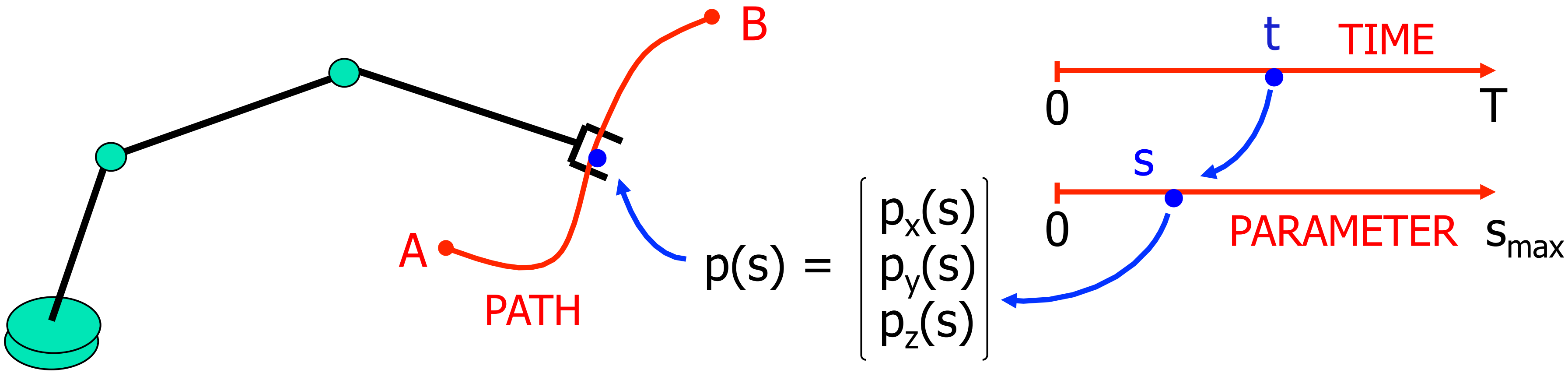
=

GEOMETRIC PATH — parameterized by s: $p=p(s)$ (e.g., s is the arc length)

+

TIMING LAW — describes the time evolution of $s=s(t)$

$\Rightarrow p(s(t))$



example: TASK planner provides A, B
TRAJECTORY planner generates p(t)

Trajectory planning
operative sequence

TASK planning

1

- sequence of pose points ("knots") in Cartesian space

  interpolation in Cartesian space

- Cartesian geometric path (position + orientation): $p = p(s)$

  path sampling and kinematic inversion

2

- sequence of "knots" in joint space

  interpolation in joint space

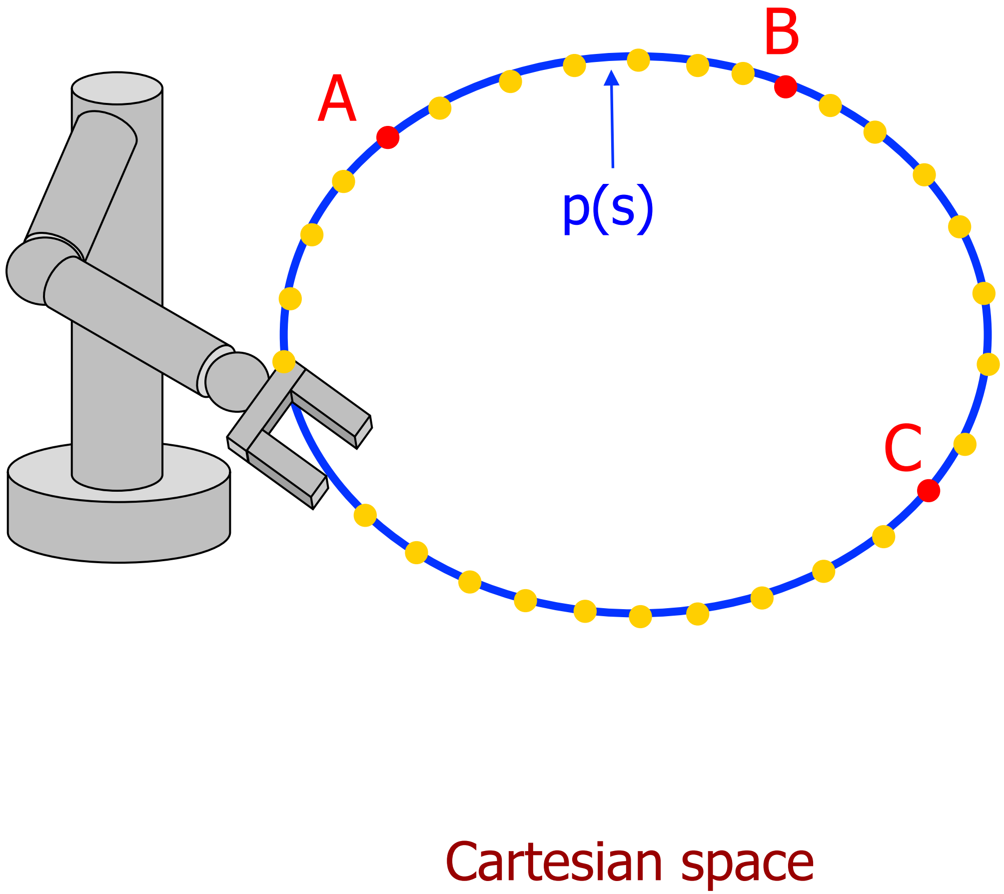
- geometric path in joint space: $q = q(\lambda)$

