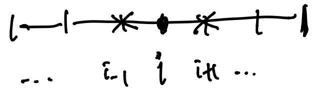
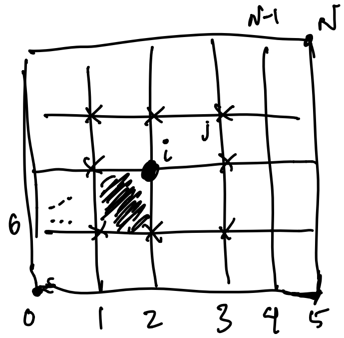
WEEK 5

Global vectors and Matrices



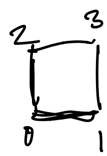
$$\vec{F} = \begin{bmatrix} \int_\Omega f \varphi_0 \, dx \\ \int_\Omega f \varphi_1 \, dx \\ \vdots \\ \int_\Omega f \varphi_N \, dx \end{bmatrix} \qquad M = \begin{bmatrix} \int_\Omega \varphi_0 \varphi_0 & \int_\Omega \varphi_0 \varphi_1 & \cdots & \int_\Omega \varphi_0 \varphi_N \\ \int \varphi_1 \varphi_0 & \cdots & & \\ \vdots & & & \\ \int \varphi_N \varphi_0 & \cdots & & \int_\Omega \varphi_N \varphi_N \end{bmatrix}$$



$$[M]_i = \left[\int_\Omega \varphi_i \varphi_0 \;\; \int_\Omega \varphi_i \varphi_1 \; \cdots \; \int_\Omega \varphi_i^2 \; \cdots \; \int_\Omega \varphi_i \varphi_j \; \cdots \; \int_\Omega \varphi_i \varphi_N \right]$$

NOTE: In this case, only 9 entries are non-zero

* We break up integrals in elements and Matrices.



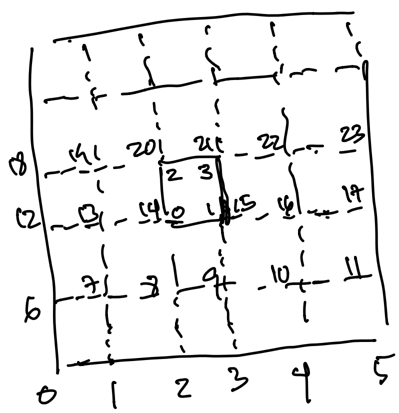
$$\int_k f \varphi_i \, dx \;; \; i = 0, \dots, 3 \implies f^e(\varphi) = \begin{bmatrix} \int_e f \varphi_0 \\ \int_e f \varphi_1 \\ \vdots \\ \int_e f \varphi_3 \end{bmatrix}$$

- what about the mass matrix?

$$M^e \in \mathbb{R}^{4 \times 4} \qquad M = \begin{bmatrix} \int_e \varphi_0 \varphi_0 & \int_e \varphi_0 \varphi_1 & \int_e \varphi_0 \varphi_2 & \int_e \varphi_0 \varphi_3 \\ \vdots & & & \\ \int_e \varphi_3 \varphi_0 & \cdots & & \int_e \varphi_3 \varphi_3 \end{bmatrix}$$

- <u>Question</u>: Is the element mass matrix dense?

* we need to combine the element based vectors and matrices to form the global vectors and matrices



given M^e_{ij} ; $i,j = \{0,1,2,3\}$

we need to sum or assemble it onto the global Matrix M

$$M^e = \begin{bmatrix} M^e_{00} & M^e_{01} & M^e_{02} & M^e_{03} \\ M_{10} & M_{11} & \cdots & M_{13} \\ \vdots & \vdots & & \end{bmatrix} \qquad M = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

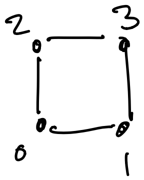
$$F^e = \begin{bmatrix} F^e_0 \\ F^e_1 \\ F^e_2 \\ F^e_3 \end{bmatrix} \qquad F = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

* We need a map that gives the global index for a given element and a given local DoF.

* F.E. Loop:

- we need to know all quantities in the reference element.
- we need to compute integrals (using the reference elements)
- we need to assemble element based operators $\rightarrow$ global.

