WEEK 4

* Approximation theory:

- What is the error we make by using polynomials?

$$\|f(x) - f_h(x)\|_{L^2([a,b])} \leq c(b-a)^{p+1}$$

* Approx. of PDEs:

- We consider weak formulations to reduce the restrictions on the soln.
- weak form → discrete weak form:

  plug u_h and consider the test function $\varphi \in V_h$

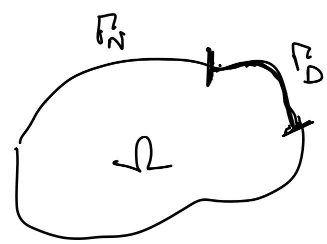
- Discrete weak form → Lin. algebra: $\begin{cases} \varphi = \varphi_i \in B = \{\varphi_j\}_{j=1}^{\dim(V_h)} \\ u_h = \sum_j U_j \varphi_j(x) \end{cases}$

* Example:

$$U - \Delta U = 0 \quad \forall x \in \Omega$$

$$U = U_D \quad \forall x \in \Gamma_D$$

$$\partial_{\vec{n}} U = b(x) \quad \forall x \in \Gamma_N$$



- weak form:

$$\int_\Omega v\varphi\,dx + \int_\Omega \nabla v\cdot\nabla\varphi\,dx - \int_{\partial\Omega} \overbrace{(\nabla v\cdot\vec{n})}^{=\partial_{\vec{n}} v = b(x)}\varphi\,ds = 0 \quad \forall \varphi\in V$$

$$V = \{v\in H^1(\Omega) \mid v=0 \ \forall x\in\Gamma_D\}$$

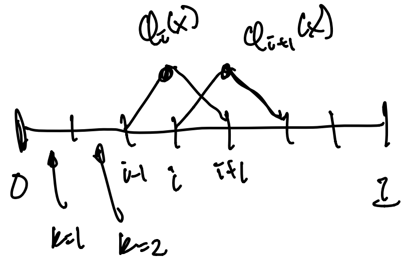
- Discrete weak form: plug $v=v_h$ and $\varphi\in V_h$.
- Linear algebra problem: plug $v_h=\sum_j v_j\varphi_j$ and $\varphi=\varphi_i$

$$\sum_j v_j \underbrace{\Big\{ \underbrace{\int_\Omega \varphi_i\varphi_j\,dx}_{=M_{ij}} + \underbrace{\int_\Omega \nabla\varphi_i\cdot\nabla\varphi_j\,dx}_{=S_{ij}} \Big\}}_{=A} = \underbrace{\int_{\partial\Omega} b(x)\varphi_i\,ds}_{=:F(\varphi_i)=F_i}$$

$$\Longrightarrow \boxed{Av=F}$$

- Assume $\Omega=[0,1]$ and $p=1$;
  - How do the shape functions look like?
  - consider a discretization of Ω
  - compute the matrices.

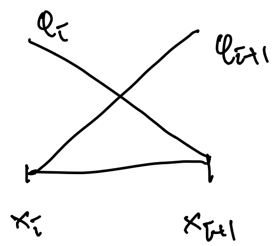
- Shape functions:



} Discretization of Ω

- <u>Mass Matrix</u>:

$$M_{i,i+1} = \int_\Omega \varphi_i \varphi_{i+1}\, dx = \cancelto{0}{\int_0^{x_i} \varphi_i \varphi_{i+1}\, dx} + \int_{x_i}^{x_{i+1}} \varphi_i \varphi_{i+1}\, dx + \cancelto{0}{\int_{x_{i+1}}^{1} \varphi_i \varphi_{i+1}\, dx}$$



$$\varphi_i(x) = \frac{1}{h}\left\{h - (x - x_i)\right\}$$

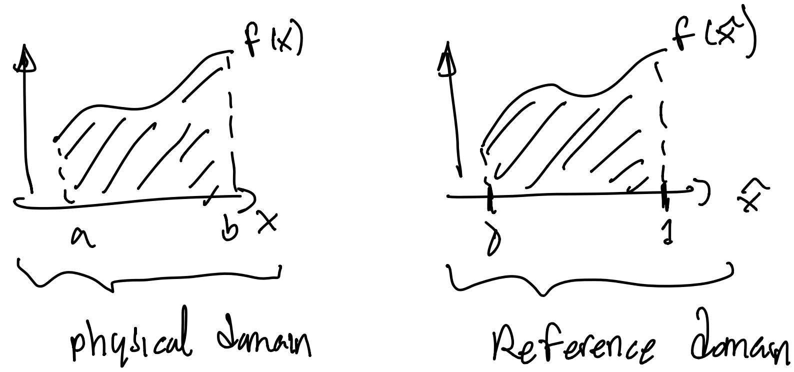
$$\varphi_{i+1}(x) = \frac{1}{h}(x - x_i)$$

$$\int_{x_i}^{x_{i+1}} \varphi_i \varphi_{i+1}\, dx = \int_{x_i}^{x_{i+1}} \frac{1}{h^2}\left[h - (x - x_i)\right](x - x_i)\, dx$$

$$= \frac{1}{h^2}\left\{ h \int_{x_i}^{x_{i+1}} (x - x_i)\, dx - \int_{x_i}^{x_{i+1}} (x - x_i)^2\, dx \right\}$$

$$= \frac{1}{h} \frac{(x - x_i)^2}{2}\Bigg|_{x_i}^{x_{i+1}} - \frac{1}{h^2}\frac{(x - x_i)^3}{3}\Bigg|_{x_i}^{x_{i+1}} = \frac{h}{2} - \frac{h}{3} = \frac{h}{6}$$

* <u>NOTE</u>: Instead of taking these integrals in every element, I can simply take them in a "reference element."