analytic inversion

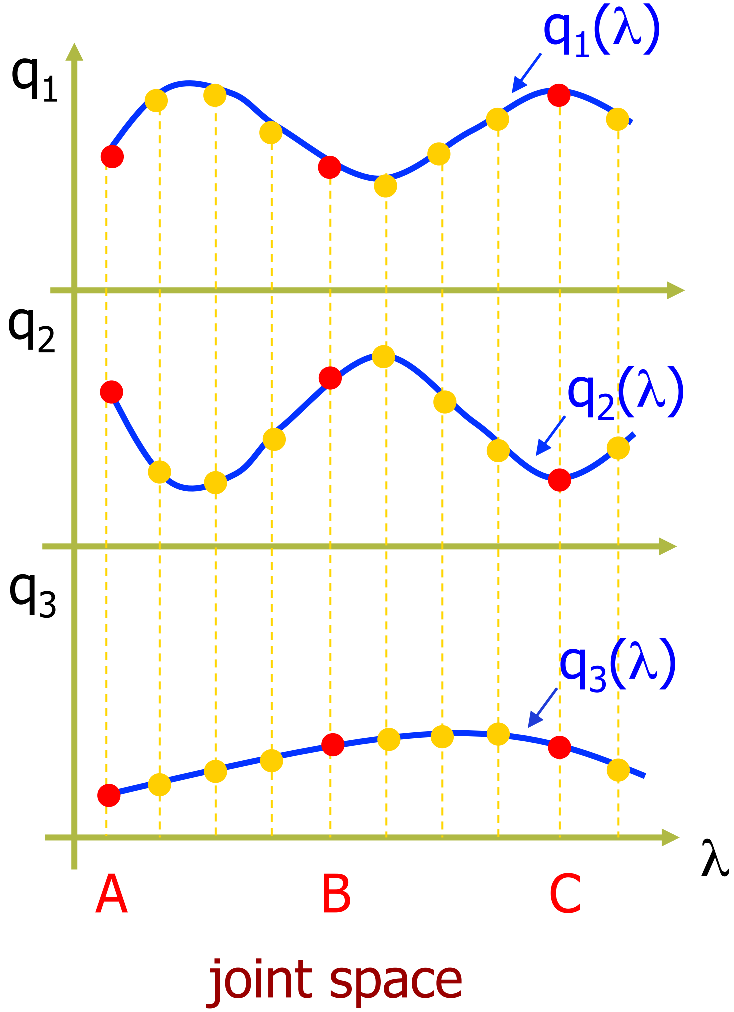
additional issues to be considered in the planning process

- obstacle avoidance
- on-line/off-line computational load
- sequence 2 is more "dense" than 1

Example



A
B
C
p(s)
Cartesian space



q_1
$q_1(\lambda)$
q_2
$q_2(\lambda)$
q_3
$q_3(\lambda)$
λ
A
B
C
joint space

Cartesian vs. joint trajectory planning

- planning in Cartesian space
  - allows a more direct visualization of the generated path
  - obstacle avoidance, lack of "wandering"
- planning in joint space
  - does not need on-line kinematic inversion
- issues in kinematic inversion
  - $\dot{q}$ e $\ddot{q}$ (or higher-order derivatives) may also be needed
    - Cartesian task specifications involve the geometric path, but also bounds on the associated timing law
  - for redundant robots, choice among ∞^{n-m} inverse solutions, based on optimality criteria or additional auxiliary tasks
  - off-line planning in advance is not always feasible
    - e.g., when interaction with the environment occurs or sensor-based motion is needed

