$\boxed{\text{WEEK 7}}$

$\boxed{\text{SUPG}}$

$$\int_\Omega \left[\frac{\partial u_h}{\partial t} + \nabla\cdot\vec{f}(u_h)\right]\varphi\,dx + \sum_k \nu_k \int_k \left[\frac{\partial u_h}{\partial t} + \nabla\cdot\vec{f}(u_h)\right] f'(u_h)\cdot\nabla\varphi\,dx = 0$$

$$\Rightarrow \sum_k \int_k \left[\frac{\partial u_h}{\partial t} + \nabla\cdot\vec{f}(u_h)\right]\underbrace{\left[\varphi + \nu_k f'(u_h)\cdot\nabla\varphi\right]}_{=:\,\psi} dx$$

$$\boxed{\sum_k \int_k \left[\frac{\partial u_h}{\partial t} + \nabla\cdot\vec{f}(u_h)\right]\psi\,dx}$$

- The method is a Petrov–Galerkin Finite element because the test and trial spaces are NOT the same:

$$u_h = \sum_j u_j \varphi_j \in V_h \quad\text{and}\quad \psi \notin V_h$$

* For advection:

$$\sum_k \int_k \left(\frac{\partial u_h}{\partial t} + \vec{a}\cdot\nabla u_h\right)\left(\varphi + \nu_k \vec{a}\cdot\nabla\varphi\right)dx = 0$$

$$\int_\Omega \left(\frac{\partial u_h}{\partial t} + a\cdot\nabla u_h\right)\varphi\,dx + \sum_k \nu_k \underbrace{\int_k (\vec{a}\cdot\nabla u_h)(\vec{a}\cdot\nabla\varphi)\,dx}_{(*)} = 0$$

(*) Is a 2nd order operator that acts only in the direction of the flow; in the "stream upwind" direction.

Consider for example

- If the solution $U_h(x)$ is smooth in the direction of the flow then we don't need a lot of dissipation.

- If U_h has a strong gradient perpendicular to $\vec{a}$, then we don't need to dissipate this gradient.

FIRST-ORDER (ALGEBRAIC) METHODS:

$$\partial_t U + a U_x = 0 \Rightarrow \partial_t \int_\Omega U_h \varphi \, dx + \int_\Omega (aU_h)_x \varphi \, dx = 0$$

$$\partial_t \int_\Omega U_h \varphi_i \, dx = \partial_t \int_\Omega \sum_j U_j \varphi_j \varphi_i \, dx = \sum_j \dot{U}_j \overbrace{\int_\Omega \varphi_i \varphi_j \, dx}^{= m_{ij}}$$

$$\simeq m_i \dot{U}_i \simeq m_i \left(\frac{U_i^{n+1} - U_i^n}{\Delta t} \right)$$

$$\int_\Omega (a U_h)_x \varphi_i \, dx = \sum_j U_j \, a \underbrace{\int_\Omega \left(\frac{\partial \varphi_j}{\partial x} \right) \varphi_i \, dx}_{= c_{ij}} = a \sum_j c_{ij} U_j$$

Then,

$$m_i \left(\frac{v_i^{n+1} - v_i^n}{\Delta t} \right) + a \sum_j c_{ij} v_j^n = 0$$

- The idea is to add another operator to get:

$$m_i \left(\frac{v_i^{n+1} - v_i^n}{\Delta t} \right) + \sum_j a c_{ij} v_j^n - \sum_j d_{ij} v_j^n = 0$$

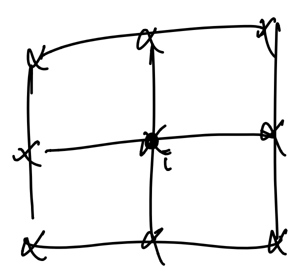
where d_{ij} has the following properties:

$$d_{ij} \geq 0, \qquad d_{ij} \geq |a c_{ij}| \quad \forall i = j$$

$$d_{ij} = d_{ji} \qquad \sum_j d_{ij} = 0$$

- Then, the following is true: (assuming Δt is small enough)

$$v_i^{min} = \min_{j \in N_i} v_j^n \leq v_i^{n+1} \leq \max_{j \in N_i} v_j^n = v_i^{max}$$



N_i

- Consider

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{m_i} \sum_j \left(a c_{ij} - d_{ij}\right) v_j^n$$

$$= \sum_j R_{ij} v_j^n, \qquad R_{ij} = \delta_{ij} - \frac{\Delta t}{m_i}\left(a c_{ij} - d_{ij}\right)$$

- Note that $\quad a c_{ij} - d_{ij} \leq 0 \quad \forall i \neq j \implies R_{ij} \geq 0 \ \forall i \neq j$

$$R_{ii} = 1 - \frac{\Delta t}{m_i}\left(a c_{ii} - d_{ii}\right) \geq 0$$ provided Δt is small enough

- Note that $\quad \displaystyle\sum_j a c_{ij} - d_{ij} = a \underbrace{\sum_j \int_\Omega \left(\frac{\partial \varphi_j}{\partial x}\right) \varphi_i \, dx}_{=0} - \underbrace{\sum_j d_{ij}}_{=0} = 0$

$$\implies \sum_j R_{ij} = 1.$$

- Then, $\quad R_{ij} \geq 0$ and $\displaystyle\sum_j R_{ij} = 1.$

$$\implies \quad v_i^{n+1} = \sum_j R_{ij} v_j^n \leq \sum_j R_{ij} v_i^{max} = v_i^{max} \underbrace{\sum_j R_{ij}}_{=1}$$

$$\implies v_i^{n+1} \leq v_i^{max}$$