AMCS 394E: Contemp. Topics in Computational Science.
Computing with the finite element method

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Solving hyperbolic conservation laws

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$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0,$$

where $u \in \mathbb{R}^d$ is a vector of conserved variables and $\mathbf{f}(u)$ is a flux function.

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Main problems when solving hyperbolic conservation laws

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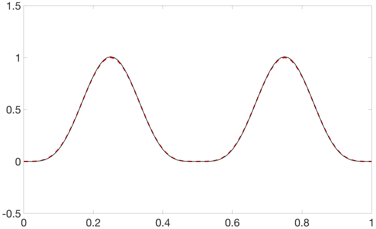

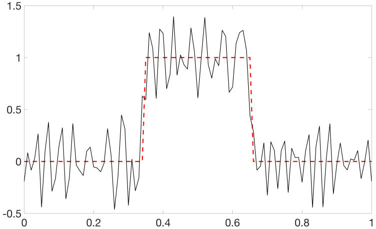

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* We must guarantee their speed of propagation is correct.
* Near shocks, the numerical solution might be more oscillatory.
* We need to consider weak solutions.
* The weak formulation admits multiple solutions.

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Then mass conservation means:

$$\frac{\partial}{\partial_t} \int_\Omega u_h d\mathbf{x} = 0 \iff \int_\Omega u_h(\mathbf{x}, t) d\mathbf{x} = \int_\Omega u_h(\mathbf{x}, 0) d\mathbf{x}.$$

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Integrate by parts to get

$$\partial_t \int_\Omega u\phi d\mathbf{x} = \int_\Omega \mathbf{f}(u) \cdot \nabla\phi d\mathbf{x} + \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{f}(u)\phi d\mathbf{x}$$

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Since $\phi = 1$ is in the space V_h, we can use it in the formulation to get

$$\partial_t \int_\Omega u d\mathbf{x} = \underbrace{\int_\Omega \mathbf{f}(u) \cdot \nabla\phi d\mathbf{x}}_{=0}$$

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In general, we must guarantee

$$\partial_t \int_\Omega u d\mathbf{x} = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{f}(u) d\mathbf{x}$$

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* Combinations like enriched Galerkin FEs DG^0-$CG^p(\Omega)$.

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* Methods based on a (non)linear weak formulation of a dissipative term.
* Algebraic methods based on flux corrections.

Artificial dissipation via continuous Galerkin FEs

The alternatives to impose artificial dissipation via continuous Galerkin finite elements are vast.

The idea is to consider

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = \nabla \cdot c(u) \nabla u,$$

where $c(u) = \mathbb{O}(h^p)$. That is, $c(u) \to 0$ as $h \to 0$.

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* The linearity of $c(u)$.

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A linear and first-order dissipative discretization is

$$\partial_t \int_\Omega u\phi d\mathbf{x} + \int_\Omega \nabla \cdot \mathbf{f}(u)\phi d\mathbf{x} + \sum_K \frac{\Delta x ||\mathbf{f}'(u)||_{L^\infty(K)}}{2} \int_K \nabla u \cdot \nabla \phi d\mathbf{x} = 0,$$

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For the 1D advection equation, we get

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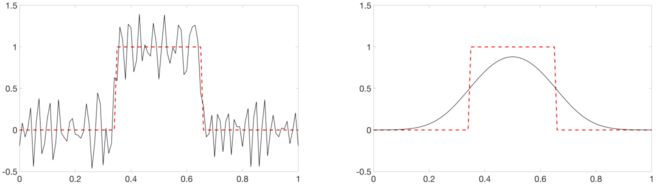

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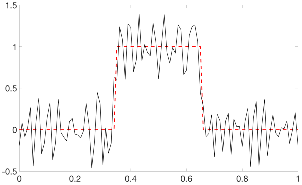
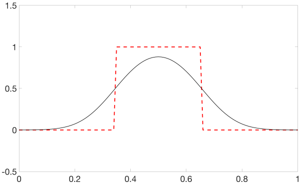


Too oscillatory!



Too dissipated!