Path and timing law

- after choosing a path, the trajectory definition is completed by the choice of a timing law

$$p = p(s) \quad \Rightarrow s = s(t) \qquad \text{(Cartesian space)}$$

$$q = q(\lambda) \quad \Rightarrow \lambda = \lambda(t) \qquad \text{(joint space)}$$

  - if $s(t) = t$, path parameterization is the natural one given by time
- the timing law
  - is chosen based on task specifications (stop in a point, move at constant velocity, and so on)
  - may consider optimality criteria (min transfer time, min energy,…)
  - constraints are imposed by actuator capabilities (max torque, max velocity,…) and/or by the task (e.g., max acceleration on payload)

note: on parameterized paths, a space-time decomposition takes place

e.g., in Cartesian space

$$\dot{p}(t) = \frac{dp}{ds}\dot{s} \qquad \ddot{p}(t) = \frac{dp}{ds}\ddot{s} + \frac{d^2p}{ds^2}\dot{s}^2$$

Trajectory classification

- space of definition
  - Cartesian, joint
- task type
  - point-to-point (PTP), multiple points (knots), continuous, concatenated
- path geometry
  - rectilinear, polynomial, exponential, cycloid, …
- timing law
  - bang-bang in acceleration, trapezoidal in velocity, polynomial, …
- coordinated or independent
  - motion of all joints (or of all Cartesian components) start and ends at the same instants (say, t=0 and t=T) = single timing law

  or
  - motions are timed independently (according to the requested displacement and robot capabilities) – mostly only in joint space

Relevant characteristics

- computational efficiency and memory space
  - e.g., store only the coefficients of a polynomial function
- predictability (vs. "wandering" out of the knots) and accuracy (vs. "overshoot" on final position)
- flexibility (allowing concatenation, over-fly, ...)
- continuity (in space and in time)

(at least C^1, but also up to jerk = third derivative in time)

A robot trajectory with bounded jerk

video

Trajectory planning in joint space

- $q = q(t)$ in time or $q = q(\lambda)$ in space (then with $\lambda = \lambda(t)$)
- it is sufficient to work component-wise (q_i in vector q)
- an implicit definition of the trajectory, by solving a problem with specified boundary conditions in a given class of functions
- typical classes: polynomials (cubic, quintic,…), (co)sinusoids, clothoids, …
- imposed conditions
  - passage through points = interpolation
  - initial, final, intermediate velocity (or geometric tangent for paths)
  - initial, final acceleration (or geometric curvature)
  - continuity up to the k-th order time (or space) derivative: class $\mathbf{C}^k$

many of the following methods and remarks can be directly applied also to Cartesian trajectory planning (and vice versa)!

Cubic polynomial in space

$\boxed{q(0) = q_0}$ $\boxed{q(1) = q_1}$ $\boxed{q'(0) = v_0}$ $\boxed{q'(1) = v_1}$ ← 4 conditions

$$q(\lambda) = q_0 + \Delta q\,\underbrace{[a\lambda^3 + b\lambda^2 + c\lambda + d]}$$

$\Delta q = q_1 - q_0$

$\lambda \in [0,1]$

4 coefficients → "doubly normalized" polynomial $q_N(\lambda)$

$q_N(0) = 0 \Leftrightarrow d = 0$ $\qquad$ $q_N(1) = 1 \Leftrightarrow a + b + c = 1$

$q_N'(0) = dq_N/d\lambda|_{\lambda=0} = c = v_0/\Delta q$ $\qquad$ $q_N'(1) = dq_N/d\lambda|_{\lambda=1} = 3a + 2b + c = v_1/\Delta q$

special case: $v_0 = v_1 = 0$ (zero tangent)

$q_N'(0) = 0 \Leftrightarrow c = 0$

$$\left.\begin{array}{l} q_N(1) = 1 \Leftrightarrow a + b = 1 \\ q_N'(1) = 0 \Leftrightarrow 3a + 2b = 0 \end{array}\right\} \Leftrightarrow \begin{array}{l} a = -2 \\ b = 3 \end{array}$$

