Projection of (non-)smooth functions

Problem

Consider the domain $\Omega = [0,1]^2$. Our objective is project

$$f(\mathbf{x}) = \sin^4(2\pi x)\sin^4(2\pi y),$$
$$g(\mathbf{x}) = \begin{cases} 1, & \text{if } x \in [0.35, 0.65] \text{ and } y \in [0.35, 0.65], \\ 0, & \text{otherwise} \end{cases}$$

to the finite element space

$$V_h = \{v \in C^0(\Omega) : v|_K \in \mathbb{P}^p(K)\},$$

where $p = 1, 2, 3$ and 4, and to verify the order of convergence using the L_2-norm.

Weak formulation

The discrete weak formulation is given as follows. Find $f_h(\mathbf{x}), g_h(\mathbf{x}) \in V_h$ such that

$$\int_\Omega f_h(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \int_\Omega f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}, \quad \forall\varphi(\mathbf{x}) \in V_h,$$
$$\int_\Omega g_h(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} = \int_\Omega g(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}, \quad \forall\varphi(\mathbf{x}) \in V_h.$$

Since $f_h(\mathbf{x}) \in V_h$, $f_h(\mathbf{x}) = \sum_j F_j\varphi_j(\mathbf{x})$. Therefore,

$$\sum_j F_j \underbrace{\int_\Omega \varphi_i(\mathbf{x})\varphi_j(\mathbf{x})d\mathbf{x}}_{=m_{ij}} = \underbrace{\int_\Omega f(\mathbf{x})\varphi_i(\mathbf{x})d\mathbf{x}}_{=r_i},$$

and similarly for $g_h(\mathbf{x})$.

Numerical results

Let us solve the problem with different polynomial spaces. The summary of a convergence test for the projection of $f(\mathbf{x})$ and $g(\mathbf{x})$ is shown in Tables 1 and 2, respectively.

Cells	$p=1$		$p=2$		$p=3$		$p=4$	
	E_2	rate	E_2	rate	E_2	rate	E_2	rate
1536	4.85e-03	–	1.06e-04	–	2.26e-06	–	4.56e-08	–
6144	1.21e-03	2.00	1.32e-05	3.00	1.41e-07	4.00	1.42e-09	5.00
24576	3.03e-04	2.00	1.65e-06	3.00	8.82e-09	4.00	4.45e-11	5.00
98304	7.59e-05	2.00	2.07e-07	3.00	5.52e-10	4.00	1.51e-12	4.88
393216	1.90e-05	2.00	2.58e-08	3.00	3.47e-11	3.99	9.51e-12	-2.65

Table 1: Convergence results of the projection of $f(\mathbf{x})$.

Cells	$p=1$		$p=2$		$p=3$		$p=4$	
	E_2	rate	E_2	rate	E_2	rate	E_2	rate
1536	4.85e-03	–	1.06e-04	–	2.26e-06	–	4.56e-08	–
6144	1.21e-03	2.00	1.32e-05	3.00	1.41e-07	4.00	1.42e-09	5.00
24576	3.03e-04	2.00	1.65e-06	3.00	8.82e-09	4.00	4.45e-11	5.00
98304	7.59e-05	2.00	2.07e-07	3.00	5.52e-10	4.00	1.51e-12	4.88
393216	1.90e-05	2.00	2.58e-08	3.00	3.47e-11	3.99	9.51e-12	-2.65

Table 2: Convergence results of the projection of $g(\mathbf{x})$.

References

- For the general structure of a similar code, see

  https://www.dealii.org/9.0.0/doxygen/deal.II/step40.html.

- To create simple domains, see

  https://www.dealii.org/9.0.0/doxygen/deal.II/step_1.html,

  https://www.dealii.org/current/doxygen/deal.II/namespaceGridGenerator.html.

- To generate a triangulation, see

  https://www.dealii.org/current/doxygen/deal.II/classTriangulation.html,

  https://www.dealii.org/current/doxygen/deal.II/classparallel_1_1distributed_1_1Triangulation.html.

- To generate a finite element space, see

  https://www.dealii.org/current/doxygen/deal.II/group__fe.html,

  https://www.dealii.org/current/doxygen/deal.II/classFE__Q.html.

- To generate a DoF handler, see

  https://www.dealii.org/current/doxygen/deal.II/group__dofs.html,

  https://www.dealii.org/current/doxygen/deal.II/classDoFHandler.html.

- To compute error norms, see

  https://www.dealii.org/current/doxygen/deal.II/step_7.html and

integrate_difference in
https://www.dealii.org/current/doxygen/deal.II/namespaceVectorTools.html.

- To generate the convergence tables, see
  https://www.dealii.org/current/doxygen/deal.II/step_7.html,
  https://www.dealii.org/current/doxygen/deal.II/classConvergenceTable.html.