Solving the Navier-Stokes equations via a projection scheme

Consider the incompressible Navier-Stokes equations in non-conservative form

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \mu\mathit{\Delta}\mathbf{u} + \nabla p = \mathbf{f}, \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1b}$$

where ρ and μ are the density and viscosity of the fluid, $\mathbf{u} = [u, v, w]^T$ is the velocity, p is the pressure and $\mathbf{f}$ is an external force. Our objective is to solve a two-dimensional version of (1) via the second-order projection scheme by [1]. The algorithm is given as follows:

$$\rho\left(\frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\mathit{\Delta}t}\right) + \rho\left(\mathbf{u}^*\cdot\nabla\mathbf{u}^{n+1}\right) - \mu\mathit{\Delta}\mathbf{u}^{n+1} + \nabla p^* = \mathbf{f}^{n+1}, \qquad \mathbf{u}^{n+1}|_{\partial\Omega} = 0,$$

where

$$\mathbf{u}^* = 2\mathbf{u}^n - \mathbf{u}^{n-1}, \qquad p^* = p^n + \frac{4}{3}\phi^n - \frac{1}{3}\phi^{n-1}.$$

Here ϕ is an auxiliary variable that represents an increment in pressure and is given by

$$\mathit{\Delta}\phi^{n+1} = \frac{3\rho}{2\mathit{\Delta}t}\nabla\cdot\mathbf{u}^{n+1}, \quad \nabla\phi^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0,$$

Finally, the pressure is updated as follows:

$$p^{n+1} = p^n + \phi^{n+1}.$$

1 First step: solving the heat equation via BDF2

Let us start by solving

$$\rho\left(\frac{\partial u}{\partial t}\right) - \mu\mathit{\Delta}u = f(\mathbf{x}),$$

$$u(\mathbf{x}) = 0, \qquad \mathbf{x}\in\partial\Omega,$$

via second-order backward differences. That is, we consider

$$\rho\left(\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\mathit{\Delta}t}\right) - \mu\mathit{\Delta}u^{n+1} = f^{n+1}, \qquad u^{n+1}|_{\partial\Omega} = 0.$$

We can use our implementation for the heat equation.

Weak formulation

The discrete weak formulation is given by

$$\rho \int_{\Omega} \left(\frac{3u_h^{n+1} - 4u_h^n + u_h^{n-1}}{2\Delta t} \right) \varphi d\mathbf{x} + \mu \int_{\Omega} \nabla u_h^{n+1} \cdot \nabla \varphi d\mathbf{x} = \int_{\Omega} f\left(\mathbf{x}, t^{n+1}\right) \varphi d\mathbf{x}, \qquad \forall \varphi \in V_h. \quad (2)$$

Using the fact that $u_h(\mathbf{x}) = \sum_j U_j \varphi(\mathbf{x})$, we get

$$\sum_j U_j^{n+1} \underbrace{\int_{\Omega} \left[\varphi_i \varphi_j + \frac{2\Delta t \mu}{3\rho} \left(\nabla \varphi_i \cdot \nabla \varphi_j \right) \right] d\mathbf{x}}_{= M_{ij} + \frac{2\Delta t \mu}{3\rho} S_{ij}} = \underbrace{\int_{\Omega} \left(\frac{4}{3} u_h^n - \frac{1}{3} u_h^{n-1} \right) \varphi d\mathbf{x} + \frac{2\Delta t}{3\rho} \int_{\Omega} f(\mathbf{x}, t^{n+1}) \varphi_i d\mathbf{x}}_{= r_i^{n+1}}$$

Therefore, we get

$$\left(M + \frac{2\Delta t \mu}{3\rho} S \right) U^{n+1} = R^{n+1}.$$

Numerical experiment

Consider $\rho = 1$, $\mu = 1$ and

$$\begin{aligned} u(\mathbf{x}, 0) &= u_0(\mathbf{x}) = \sin(2\pi x)\sin(2\pi y), \\ f(\mathbf{x}, t) &= 2\pi \left[4\pi \cos(2\pi t) - \sin(2\pi t) \right] \sin(2\pi x)\sin(2\pi y), \end{aligned}$$

and solve the equation up to $t = 1$. The exact solution is given by

$$u = \cos(2\pi t)\sin(2\pi x)\sin(2\pi y).$$

The results of a convergence test are shown in Table 1.

