Homework 3

Exercise 1: projection of a smooth function to C^0 finite element spaces

Consider the function

$$f(x) = \sin^4(2\pi x)\sin^4(2\pi y), \quad \forall (x,y) \in [0,1]^2,$$

and project it to a finite element space using triangular elements and piecewise linear polynomials. Consider $N_{\text{el}} = 100, 400, 1600, 6400$ elements. For each refinement level compute the errors

$$E_1 := \int_0^1 |f(x) - f_h(x)| dx, \qquad E_2 := \sqrt{\int_0^1 (f(x) - f_h(x))^2 dx},$$

where $f_h(x)$ is the projection. Report the four plots and the convergence table.

Exercise 2: projection of a non-smooth function to C^0 finite element spaces

Repeat exercise 1 but considering

$$f(x) = \begin{cases} 1, & \text{if } x \in [0.35, 0.65] \text{ and } y \in [0.35, 0.65], \\ 0, & \text{otherwise} \end{cases}$$

instead.

Exercise 3: solve the Poisson equation

Consider the two-dimensional Poisson equation

$$\begin{aligned} -\Delta u &= f, \qquad && \Omega = [0,1]^2, \\ u &= 0, && \forall x \in \partial\Omega. \end{aligned}$$

where

$$f = \begin{cases} 1, & \text{if } y > 0.5 + 0.25\sin(4\pi x), \\ -1, & \text{if } y \le 0.5 + 0.25\sin(4\pi x)). \end{cases}$$

Consider a finite element space using triangular elements and piecewise linear polynomials. Solve the equation using $N_{\text{el}} = 100, 400, 1600, 6400$ elements. Report the four plots.

Exercise 4: solve the Eikonal equation in a unit square

In this exercise, you can use either quadrilateral or triangular elements with piecewise bilinear or linear, respectively, polynomials. The domain is $\Omega = [0,1]^2$. Consider

$$\tilde{u}(x,y) = \begin{cases} h, & \text{if } \sqrt{(x-0.5)^2 + (y-0.5)^2} \leq 0.25, \\ -h, & \text{if } \sqrt{(x-0.5)^2 + (y-0.5)^2} > 0.25, \end{cases}$$

where $h = \Delta x = \Delta y$ is the mesh size. Use the lumped mass matrix to project $\tilde{u}$ into the finite element space to obtain

$$\tilde{u}_h(x,y) = \sum_j \tilde{U}_j \phi_j(x,y).$$

Consider the following discrete weak formulation:

$$\alpha m_i^{\Gamma(\tilde{u}_h)}(U_i - \tilde{U}_i) + \int_\Omega S_\epsilon(\tilde{u}_h)\left[|\nabla u_h| - 1\right]\phi d\mathbf{x} + ch\int_\Omega \nabla u_h \cdot \nabla \phi d\mathbf{x} = 0, \qquad \forall \phi \in V_h \tag{1}$$

where $\alpha = 10^{10}$, $c = 0.5$, $m_i^{\Gamma(\tilde{u}_h)} = \int_\Omega \delta_\epsilon(\tilde{u}_h)\phi_i d\mathbf{x}$, and

$$\begin{aligned} H_\epsilon(u) &= \begin{cases} 0, & \text{if } u \leq -\epsilon, \\ \frac{1}{2}\left[1 + \frac{u}{\epsilon} + \frac{1}{\pi}\sin\left(\frac{\pi u}{\epsilon}\right)\right], & \text{if } -\epsilon \leq u \leq \epsilon, \\ 1, & \text{if } u \geq \epsilon, \end{cases} \\ S_\epsilon(u) &= 2H_\epsilon(u) - 1, \\ \delta_\epsilon(u) &= H'_\epsilon(u), \\ \epsilon &= 0.1h. \end{aligned}$$

Solve (1) using $N_{\text{el}} = 400, 1600, 6400$ elements. Report the three plots.