Cubic polynomial in time

$\boxed{q(0) = q_{in}}$ $\boxed{q(T) = q_{fin}}$ $\boxed{\dot{q}(0) = v_{in}}$ $\boxed{\dot{q}(T) = v_{fin}}$ ← 4 conditions

$$q(\tau) = q_{in} + \Delta q\,\underbrace{[a\tau^3 + b\tau^2 + c\tau + d]}$$

$\Delta q = q_{fin} - q_{in}$

$\tau = t/T,\ \tau \in [0,1]$

4 coefficients → "doubly normalized" polynomial $q_N(\tau)$

$q_N(0) = 0 \Leftrightarrow d = 0$ $q_N(1) = 1 \Leftrightarrow a + b + c = 1$

$q_N'(0) = dq_N/d\tau|_{\tau=0} = c = v_{in}T/\Delta q$ $q_N'(1) = dq_N/d\tau|_{\tau=1} = 3a + 2b + c = v_{fin}T/\Delta q$

special case: $v_{in} = v_{fin} = 0$ (rest-to-rest)

$q_N'(0) = 0 \Leftrightarrow c = 0$

$$\left.\begin{array}{l} q_N(1) = 1 \Leftrightarrow a + b = 1 \\ q_N'(1) = 0 \Leftrightarrow 3a + 2b = 0 \end{array}\right\} \Leftrightarrow \begin{array}{l} a = -2 \\ b = 3 \end{array}$$

Quintic polynomial

$$\boxed{q(\tau) = a\tau^5 + b\tau^4 + c\tau^3 + d\tau^2 + e\tau + f}$$ 6 coefficients

$\tau \in [0, 1]$

allows to satisfy 6 conditions, for example (in normalized time $\tau = t/T$)

$\boxed{q(0) = q_0}$ $\boxed{q(1) = q_1}$ $\boxed{q'(0) = v_0 T}$ $\boxed{q'(1) = v_1 T}$ $\boxed{q''(0) = a_0 T^2}$ $\boxed{q''(1) = a_1 T^2}$

$$\begin{aligned} q(\tau) = {} & (1 - \tau)^3 [q_0 + (3q_0 + v_0 T)\tau + (a_0 T^2 + 6v_0 T + 12q_0)\tau^2/2] \\ & + \tau^3 [q_1 + (3q_1 - v_1 T)(1-\tau) + (a_1 T^2 - 6v_1 T + 12q_1)(1 - \tau)^2/2] \end{aligned}$$

special case: $v_0 = v_1 = a_0 = a_1 = 0$

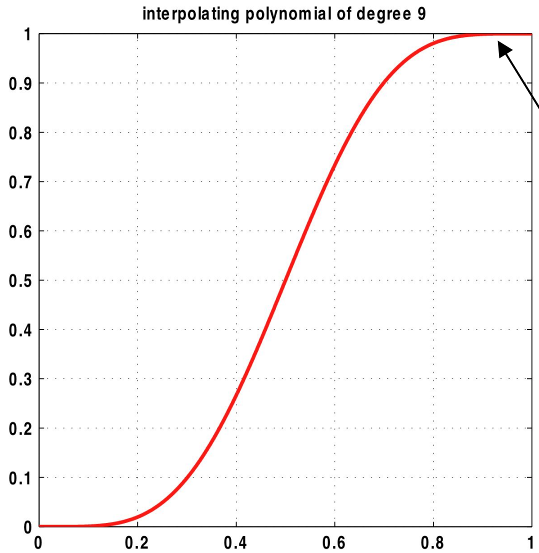
$$\boxed{q(\tau) = q_0 + \Delta q\, [6\tau^5 - 15\tau^4 + 10\tau^3]}$$ $\Delta q = q_1 - q_0$

Higher-order polynomials

- a suitable solution class for satisfying symmetric boundary conditions (in a PTP motion) that impose zero values on higher-order derivatives
  - the interpolating polynomial is always of odd degree
  - the coefficients of such (doubly normalized) polynomial are always integers, alternate in sign, sum up to unity, and are zero for all terms up to the power = (degree-1)/2
- in all other cases (e.g., for interpolating a large number N of points), their use is not recommended
  - N-th order polynomials have N-1 maximum and minimum points
  - oscillations arise out of the interpolation points (wandering)