Cells	DoFs	E_2	rate
256	289	6.76e-03	–
1024	1089	1.66e-03	2.03
4096	4225	4.07e-04	2.03
16384	16641	1.01e-04	2.01
65536	66049	2.53e-05	2.00

Table 1: Convergence test for the heat equation.

2 Second step: solving two heat equations via BDF2

Now let us solve

$$\begin{aligned} \rho \left(\frac{\partial u}{\partial t} \right) - \mu \Delta u &= f(\mathbf{x}, t), \qquad u(\mathbf{x}, t) = u_B(\mathbf{x}, t), \qquad \mathbf{x} \in \partial\Omega, \\ \rho \left(\frac{\partial v}{\partial t} \right) - \mu \Delta v &= g(\mathbf{x}, t), \qquad v(\mathbf{x}, t) = v_B(\mathbf{x}, t), \qquad \mathbf{x} \in \partial\Omega, \end{aligned}$$

via second-order backward differences. That is, we consider

$$\rho\left(\frac{3u^{n+1}-4u^n+u^{n-1}}{2\Delta t}\right)-\mu\Delta u^{n+1}=f^{n+1}, \qquad u^{n+1}|_{\partial\Omega}=u_B(\mathbf{x},t^{n+1}),$$
$$\rho\left(\frac{3v^{n+1}-4v^n+v^{n-1}}{2\Delta t}\right)-\mu\Delta v^{n+1}=g^{n+1}, \qquad v^{n+1}|_{\partial\Omega}=v_B(\mathbf{x},t^{n+1}).$$

We need to duplicate the implementation to solve for each variable u and v. Later, we will couple these two equations via the nonlinear term in the Navier-Stokes equations.

Numerical experiment

Test the implementation and verify we get second-order convergence for each component. Consider $\rho = 1$, $\mu = 1$,

$$u(\mathbf{x},0)=u_0(\mathbf{x})=\sin(x)\sin(y), \qquad v(\mathbf{x},0)=v_0(\mathbf{x})=\cos(x)\cos(y),$$

and

$$f(\mathbf{x},t)=\underbrace{\cos(t+y)\sin(x)}_{\text{time derivative}}+\underbrace{2\mu\sin(x)\sin(t+y)}_{\text{viscosity}},$$
$$g(\mathbf{x},t)=\underbrace{-\cos(x)\sin(t+y)}_{\text{time derivative}}+\underbrace{2\mu\cos(x)\cos(t+y)}_{\text{viscosity}}$$

Solve the equation up to $t = 1$. The exact solution is given by

$$u=\sin(x)\sin(y+t), \qquad v=\cos(x)\cos(y+t).$$

The results of a convergence test are shown in Table 2.

Cells	DoFs	$E_2(u)$	rate	$E_2(v)$	rate
256	289	3.16E-04	–	1.57E-04	–
1024	1089	7.90E-05	2.00	3.93E-05	2.00
4096	4225	1.97E-05	2.00	9.88E-06	1.99
16384	16641	4.93E-06	2.00	2.47E-06	2.00
65536	66049	1.23E-06	2.00	6.17E-07	2.00

Table 2: Convergence test for the heat equation.

3 Third step: include the nonlinear term

By considering the nonlinear term

$$\rho\mathbf{u}^*\cdot\nabla\mathbf{u}^{n+1}=\rho\begin{bmatrix}u^*\\v^*\end{bmatrix}\cdot\begin{bmatrix}u_x^{n+1} & u_y^{n+1}\\ v_x^{n+1} & v_y^{n+1}\end{bmatrix}=\rho\begin{bmatrix}u^*u_x^{n+1}+v^*u_y^{n+1}\\ u^*v_x^{n+1}+v^*v_y^{n+1}\end{bmatrix},$$

we get

$$
\begin{aligned}
\rho\left(\frac{3u^{n+1}-4u^{n}+u^{n-1}}{2\Delta t}\right)+\rho\left(u^{*}u_{x}^{n+1}+v^{*}u_{y}^{n+1}\right)-\mu\Delta u^{n+1}&=f^{n+1}, \qquad u^{n+1}|_{\partial\Omega}=u_B(\mathbf{x},t^{n+1}),\\
\rho\left(\frac{3v^{n+1}-4v^{n}+v^{n-1}}{2\Delta t}\right)+\rho\left(u^{*}v_{x}^{n+1}+v^{*}v_{y}^{n+1}\right)-\mu\Delta v^{n+1}&=g^{n+1}, \qquad v^{n+1}|_{\partial\Omega}=v_B(\mathbf{x},t^{n+1}).
\end{aligned}
$$

