Projection of (non-)smooth functions

Problem

Consider the domain $\Omega = [0,1]^2$. Our objective is project

$$f(\mathbf{x}) = \sin^4(2\pi x)\sin^4(2\pi y),$$

to the finite element space

$$V_h = \{v \in C^0(\Omega) : v|_K \in \mathbb{P}^p(K)\},$$

where $p = 1, 2, 3$ and 4, and to verify the order of convergence using the L_2-norm.

Weak formulation

The discrete weak formulation is given as follows. Find $f_h(\mathbf{x}) \in V_h$ such that

$$\int_\Omega f_h(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \int_\Omega f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}, \quad \forall\varphi(\mathbf{x}) \in V_h.$$

Since $f_h(\mathbf{x}) \in V_h$, $f_h(\mathbf{x}) = \sum_j F_j\varphi_j(\mathbf{x})$. Therefore,

$$\sum_j F_j \underbrace{\int_\Omega \varphi_i(\mathbf{x})\varphi_j(\mathbf{x})d\mathbf{x}}_{=m_{ij}} = \underbrace{\int_\Omega f(\mathbf{x})\varphi_i(\mathbf{x})d\mathbf{x}}_{=r_i}.$$

Numerical results

Let us solve the problem with different polynomial spaces. The summary of a convergence test for the projection of $f(\mathbf{x})$ is shown in Table 1.

References

- For the general structure of a similar code, see
  https://www.dealii.org/9.0.0/doxygen/deal.II/step40.html.
- To create simple domains, see
  https://www.dealii.org/9.0.0/doxygen/deal.II/step_1.html,
  https://www.dealii.org/current/doxygen/deal.II/namespaceGridGenerator.html.

Cells	$p=1$		$p=2$		$p=3$		$p=4$	
	E_2	rate	E_2	rate	E_2	rate	E_2	rate
256	5.04e-03	–	9.68e-04	–	4.83e-05	–	4.60e-06	–
1024	5.83e-04	3.11	1.54e-04	2.65	1.49e-06	5.02	1.87e-07	4.62
4096	7.14e-05	3.03	2.07e-05	2.89	4.64e-08	5.01	6.28e-09	4.90
16384	8.89e-06	3.01	2.64e-06	2.97	1.45e-09	5.00	2.00e-10	4.97
65536	1.11e-06	3.00	3.32e-07	2.99	4.52e-11	5.00	6.27e-12	4.99

Table 1: Convergence results of the projection of $f(\mathbf{x})$.

- To generate a triangulation, see

  https://www.dealii.org/current/doxygen/deal.II/classTriangulation.html,

  https://www.dealii.org/current/doxygen/deal.II/classparallel_1_1distributed_1_1Triangulation.html.

- To generate a finite element space, see

  https://www.dealii.org/current/doxygen/deal.II/group__fe.html,

  https://www.dealii.org/current/doxygen/deal.II/classFE__Q.html.

- To generate a DoF handler, see

  https://www.dealii.org/current/doxygen/deal.II/group__dofs.html,

  https://www.dealii.org/current/doxygen/deal.II/classDoFHandler.html.

- To compute error norms, see

  https://www.dealii.org/current/doxygen/deal.II/step_7.html and

  integrate_difference in

  https://www.dealii.org/current/doxygen/deal.II/namespaceVectorTools.html.

- To generate the convergence tables, see

  https://www.dealii.org/current/doxygen/deal.II/step_7.html,

  https://www.dealii.org/current/doxygen/deal.II/classConvergenceTable.html.