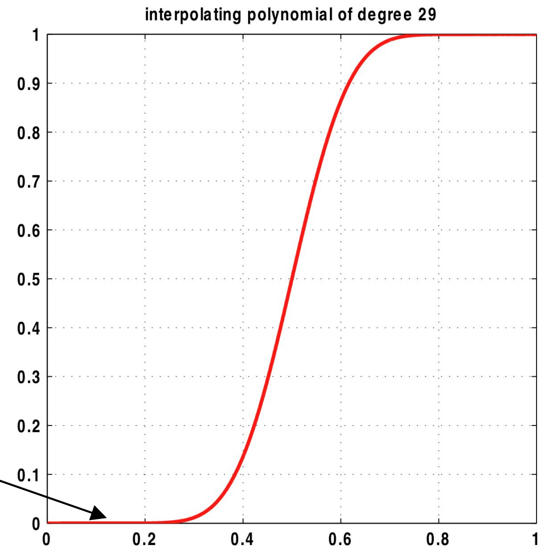
Numerical examples

9th degree



4 derivatives are zero

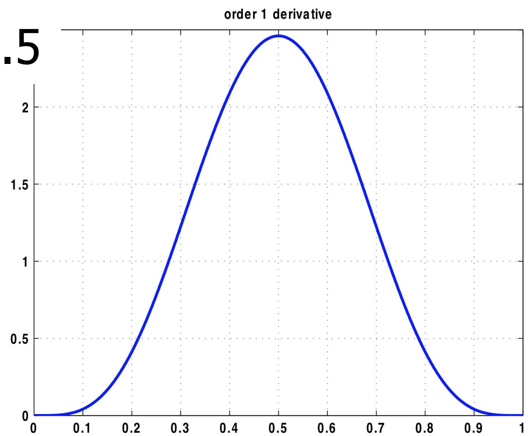
14 derivatives are zero!



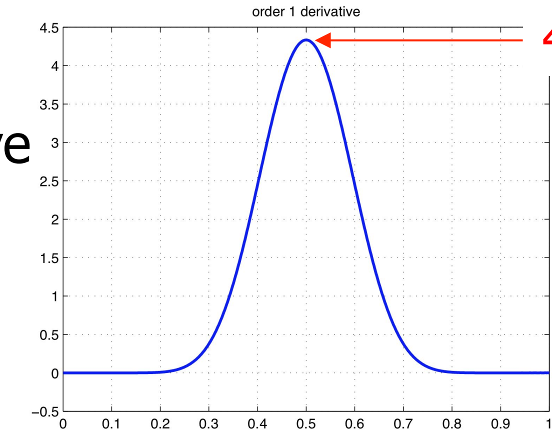
29th degree

no overshoot nor wandering

2.5



normalized first derivative (velocity in time)

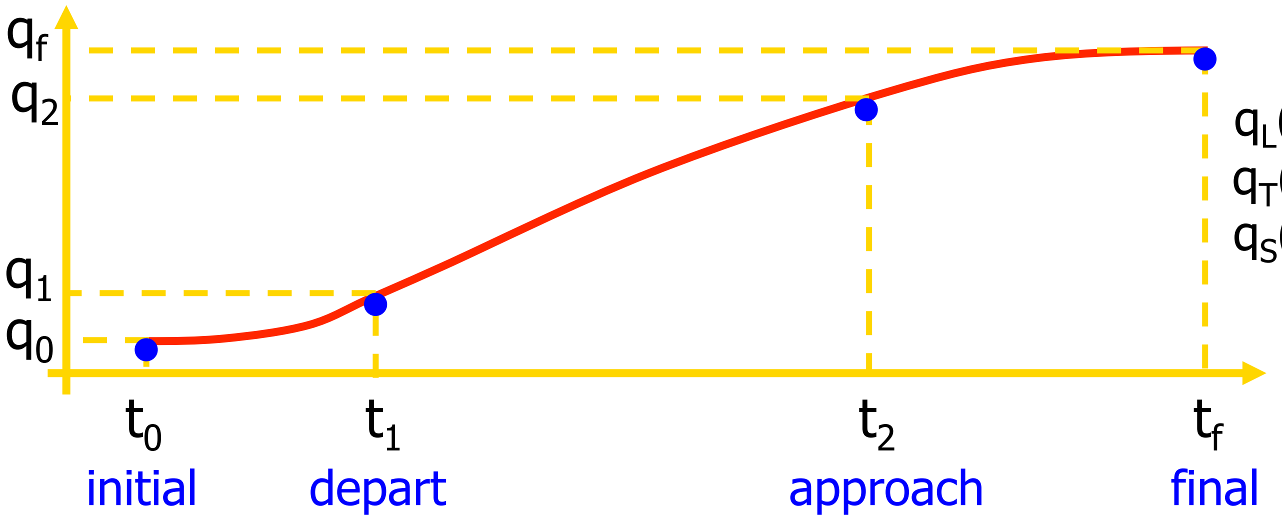


4.5!!