Weak formulation

The discrete weak formulation changes in the definition of the matrices. Now we have

$$
\begin{aligned}
\sum_j U_j^{n+1}\int_\Omega\left[\varphi_i\varphi_j+\frac{2\Delta t}{3}\left(u_h^*\partial_x\varphi_j+v_h^*\partial_y\varphi_j\right)\varphi_i+\frac{2\Delta t\mu}{3\rho}\left(\nabla\varphi_i\cdot\nabla\varphi_j\right)\right]d\mathbf{x}&=r_i^{(u),n+1},\\
\sum_j V_j^{n+1}\int_\Omega\left[\varphi_i\varphi_j+\frac{2\Delta t}{3}\left(u_h^*\partial_x\varphi_j+v_h^*\partial_y\varphi_j\right)\varphi_i+\frac{2\Delta t\mu}{3\rho}\left(\nabla\varphi_i\cdot\nabla\varphi_j\right)\right]d\mathbf{x}&=r_i^{(v),n+1}.
\end{aligned}
$$

Numerical experiment

Test the implementation and verify we get second-order convergence for each component. Consider $\rho=1$, $\mu=1$,

$$
u(\mathbf{x},0)=u_0(\mathbf{x})=\sin(x)\sin(y), \qquad v(\mathbf{x},0)=v_0(\mathbf{x})=\cos(x)\cos(y),
$$

and

$$
\begin{aligned}
f(\mathbf{x},t)&=\underbrace{\cos(t+y)\sin(x)}_{\text{time derivative}}+\underbrace{2\mu\sin(x)\sin(t+y)}_{\text{viscosity}}+\underbrace{\cos(x)\sin(x)}_{\text{nonlinearity}},\\
g(\mathbf{x},t)&=\underbrace{-\cos(x)\sin(t+y)}_{\text{time derivative}}+\underbrace{2\mu\cos(x)\cos(t+y)}_{\text{viscosity}}-\underbrace{\frac{1}{2}\sin(2(t+y))}_{\text{nonlinearity}},
\end{aligned}
$$

Solve the equation up to $t=1$. The exact solution is given by

$$
u=\sin(x)\sin(y+t), \qquad v=\cos(x)\cos(y+t).
$$

The results of a convergence test are shown in Table 3.

Cells	DoFs	$E_2(u)$	rate	$E_2(v)$	rate
256	289	3.00E-04	–	1.69E-04	–
1024	1089	7.50E-05	2.00	4.22E-05	2.00
4096	4225	1.87E-05	2.00	1.06E-05	2.00
16384	16641	4.68E-06	2.00	2.64E-06	2.00

Table 3: Convergence test including the nonlinear terms.

4 Fourth step: solve the Poisson equation in the algorithm

At each time step, after solving the momentum equation, we need to solve

$$-\Delta\phi^{n+1} = -\frac{3\rho}{2\Delta t}\nabla\cdot\mathbf{u}^{n+1}, \qquad \nabla\phi^{n+1}\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

Weak formulation

The discrete weak formulation is given as follows:

$$\int_{\Omega}\nabla\phi_h^{n+1}\cdot\nabla\varphi d\mathbf{x} - \underbrace{\int_{\partial\Omega}(\nabla\phi_h^{n+1}\cdot\mathbf{n})\varphi d\mathbf{x}}_{=0} = -\frac{3\rho}{2\Delta t}\int_{\Omega}(\nabla\cdot\mathbf{u}_h^{n+1})\varphi d\mathbf{x}.$$

Remark: we will use a different space for the pressure; therefore, in the implementation, we must create a new space, DoFHandler, index sets for locally relevant and locally owned DoFs, etc.

