Poisson problem in a 2D annulus

Problem

Our objective is to solve

$$\begin{aligned} -\Delta u(\mathbf{x}) &= f(\mathbf{x}) = -\frac{1}{r^2}\left[\pi r\cos(\pi r) - (\pi^2 r^2 + \omega^2)\sin(\pi r)\right]\cos(\omega\theta), && \forall \mathbf{x} \in \Omega, \\ u(\mathbf{x}) &= u_I(\mathbf{x}) = \sin(0.25\pi)\cos(\omega\theta), && \forall \mathbf{x} \in \Gamma_I, \\ \partial_r u &= -\pi\cos(\omega\theta), && \forall \mathbf{x} \in \Gamma_O, \end{aligned}$$

where $\omega = 2$ and $(r(\mathbf{x}), \theta(\mathbf{x}))$ are the polar coordinates. The domain is given by

$$\begin{aligned} \Omega &= \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 1\} \setminus \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \leq 0.25\}, \\ \Gamma_I &= \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} = 0.25\}, \\ \Gamma_O &= \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} = 1\}. \end{aligned}$$

The exact solution, shown in Figure 1, is

$$u(\mathbf{x}) = \sin(\pi r)\cos(\omega\theta).$$

Note that we want to impose the outer boundary via $\partial_r u = -\pi\cos(\omega\theta)$ (not via Dirichlet boundary conditions).

Weak formulation

The discrete weak formulation is given as follows. Find $u_h \in V_h^{\Gamma_I}$ such that

$$\int_\Omega \nabla u_h \cdot \nabla\varphi d\mathbf{x} = \int_\Omega f(\mathbf{x})\varphi d\mathbf{x} - \pi \int_{\Gamma_O} \cos(\omega\theta)\varphi d\mathbf{s}, \qquad \forall \varphi \in V_h$$

where

$$\begin{aligned} V_h &= \{v \in C^0(\Omega) : v_K \in \mathbb{P}^p(K), v(\mathbf{x}) = 0\ \forall \mathbf{x} \in \Gamma_I\}, \\ V_h^{\Gamma_I} &= \{v \in C^0(\Omega) : v_K \in \mathbb{P}^p(K), v(\mathbf{x}) = u_I(\mathbf{x})\ \forall \mathbf{x} \in \Gamma_I\}. \end{aligned}$$

Numerical results

Let us solve the problem with different polynomial spaces. As a reference, we first use the exact solution to impose the outer boundary strongly. The results of a convergence study are summarized in Table 1. We obtain the expected convergence rates.