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We can consider combinations. For instance,

SUPG + shock capturing

High-order artificial dissipation via SUPG

The SUPG method is given as follows:

$$\int_\Omega \left(\frac{\partial u_h}{\partial t} + \nabla \cdot \mathbf{f}(u_h)\right) \varphi d\mathbf{x} + \sum_K s_K(u_h, \varphi) = 0,$$

where

$$s_K(u_h, \varphi) = \nu_K \int_K \left(\frac{\partial u_h}{\partial t} + \nabla \cdot \mathbf{f}(u_h)\right) \left(\mathbf{f}'(u) \cdot \nabla \varphi\right) d\mathbf{x}.$$

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Here ν_K is a coefficient given as follows

$$\nu_K = \frac{c h_K}{||\mathbf{f}'(u_h)||_{L^\infty(K)}}$$

where c is a user defined constant.

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If the solution is non-smooth, we need to consider instead

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Integrate over the domain and integrate by parts the RHS:

$$\int_\Omega \left[\frac{\partial \eta(u)}{\partial t} + \nabla \cdot \mathbf{q}(u)\right] d\mathbf{x} = -\epsilon \int_\Omega \eta''(u) |\nabla u|^2 d\mathbf{x} \leq 0$$

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$$\int_\Omega \Big[\underbrace{\frac{\partial \eta(u)}{\partial t} + \nabla \cdot \mathbf{q}(u)}_{=R(u)} \Big] d\mathbf{x} \leq 0$$

High-order artificial dissipation via Entropy Viscosity (EV)

The idea of the EV method is to add dissipation
if some local measurement of $R(u)$ is non-zero.

High-order artificial dissipation via Entropy Viscosity (EV)

One version of the EV method is given as follows:

$$\int_{\Omega}\left(\frac{\partial u_h}{\partial t}+\nabla\cdot\mathbf{f}(u_h)\right)\varphi d\mathbf{x}+\sum_{K}s_K^{\mathrm{EV}}(u_h,\varphi)=0,$$

where

$$s_K^{\mathrm{EV}}(u_h,\varphi)=\nu_K^{\mathrm{EV}}\int_K\nabla u_h\cdot\nabla\varphi d\mathbf{x}$$

Here

$$\nu_K^{\mathrm{EV}}=\min\left(\frac{c_1 h_K||\mathbf{f}'(u_h)||_{L^\infty(K)}}{2},\ \frac{c_2 h_K^2||R(u_h)||_{L^\infty(K)}}{||\eta(u_h)-\bar{\eta}(u_h)||_{L^\infty(\Omega)}}\right)$$

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which guarantees

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$$\min_x u_h(\mathbf{x}, t = 0) \leq U_i^{n+1} \leq \max_x u_h(\mathbf{x}, t = 0)$$

Second order Flux Corrected Transport (FCT)

Consider a low-order solution; e.g.,

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$$U_i^H = U_i^L + \frac{1}{m_i}\sum_j f_{ij},$$

Here $f_{ij} := -\Delta t d_{ij}(U_j^n - U_i^n)$ is an anti-diffusive flux.

Algebraic methods for hyperbolic conservation laws

The idea with the FCT method is to add flux limiters $0 \leq \alpha_{ij} \leq 1$ that guarantee

$$U_i^{\min} = \min_{j \in N_i} U_j^n \leq U_i^{n+1} \leq \max_{j \in N_i} U_j^n = U_i^{\max},$$

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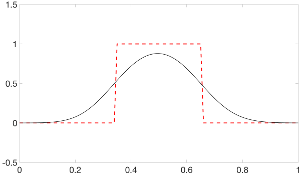
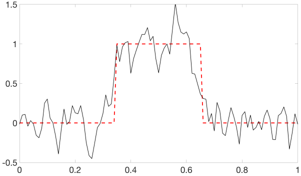
Consider the 1D advection equation $u_t + u_x = 0$.