peaking at midpoint

4-3-4 polynomials

three phases (Lift off, Travel, Set down) in a pick-and-place operation in time



$q_L(t)$ = 4th order polynomial
$q_T(t)$ = 3rd order polynomial
$q_S(t)$ = 4th order polynomial

14 coefficients

boundary conditions

$q(t_0) = q_0 \quad q(t_1^-) = q(t_1^+) = q_1 \quad q(t_2^-) = q(t_2^+) = q_2 \quad q(t_f) = q_f$ } 6 passages

$\dot{q}(t_0) = \dot{q}(t_f) = 0 \qquad \ddot{q}(t_0) = \ddot{q}(t_f) = 0$ } 4 initial/final velocity/acceleration

$\dot{q}(t_i^-) = \dot{q}(t_i^+) \qquad \ddot{q}(t_i^-) = \ddot{q}(t_i^+) \quad i = 1,2$ } 4 continuity

Interpolation using splines

- problem

  interpolate N knots, with continuity up to the second derivative

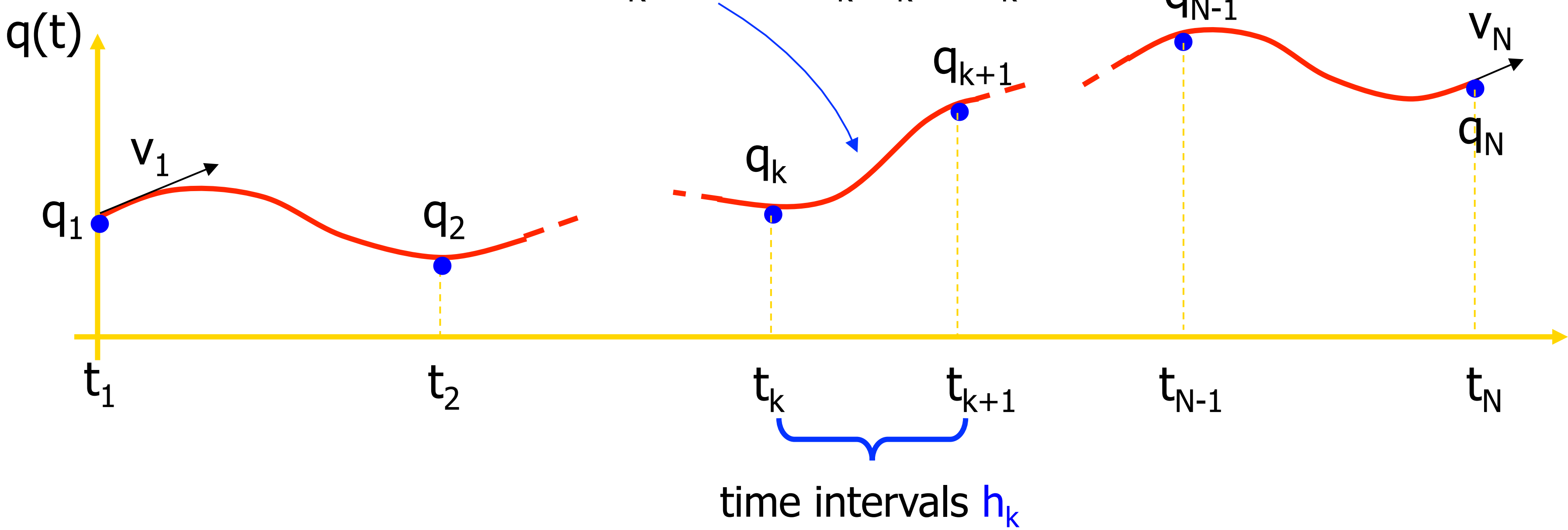
- solution

  spline: N-1 cubic polynomials, concatenated so as to pass through N knots and being continuous up to the second derivative at the N-2 internal knots