* Basis functions for quad-ref. element:



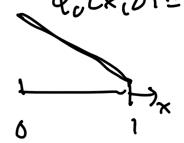
$\varphi_0(x,y) = ax + by + cxy + d$

with $\varphi_0(0,0) = 1$, $\varphi_0(0,1) = \varphi_0(1,1) = \varphi_0(1,0) = 0$

- consider $y=0$:

$$\varphi_0(x,0) = ax + d$$

$\varphi_0(x,0) = 1 - x$

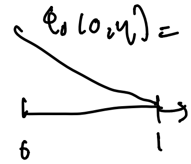


$a = -1$
$d = 1$

- consider $x=0$:

$$\varphi_0(0,y) = by + d$$

$\varphi_0(0,y) = 1 - y$



$b = -1$

- consider $\varphi_0(1,1) = 0$

$$\Rightarrow -1 - \cancel{1} + c \cancel{+1} = 0 \Rightarrow c = 1$$

- Then $\varphi_0(x,y) = 1 - x - y + xy = x(y-1) - (y-1)$

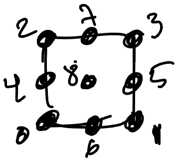
$$= (x-1)(y-1) = \underbrace{(1-x)}_{=\varphi_0(x)}\underbrace{(1-y)}_{\varphi_0(y)}$$

- similarly, $\varphi_1(x,y) = \varphi_1(x)\,\varphi_0(y)$

$\varphi_2(x,y) = \varphi_0(x)\,\varphi_1(y)$

$\varphi_3(x,y) = \varphi_1(x)\,\varphi_1(y)$

- for quadratics: $\varphi_0(x,y) = ax^2 + bx^2y + cx^2y^2 + dy^2 + ey^2x$
$+ fx + gxy + hy + i$

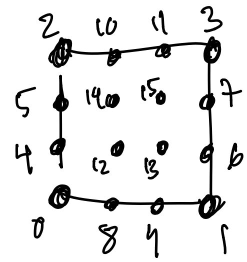


$$\varphi_0(x) = 2(1-x)(1/2 - x)$$
$$\varphi_1(x) = -2x(1/2 - x)$$
$$\varphi_2(x) = 4x(1-x)$$

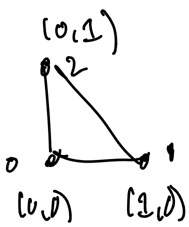
$$\varphi_0(x,y) = \varphi_0(x)\,\varphi_0(y) \qquad \varphi_4 = \varphi_0(x)\,\varphi_2(y)$$
$$\varphi_1(x,y) = \varphi_0(x)\,\varphi_1(y) \qquad \varphi_5 = \varphi_1(x)\,\varphi_2(y)$$
$$\varphi_2(x,y) = \varphi_0(x)\,\varphi_1(y) \qquad \varphi_6 = \varphi_2(x)\,\varphi_0(y)$$
$$\varphi_3(x,y) = \varphi_1(x)\,\varphi_1(y) \qquad \varphi_7 = \varphi_2(x)\,\varphi_1(y)$$
$$\varphi_8 = \varphi_2(x)\,\varphi_2(y)$$

- Look at the plots in Matlab.

- <u>for Cubics:</u>



Shape functions for triangular-reference element:



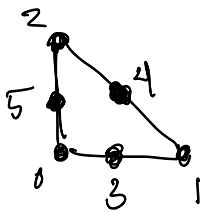
$$\varphi_0(x,y) = ax + by + d$$
$$\varphi_0(0,0) = 1 = d$$
$$\varphi_0(1,0) = a + d = 0 \Rightarrow a = -1$$
$$\varphi_0(0,1) = b + d = 0 \Rightarrow b = -1$$

$$\Rightarrow \varphi_0(x,y) = 1 - x - y$$

$$\varphi_1(x,y) = ax + by + d$$
$$\varphi_1(0,0) = d = 0, \quad \varphi_1(0,1) = b = 0, \quad \varphi_1(1,0) = a = 1$$

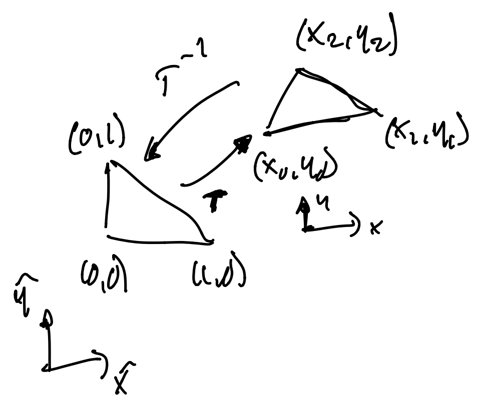
$$\Rightarrow \varphi_1(x,y) = x, \quad \varphi_2(x,y) = y$$

• for quadratics: $e_0(x,y) = ax^2 + by^2 + cxy + dx + ey + f$



$\vdots$

*** Transformation from physical to reference elements:**



$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix} = \overbrace{\begin{bmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{bmatrix}}^{=T} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\text{Affine Transformation.}}$$

Note that:

$$J := \nabla_{\hat{x}} \begin{bmatrix} x \\ y \end{bmatrix} = T$$

} Transf. matrix is the Jacobian $\nabla_{\hat{x}} \begin{bmatrix} x \\ y \end{bmatrix}$

• Then,

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = J \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \; ; \; \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = J^{-1} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}}$$

• given $A \in \mathbb{R}^{n \times n}$, what is $|A|$?