5 Fifth step: combine momentum and Poisson equations

Up to this point, we are solving a nonlinear advection-diffusion-reaction equation to obtain u and v and a Poisson equation to obtain ϕ. Now, we need to combine these equations to solve the Navier-Stokes equations. We get

$$\rho\left(\frac{3u^{n+1}-4u^n+u^{n-1}}{2\Delta t}\right)+\rho\left(u^*u_x^{n+1}+v^*u_y^{n+1}\right)+p_x^n+\frac{4}{3}\phi_x^n-\frac{1}{3}\phi_x^{n-1}-\mu\Delta u^{n+1}=f^{n+1},$$
$$\rho\left(\frac{3v^{n+1}-4v^n+v^{n-1}}{2\Delta t}\right)+\rho\left(u^*v_x^{n+1}+v^*v_y^{n+1}\right)+p_y^n+\frac{4}{3}\phi_y^n-\frac{1}{3}\phi_y^{n-1}-\mu\Delta v^{n+1}=g^{n+1},$$

Weak formulation

The discrete weak formulation changes in the right hand side. Now we have

$$\sum_j U_j^{n+1}\int_{\Omega}[\dots]\,d\mathbf{x} = r_i^{(u),n+1} = \dots - \frac{2\Delta t}{3\rho}\int_{\Omega}\left(p_x^n+\frac{4}{3}\phi_x^n-\frac{1}{3}\phi_x^{n-1}\right)\varphi d\mathbf{x},$$
$$\sum_j V_j^{n+1}\int_{\Omega}[\dots]\,d\mathbf{x} = r_i^{(v),n+1} = \dots - \frac{2\Delta t}{3\rho}\int_{\Omega}\left(p_y^n+\frac{4}{3}\phi_y^n-\frac{1}{3}\phi_y^{n-1}\right)\varphi d\mathbf{x},$$

Remark: for stability reasons we will consider the so called Taylor-Hood finite elements, which consist on using $\mathbb{Q}^{k+1}$ spaces for the velocity and a $\mathbb{Q}^k$ space for the pressure. In particular, let's consider $k=1$.

Numerical experiment

Test the implementation and verify we get second-order convergence for each component. Consider $\rho = 1$, $\mu = 1$,

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) = \sin(x)\sin(y), \qquad v(\mathbf{x}, 0) = v_0(\mathbf{x}) = \cos(x)\cos(y),$$

and

$$\begin{aligned}
f(\mathbf{x}, t) &= \underbrace{\cos(t+y)\sin(x)}_{\text{time derivative}} + \underbrace{2\mu\sin(x)\sin(t+y)}_{\text{viscosity}} + \underbrace{\cos(x)\sin(x)}_{\text{nonlinearity}} - \underbrace{\sin(x)\sin(t+y)}_{\text{pressure}},\\
g(\mathbf{x}, t) &= \underbrace{-\cos(x)\sin(t+y)}_{\text{time derivative}} + \underbrace{2\mu\cos(x)\cos(t+y)}_{\text{viscosity}} - \underbrace{\frac{1}{2}\sin(2(t+y))}_{\text{nonlinearity}} + \underbrace{\cos(x)\cos(t+y)}_{\text{pressure}}
\end{aligned}$$

Solve the equation up to $t = 1$. The exact solution is given by

$$u = \sin(x)\sin(y+t), \qquad v = \cos(x)\cos(y+t), \qquad p = \cos(x)\sin(y+t).$$

The results of a convergence test are shown in Table 4.

Cells	DoFs	$E_2(u)$	rate	$E_2(v)$	rate	$E_2(p)$	rate
256	1089	9.54E-06	–	4.80e-05	–	8.90e-04	–
1024	4225	2.35E-06	2.02	1.19e-05	2.01	3.42e-04	1.38
4096	16641	5.58E-07	2.07	2.85e-06	2.07	1.44e-04	1.25
16384	66049	1.39E-07	2.00	7.11e-07	2.00	6.73e-05	1.10

Table 4: Convergence test for the projection scheme.

References

[1] J-L. Guermond, A. Salgado, A splitting method for incompressible flows with variable density based on a pressure Poisson. *J. Comput. Phys.* **228** (2009) 2834–2846.