Homework 2

Exercise 1: projection scheme for the Navier-Stokes equations

Consider the incompressible Navier-Stokes equations in non-conservative form

$$
\begin{aligned}
\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p - \mu \Delta \mathbf{u} &= \mathbf{f}, && \forall x \in \Omega,\\
\nabla\cdot\mathbf{u} &= 0, && \forall x \in \Omega,\\
\mathbf{u}\cdot\mathbf{n} &= 0, && \forall x \in \partial\Omega,
\end{aligned}
$$

where $\mathbf{u} \in \mathbb{R}^d$ is the speed, ρ and μ are the density and viscosity, p is the pressure and $\mathbf{f}$ are external forces.

The original Chorin's projection method considers the following discretization in time:

$$
\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} + (\mathbf{u}^n\cdot\nabla)\mathbf{u}^n - \mu\Delta\mathbf{u}^* = \mathbf{f}. \tag{1}
$$

Note that we ignore the pressure (think about operator splitting). The nonlinear term is treated explicitly to avoid the nonlinearity and we treat the viscous term implicitly to avoid extremely small time step restrictions. Equation (1) does not impose $\nabla\cdot\mathbf{u}^* = 0$. To fix this, the projection method considers (again think about operator splitting):

$$
\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\frac{1}{\rho}\nabla p^{n+1}. \tag{2}
$$

Take the divergence of (2) and impose $\nabla\cdot\mathbf{u}^{n+1} = 0$ to get

$$
\Delta p^{n+1} = \frac{\rho}{\Delta t}\nabla\cdot\mathbf{u}^*. \tag{3}
$$

Finally, the divergence-free velocity is given by

$$
\mathbf{u}^{n+1} = \mathbf{u}^* - \frac{\Delta t}{\rho}\nabla p^{n+1}. \tag{4}
$$

Proceed as follows:

1. Consider two discrete spaces. For the velocity and pressure use continuous piecewise bi-quadratic and bilinear polynomials (in 2D)

$$
\begin{aligned}
p_1(x,y) &= c_0 x + c_1 y + c_2 xy + c_3,\\
p_2(x,y) &= c_0 x^2 + c_1 x^2 y + c_2 x^2 y^2 + c_3 y^2 + c_4 x y^2 + c_5 x + c_6 y + c_7 xy + c_8,
\end{aligned}
$$

respectively. How many shape functions do we have for each space in the reference element? Derive the shape functions for the reference element (hint: use tensor products). Plot these shape functions (for each space).

2. Consider the strong forms (1), (3) and (4) and obtain the weak formulation, discrete weak formulation, and the linear algebra representation of the problem (do not compute the entries of the matrices).

Exercise 2: projection of a smooth function to C^0 finite element spaces

From HW1, we consider the function

$$f(x) = \sin^4(2\pi x), \quad \forall x \in [0, 1],$$

and project it to finite element spaces. Proceed as follows:

1. Consider piecewise linear and quadratic continuous polynomials.
2. Consider the reference element $[0, 1]$ and interpolatory basis functions to derive the shape functions for each space.
3. What is the weak formulation and the linear algebra problem associated with the projection?
4. Compute the entries of the mass matrix for each space.
5. Solve the system to obtain the DoF associated with the projection.
6. Plot the projected functions considering $N = 25, 50, 100, 200$ cells.
7. Compute the following (L_1 and L_2) errors

$$E_1 := \int_0^1 |f(x) - f_h(x)| dx, \qquad E_2 := \sqrt{\int_0^1 (f(x) - f_h(x))^2 dx}, \tag{5}$$

   where $f_h(x)$ is the projection. Estimate the order of convergence for each space. That is, assuming the error behaves as follows

$$E = Ch^p,$$

   where C is a constant and $h = 1/N$ is the mesh size, what is the value of p? How does the error behave for the different spaces?

Exercise 3: projection of a non-smooth function to C^0 finite element spaces

Repeat exercise 2 but considering

$$f(x) = \begin{cases} 1, & \text{if } 0.35 \le x \le 0.65, \\ 0, & \text{otherwise} \end{cases}$$

instead.

Exercise 4: advection-diffusion equation via finite differences and finite elements

Consider the one-dimensional advection diffusion equation

$$u_t + u_x - \mu u_{xx} = 0, \qquad \Omega = [0, 1], \tag{6}$$

where $\mu > 0$ is a coefficient. Consider periodic boundary conditions and the following initial condition:

$$u(x, 0) = \sin^4(2\pi x).$$

The exact solution is given by

$$u(x, t) = \frac{3}{8} - \frac{1}{2}e^{-4\mu t}\cos(2(x - t)) + \frac{1}{8}e^{-16\mu t}\cos(4(x - t)).$$

Proceed as follows:

1. Consider continuous piecewise linear polynomials and interpolatory basis functions.
2. Obtain the discrete weak formulation.
3. Identify the different matrices associated with the finite element discretization.
4. Implement and solve the equation via finite elements up to $t = 1$.
5. Compute the L_1 and L_2 errors and perform a convergence test.
6. Consider the finite difference discretization from HW1 and perform a convergence test based on the L_1 and L_2 norms.

Note: if μ is small and Δt is small enough, using forward Euler is feasible. However, forward Euler is a first order method in time and the spatial discretization is second-order. Therefore, the overall error can saturate to first-order. To avoid this problem, choose Δt small enough so that the time discretization errors are negligible.