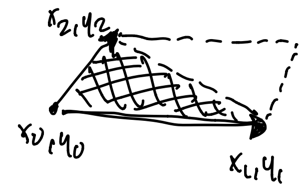
<u>ANS:</u> volume of the parallelogram form by the columns

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$$

- given dv, what is $d\hat{v}$?

consider $\int_k dv = |k| = \frac{1}{2}|J|$

$$J = \begin{bmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & x_2 - x_0 \end{bmatrix}$$



consider $\int_{\hat{k}} d\hat{v} = |\hat{k}| = \frac{1}{2}$

Then, $\int_k dv = \int_{\hat{k}} \alpha d\hat{v} \Rightarrow \frac{1}{2}|J| = \alpha/2 \Rightarrow \alpha = |J|$

$$\Longrightarrow \boxed{dv = |J| d\hat{v}}$$

- What about $\nabla f(x)$?

$$\nabla_x f(x) = (\nabla_x \hat{x}) \nabla_{\hat{x}} \hat{f}(\hat{x}) = J^{-1} \nabla_{\hat{x}} \hat{f}(\hat{x})$$

$$\Rightarrow \int_k \nabla f(x) dv = \int_{\hat{k}} J^{-1} \nabla_{\hat{x}} \hat{f}(\hat{x}) |J| d\hat{v}$$

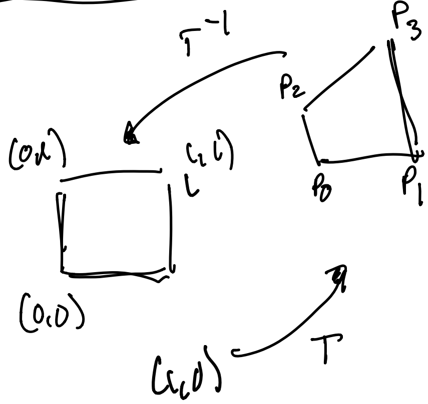
and $\int_k \nabla \varphi_i \cdot \nabla \varphi_j dv = \int_{\hat{k}} (J^{-1} \nabla_{\hat{x}} \hat{\varphi}_i) \cdot (J^{-1} \nabla_{\hat{x}} \hat{\varphi}_j) |J| d\hat{v}$

- Let's verify the <u>1D case</u>:

$J = x_1 - x_0 = h, \quad |J| = h$

$\Rightarrow \quad dx = h d\hat{x}; \quad \int_{x_0}^{x_1} \nabla \varphi_i \cdot \nabla \varphi_j dv = \frac{1}{h} \int_0^1 \nabla \hat{\varphi}_i \cdot \nabla \hat{\varphi}_j d\hat{v}$

* For QUADS:



$$\vec{x} = (1-\hat{x})(1-\hat{y})\vec{x}_0 + \hat{x}(1-\hat{y})\vec{x}_1 + \hat{x}\hat{y}\vec{x}_3 + (1-\hat{x})\hat{y}\vec{x}_2 \quad (*)$$

NOTE that this is NOT affine.

$\Rightarrow$ So we can't express:

$$\vec{x} = T\hat{x} + x_0$$

- But we can still do the transformations that we need.

- consider $(*)$:

$$T(\hat{x}) = \vec{x} = \underbrace{\underbrace{(1-\hat{x})}_{\varphi_0(\hat{x})}\underbrace{(1-\hat{y})}_{\varphi_0(\hat{y})}}_{\varphi_0(\hat{x},\hat{y})}\vec{x}_0 + \underbrace{\underbrace{\hat{x}}_{\varphi_1(\hat{x})}\underbrace{(1-\hat{y})}_{\varphi_0(\hat{y})}}_{\varphi_1(\hat{x},\hat{y})}\vec{x}_1 + \underbrace{\underbrace{\hat{x}\hat{y}}_{\varphi_1(\hat{x})\varphi_1(\hat{y})}}_{\varphi_3(\hat{x},\hat{y})}\vec{x}_3 + \underbrace{\underbrace{(1-\hat{x})\hat{y}}_{\varphi_0(\hat{x})\varphi_1(\hat{y})}}_{\varphi_2(\hat{x},\hat{y})}\vec{x}_2$$

$$\vec{x} = \sum_{j=0}^{3} \vec{X}_j\, \hat{\varphi}_j(\hat{x},\hat{y})$$ The element transformation can be expressed via the linear shape functions on $\hat{K}$.