- 4(N-1) coefficients
- 4(N-1)-2 conditions, or
  - 2(N-1) of passage (for each cubic, in the two knots at its ends)
  - N-2 of continuity for first derivative (at the internal knots)
  - N-2 of continuity for second derivative (at the internal knots)
- 2 free parameters are still left over
  - can be used, e.g., to assign initial and final derivatives, v_1 and v_N
- presented next in terms of time t, but similar in terms of space λ
  - then: first derivative = velocity, second derivative = acceleration

Building a cubic spline

$$q = \theta(t) = \{\theta_K(t),\ t \in [t_k, t_k + h_k]\}$$



$$\boxed{\theta_K(\tau) = a_{k0} + a_{k1}\,\tau + a_{k2}\,\tau^2 + a_{k3}\,\tau^3}$$

$\tau \in [0, h_k],\ \tau = t - t_k\quad (k = 1, \ldots, N\text{-}1)$

continuity conditions
for velocity and acceleration

$$\begin{aligned} \dot{\theta}_K(h_k) &= \dot{\theta}_{K+1}(0) \\ \ddot{\theta}_K(h_k) &= \ddot{\theta}_{K+1}(0) \end{aligned} \qquad k = 1, \ldots, N\text{-}2$$

An efficient algorithm

1. if all velocities v_k at internal knots were known, then each cubic in the spline would be uniquely determined by

$$\theta_K(0) = q_K = a_{K0} \qquad \begin{pmatrix} h_K^2 & h_K^3 \\ 2h_K & 3h_K^2 \end{pmatrix} \begin{pmatrix} a_{K2} \\ a_{K3} \end{pmatrix} = \begin{pmatrix} q_{K+1} - q_K - v_K h_K \\ v_{K+1} - v_K \end{pmatrix} \quad (1)$$
$$\dot{\theta}_K(0) = v_K = a_{K1}$$

2. impose the continuity for accelerations (N-2 conditions)

$$\ddot{\theta}_K(h_k) = 2\, a_{K2} + 6\, a_{K3}\, h_K = \ddot{\theta}_{K+1}(0) = 2\, a_{K+1,2}$$

3. expressing the coefficients a_{k2}, a_{k3}, $a_{k+1,2}$ in terms of the still unknown knot velocities (see step 1.) yields a linear system of equations that is always (easily) solvable

$$\begin{pmatrix} A(h) \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_{N-1} \end{pmatrix} = \begin{pmatrix} b(h,q,v_1,v_N) \end{pmatrix}$$

tri-diagonal matrix always invertible

unknown

known vector

to be substituted then back in (1)

Structure of A(h)

$$
\begin{pmatrix}
2(h_1+h_2) & h_1 & & & \\
h_3 & 2(h_2+h_3) & h_2 & & \\
 & \ddots & \ddots & \ddots & \\
 & & h_{N-2} & 2(h_{N-3}+h_{N-2}) & h_{N-3} \\
 & & & h_{N-1} & 2(h_{N-2}+h_{N-1})
\end{pmatrix}
$$

diagonally dominant matrix (for $h_k > 0$)
[the same matrix for all joints]

Structure of $b(h,q,v_1,v_N)$

$$
\begin{pmatrix}
\frac{3}{h_1 h_2}[h_1^2(q_3 - q_2) + h_2^2(q_2 - q_1)] - h_2 v_1 \\
\frac{3}{h_2 h_3}[h_2^2(q_4 - q_3) + h_3^2(q_3 - q_2)] \\
\vdots \\
\frac{3}{h_{N-3} h_{N-2}}[h_{N-3}^2(q_{N-1} - q_{N-2}) + h_{N-2}^2(q_{N-2} - q_{N-3})] \\
\frac{3}{h_{N-2} h_{N-1}}[h_{N-2}^2(q_N - q_{N-1}) + h_{N-1}^2(q_{N-1} - q_{N-2})] - h_{N-2} v_N
\end{pmatrix}
$$