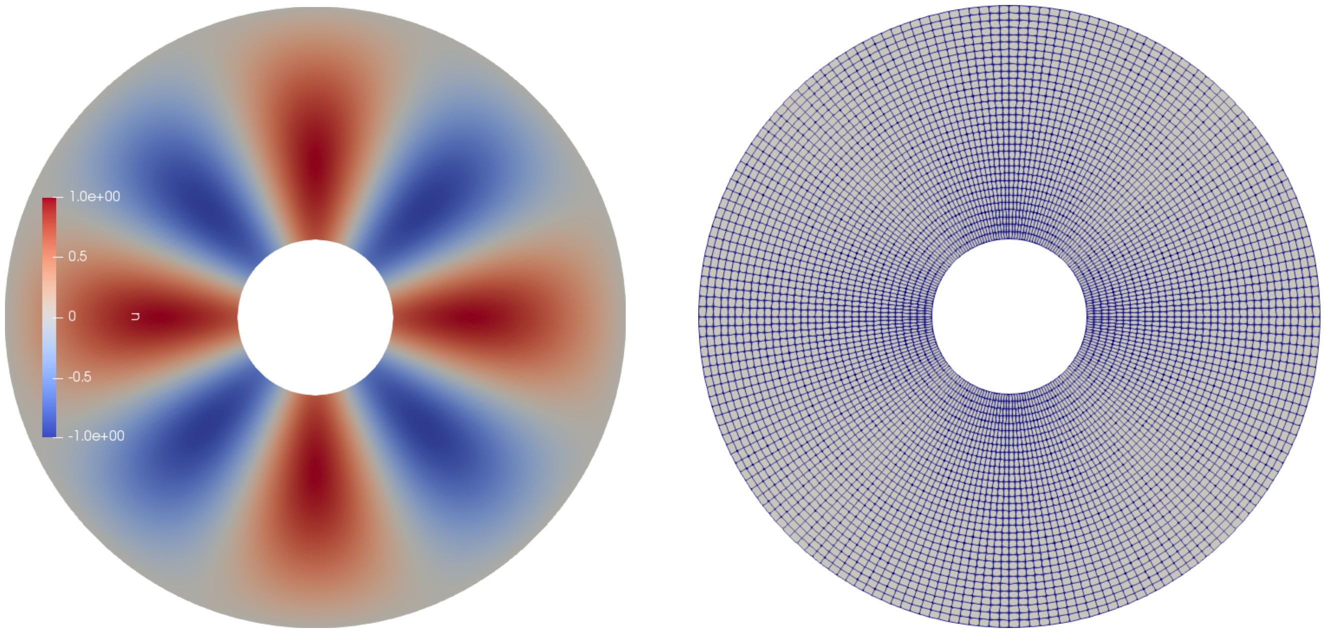

Figure 1: Left: exact solution. Right: typical mesh

Cells	$p=1$		$p=2$		$p=3$		$p=4$	
	E_2	rate	E_2	rate	E_2	rate	E_2	rate
1536	4.85e-03	–	1.06e-04	–	2.26e-06	–	4.56e-08	–
6144	1.21e-03	2.00	1.32e-05	3.00	1.41e-07	4.00	1.42e-09	5.00
24576	3.03e-04	2.00	1.65e-06	3.00	8.82e-09	4.00	4.45e-11	5.00
98304	7.59e-05	2.00	2.07e-07	3.00	5.52e-10	4.00	1.51e-12	4.88
393216	1.90e-05	2.00	2.58e-08	3.00	3.47e-11	3.99	9.51e-12	-2.65

Table 1: Convergence imposing the outer boundary strongly.

Now let us consider the outer boundary via $\partial_r u = -\pi\cos(\omega\theta)$. The results of a convergence study are summarized in Table 2. The convergence rate drops to second-order. The reason being that the condition $\partial_r u = -\pi\cos(\omega\theta)$ assumes the outer boundary is a perfect circle. However, we approximate it via straight lines. In particular, we perform the integral $\int_{\Gamma_O}\cos(\omega\theta)d\mathbf{s}$ along straight lines.

Cells	$p=1$		$p=2$		$p=3$		$p=4$	
	E_2	rate	E_2	rate	E_2	rate	E_2	rate
1536	5.16e-03	–	1.34e-04	–	7.89e-05	–	7.89e-05	–
6144	1.29e-03	2.00	2.39e-05	2.49	1.97e-05	2.00	1.97e-05	2.00
24576	3.23e-04	2.00	5.20e-06	2.20	4.91e-06	2.00	4.91e-06	2.00
98304	8.07e-05	2.00	1.25e-06	2.06	1.23e-06	2.00	1.23e-06	2.00
393216	2.02e-05	2.00	3.08e-07	2.02	3.07e-07	2.00	3.07e-07	2.00

Table 2: Convergence with Neumann boundary conditions with a Q_1 approx. of the boundary.

To recover the high-order accuracy, we can approximate the boundary (and the domain) via high-order polynomials. In deal.II this is done by using high-order element transformations. The results of a convergence test using element transformations based on polynomials of degree p are shown in Table 3.

Cells	$p=1$		$p=2$		$p=3$		$p=4$	
	E_2	rate	E_2	rate	E_2	rate	E_2	rate
1536	5.16e-03	–	7.16e-05	–	1.10e-06	–	1.33e-08	–
6144	1.29e-03	2.00	8.96e-06	3.00	6.87e-08	4.00	4.17e-10	5.00
24576	3.23e-04	2.00	1.12e-06	3.00	4.29e-09	4.00	1.30e-11	5.00
98304	8.07e-05	2.00	1.40e-07	3.00	2.68e-10	4.00	9.56e-13	3.77
393216	2.02e-05	2.00	1.75e-08	3.00	1.68e-11	4.00	2.24e-12	-1.23

Table 3: Convergence with Neumann boundary conditions with Q_p mappings.

References

- To generate circular and annular domains, see
  https://www.dealii.org/9.0.0/doxygen/deal.II/step_1.html,
  https://www.dealii.org/7.3.0/doxygen/deal.II/classGridGenerator.html.
- To attach manifold descriptions to boundaries, see
  https://www.dealii.org/9.0.0/doxygen/deal.II/group__manifold.html.
- To use high-order mappings, see
  https://www.dealii.org/current/doxygen/deal.II/classMappingQGeneric.html,
  https://www.dealii.org/current/doxygen/deal.II/classMappingQ.html.
- To see the effect of using high-order mappings in circular domains, see
  https://www.dealii.org/current/doxygen/deal.II/step_10.html.
- To output data using a high-order mapping, see the function build_patches in
  https://www.dealii.org/current/doxygen/deal.II/classDataOut.html.
- To impose Dirichlet boundary conditions via high-order mappings, see interpolate_boundary_values in
  https://www.dealii.org/current/doxygen/deal.II/namespaceVectorTools.html.
- To compute error norms based on high-order mappings, see integrate_difference in
  https://www.dealii.org/current/doxygen/deal.II/namespaceVectorTools.html.