- Let's consider (for example)

$\dfrac{\partial \varphi_i(x,y)}{\partial x}$ and transform it to $\hat{K}$.

$$\frac{\partial \hat{\varphi}_i(\hat{x},\hat{y})}{\partial \hat{x}} = \frac{\partial \varphi_i(x,y)}{\partial x} \cdot \frac{\partial x}{\partial \hat{x}} + \frac{\partial \varphi_i(x,y)}{\partial y} \cdot \frac{\partial y}{\partial \hat{x}}$$

$$\frac{\partial \hat{\varphi}_i(\hat{x},\hat{y})}{\partial \hat{y}} = \frac{\partial \varphi_i(x,y)}{\partial x} \frac{\partial x}{\partial \hat{y}} + \frac{\partial \varphi_i(x,y)}{\partial y} \cdot \frac{\partial y}{\partial \hat{y}}$$

$$\Rightarrow \begin{bmatrix} \partial \hat{\varphi}_i/\partial \hat{x} \\ \partial \hat{\varphi}/\partial \hat{y} \end{bmatrix} = \underbrace{\begin{bmatrix} \partial x/\partial \hat{x} & \partial y/\partial \hat{x} \\ \partial x/\hat{y} & \partial y/\partial \hat{y} \end{bmatrix}}_{=:\, J} \begin{bmatrix} \partial \varphi_i/\partial x \\ \partial \varphi_i/\partial y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \partial \varphi_i/\partial x \\ \partial \varphi_i/\partial y \end{bmatrix} = J^{-1} \begin{bmatrix} \partial \hat{\varphi}_i/\partial \hat{x} \\ \partial \hat{\varphi}/\partial \hat{y} \end{bmatrix} \Rightarrow \nabla \varphi = J^{-1} \nabla \hat{\varphi}(\hat{x},\hat{y})$$

- How to compute J in the code (for each element)?

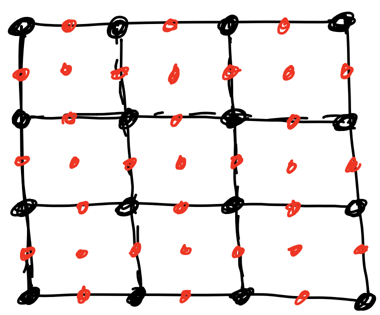
$$\frac{\partial x}{\partial \hat{x}} = \sum_{j=0}^{3} X_j \frac{\partial \hat{\varphi}_j}{\partial \hat{x}} \; ; \quad \frac{\partial y}{\partial \hat{x}} = \sum_{j=0}^{3} Y_j \frac{\partial \hat{\varphi}_j}{\partial \hat{x}}$$

$$\frac{\partial x}{\partial \hat{y}} = \sum_{j=0}^{3} X_j \frac{\partial \hat{\varphi}_j}{\partial \hat{y}} \; ; \quad \frac{\partial y}{\partial \hat{y}} = \sum_{j=0}^{3} Y_j \frac{\partial \hat{\varphi}_j}{\partial \hat{y}}$$

- How to compute dV?

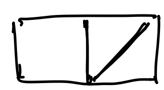
$$dV = |J| \, d\hat{V}$$

Mesh and Data Structures:



$N_{geo} = 16$, $N_{el} = 9$, $N_{faces} = 12$

$N_{geo}^e = 4$, $N_{geo}^f = 2$



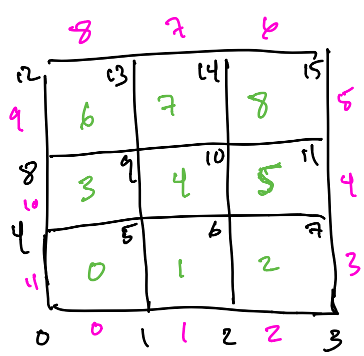
In this case, N_{geo}^e is not constant.

$N_h = 49$

$N_h^e = 9$, $N_h^f = 3$

- connectivity matrix:

$$C \in \mathbb{R}^{N_{el} \times N_{geo}^e} = \mathbb{R}^{8 \times 4}$$



local DoFs

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 4 & 5 \\ 1 & 1 & 2 & 5 & 6 \\ 2 & 2 & 3 & 6 & 7 \\ 3 & 4 & 5 & 8 & 9 \\ 4 & 5 & 6 & 9 & 10 \\ 5 & 6 & 7 & 10 & 11 \\ 6 & 8 & 9 & 12 & 13 \\ 7 & 9 & 10 & 13 & 14 \\ 8 & 10 & 11 & 14 & 15 \end{array} = C$$

- connectivity Matrix for the faces:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \\ 11 \end{array}
\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left[\begin{array}{cc} 0 & 1 \\ 1 & 2 \\ 2 & 3 \\ 3 & 7 \\ 7 & 11 \\ 11 & 15 \\ \vdots & \\ 4 & 0 \end{array}\right] \end{array} = C^f$$