Low-order solution



High-order (with mass lumping)



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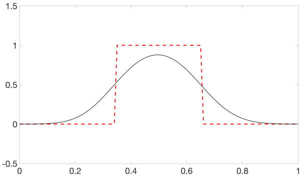

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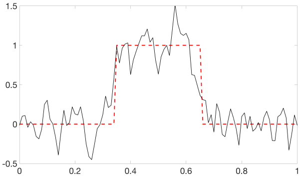
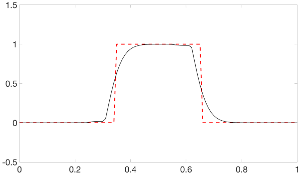


FCT with mass lumping errors



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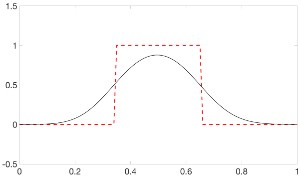
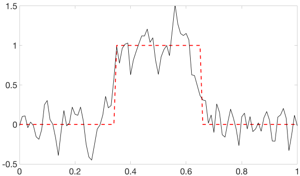
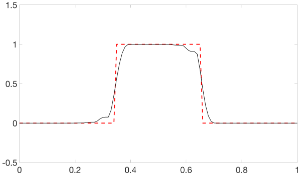


High-order (with mass lumping)

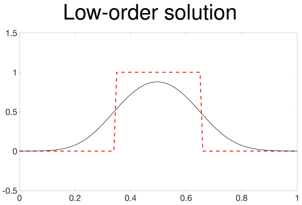


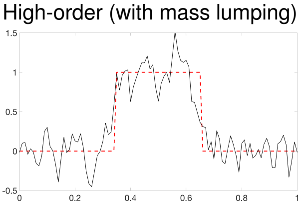
FCT with mass lumping correction

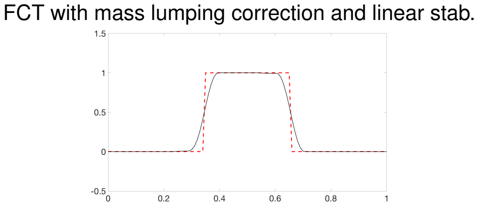


Algebraic methods for hyperbolic conservation laws

Consider the 1D advection equation $u_t + u_x = 0$ with $(\Delta x = 0.01)$.



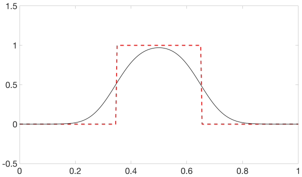




Algebraic methods for hyperbolic conservation laws

Consider the 1D advection equation $u_t + u_x = 0$ with $(\Delta x = 0.01/2)$.

Low-order solution



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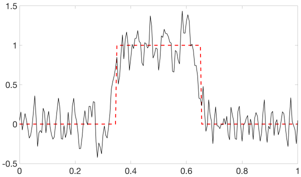
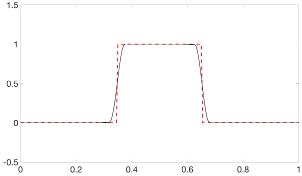


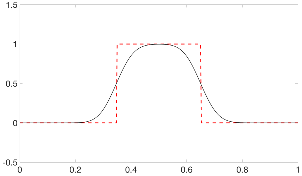
FCT with mass lumping correction and linear stab.



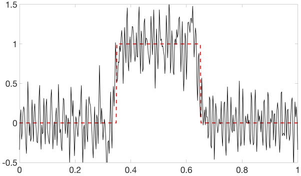
Algebraic methods for hyperbolic conservation laws

Consider the 1D advection equation $u_t + u_x = 0$ with $(\Delta x = 0.01/4)$.

Low-order solution



High-order (with mass lumping)



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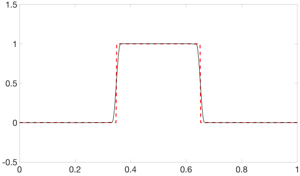

Algebraic methods for hyperbolic conservation laws

Consider the 1D advection equation $u_t + u_x = 0$ with $(\Delta x = 0.01/8)$.

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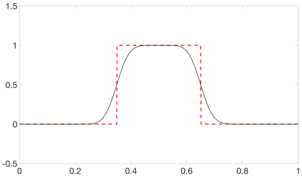
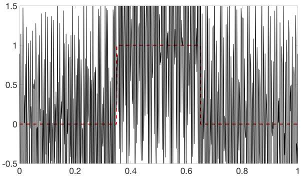


High-order (with mass lumping)



FCT with mass lumping correction and linear stab.