Properties of splines

- a spline (in space) is the solution with minimum curvature among all interpolating functions having continuous second derivative
- for cyclic tasks ($q_1 = q_N$), it is preferable to simply impose continuity of first and second derivatives (i.e., velocity and acceleration in time) at the first/last knot as "squaring" conditions
  - choosing $v_1 = v_N = v$ (for a given v) doesn't guarantee in general the continuity up to the second derivative (in time, of the acceleration)
  - in this way, the first = last knot will be handled as all other internal knots
- a spline is uniquely determined from the set of data $q_1,\ldots,q_N$, $h_1, \ldots, h_{N-1}, v_1, v_N$
- in time, the total motion occurs in $T = \Sigma_k\, h_k = t_N - t_1$
- the time intervals h_k can be chosen so as to minimize T (linear objective function) under (nonlinear) bounds on velocity and acceleration in [0,T]
- in time, the spline construction can be suitably modified when the acceleration is also assigned at the initial and final knots

A modification

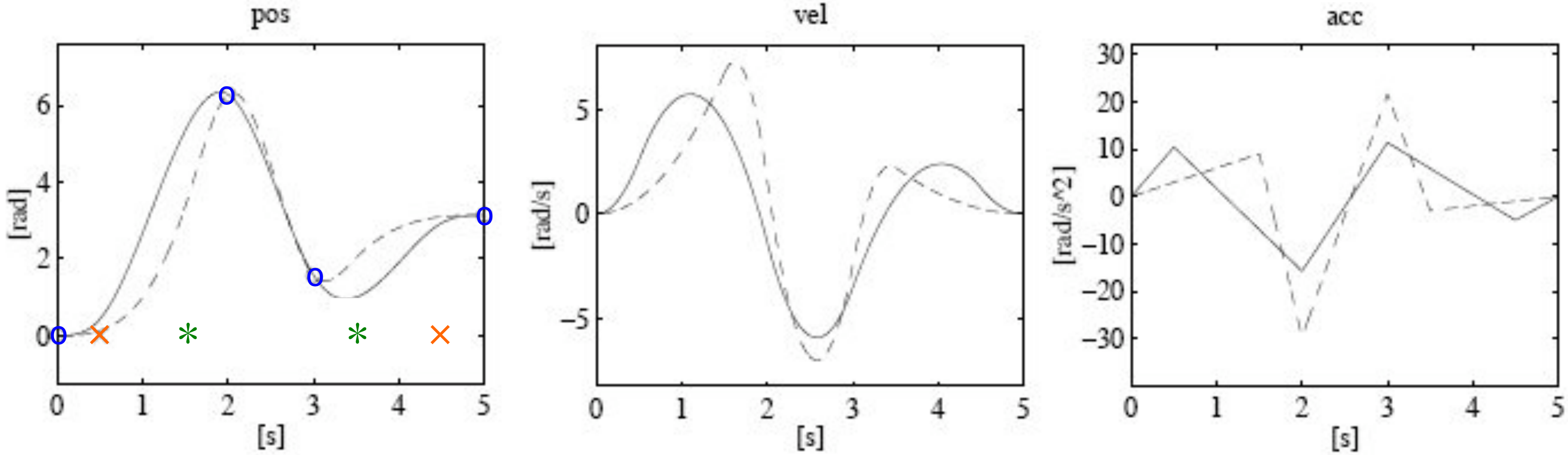
handling assigned initial and final accelerations

- two more parameters are needed in order to impose also the initial acceleration α_1 and final acceleration α_N
- two "fictitious knots" are inserted in the first and last original intervals, increasing the number of cubic polynomials from N-1 to N+1
- in these two knots only continuity conditions on position, velocity and acceleration are imposed

  $\Rightarrow$ two free parameters are left over (one in the first cubic and the other in the last cubic), which are used to satisfy the boundary conditions on acceleration
- depending on the (time) placement of the two additional knots, the resulting spline changes

A numerical example

- $N = 4$ knots (3 cubic polynomials)
  - joint values $q_1 = 0$, $q_2 = 2\pi$, $q_3 = \pi/2$, $q_4 = \pi$
  - at $t_1 = 0$, $t_2 = 2$, $t_3 = 3$, $t_4 = 5$ (thus, $h_1 = 2$, $h_2 = 1$, $h_3 = 2$)
  - boundary velocities $v_1 = v_4 = 0$
- 2 added knots to impose accelerations at both ends (5 cubic polynomials)
  - boundary accelerations $\alpha_1 = \alpha_4 = 0$
  - two placements: at $t_1' = 0.5$ and $t_4' = 4.5$ ($\times$), or $t_1'' = 1.5$ and $t_4'' = 3.5$ ($*$)



——— = placement′ --------- = placement″