# Poisson equation with Dirichlet Boundary conditions

## Problem

Our objective is to solve

$$\begin{aligned} -\Delta u &= f(\mathbf{x}) = 8\pi^2 \sin(2\pi x)\sin(2\pi y), && \forall \mathbf{x} \in \Omega = (0,1)^2, \\ u(\mathbf{x}) &= 0, && \forall \mathbf{x} \in \partial\Omega. \end{aligned}$$

The exact solution for this problem is

$$u(\mathbf{x}) = \sin(2\pi x)\sin(2\pi y).$$

## Weak formulation

The discrete weak formulation is given as follows. Find $u_h(\mathbf{x}) \in V_h$ such that

$$\int_\Omega \nabla u_h(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) d\mathbf{x} = \int_\Omega f(\mathbf{x})\varphi(\mathbf{x}) d\mathbf{x}, \qquad \forall \varphi \in V_h$$

where

$$V_h = \{v \in C^0(\Omega) : v_K \in \mathbb{P}^p(K), v(\mathbf{x}) = 0 \ \forall \mathbf{x} \in \partial\Omega\}.$$

Since $u_h(\mathbf{x}) \in V_h$, $u_h(\mathbf{x}) = \sum_j U_j \varphi(\mathbf{x})$. Therefore, we get the linear system

$$\sum_j U_j \underbrace{\int_\Omega \nabla \varphi_i(\mathbf{x}) \cdot \nabla \varphi_j(\mathbf{x}) d\mathbf{x}}_{=S_{ij}} = \underbrace{\int_\Omega f(\mathbf{x})\varphi_i(\mathbf{x}) d\mathbf{x}}_{=r_i}, \qquad \forall i = 0, \ldots, N_h - 1.$$

## Numerical results

Let us solve the problem with different polynomial spaces.

## References

- The solution for a similar problem (with adaptive mesh refinement) is given in https://www.dealii.org/current/doxygen/deal.II/step_40.html
- To impose Dirichlet boundary conditions strongly, see interpolate_boundary_values in https://www.dealii.org/current/doxygen/deal.II/namespaceVectorTools.html