- Transformation from the physical to the reference domains

$$\left\{ \begin{array}{l} x = \hat{x}(b-a) + a \\ \Rightarrow \hat{x} = \dfrac{x-a}{b-a} \end{array} \right.$$

- Then

$$\int_a^b f(x)\,dx = \int_0^1 f(\hat{x})(b-a)\,d\hat{x} = (b-a)\int_0^1 f(\hat{x})\,d\hat{x}$$

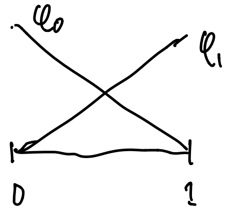
$$\hat{x}(a) = 0, \quad dx = (b-a)\,d\hat{x}$$
$$\hat{x}(b) = 1$$

$$\int_a^b \frac{\partial f(x)}{\partial x}\,dx = \int_0^1 \frac{1}{b-a}\frac{\partial f(\hat{x})}{\partial \hat{x}}(b-a)\,d\hat{x} = \int_0^1 \frac{\partial f(\hat{x})}{\partial \hat{x}}\,d\hat{x}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} = \frac{\partial}{\partial \hat{x}}\frac{1}{(b-a)}$$

$$\int_a^b \frac{\partial f(x)}{\partial x}\cdot\frac{\partial g(x)}{\partial x}\,dx = \frac{1}{b-a}\int_0^1 \frac{\partial f(\hat{x})}{\partial \hat{x}}\cdot\frac{\partial g(\hat{x})}{\partial \hat{x}}\,d\hat{x}$$

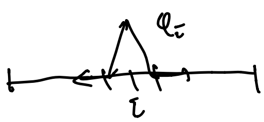
- Compute again $M_{i\,i+1} = \int_{x_i}^{x_{i+1}} \varphi_i \varphi_{i+1}\, dx = \underbrace{(x_{i+1} - x_i)}_{=h} \int_0^1 \varphi_0 \varphi_1\, dx$



$$= h \int_0^1 (1-x)\,x\, dx = h \left\{ \frac{x^2}{2} - \frac{x^3}{3} \right\}\Big|_0^1 = \frac{h}{6}$$

- By symmetry $M_{i\,i-1} = M_{i\,i+1} = h/6$

- What about M_{ii}? $M_{ii} = \int_{x_{i-1}}^{x_i} \varphi_i^2\, dx + \int_{x_i}^{x_{i+1}} \varphi_i^2\, dx$



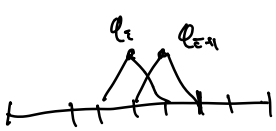
$$= h \int_0^1 x^2\, dx + h \int_0^1 (1-x)^2\, dx$$

$$= h\, x^3/3 \Big|_0^1 - h \frac{(1-x)^3}{3} \Big|_0^1 = \frac{h}{3} + \frac{h}{3} = \frac{2h}{3}$$

- So we have: $(M)_i = [0 \dots 0 \;\; 1/6 \;\; 2/3 \;\; 1/6 \;\; 0 \dots 0]\, h$

- Stiffness Matrix:

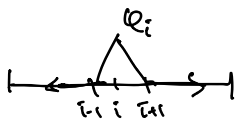
$$S_{i\,i+1} = \int_0^1 \frac{\partial \varphi_i}{\partial x} \cdot \frac{\partial \varphi_{i+1}}{\partial x}\, dx = \int_{x_i}^{x_{i+1}} (\dots)\, dx = \frac{1}{h} \int_0^1 \overbrace{\frac{\partial \varphi_0}{\partial x}}^{=-1} \cdot \overbrace{\frac{\partial \varphi_1}{\partial x}}^{=1}\, dx$$



$$= \frac{1}{h} (-1)\, x \Big|_0^1 = -\frac{1}{h} \Rightarrow S_{i\,i+1} = \frac{-1}{h}$$

- and by symmetry: $S_{i\,i-1} = \frac{-1}{h}$

• What about S_{ii}?



$$S_{ii} = \int_\Omega \left(\frac{\partial \varphi_i}{\partial x}\right)^2 dx = \int_{x_{i-1}}^{x_i} \left(\frac{\partial \varphi_i}{\partial x}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{\partial \varphi_i}{\partial x_i}\right)^2 dx$$

$$= \frac{1}{h}\int_0^1 (1)^2 dx + \frac{1}{h}\int_0^1 (-1)^2 dx = \frac{2}{h}$$

• So we have: $\quad (S)_i = [0 \dots 0 \ -1 \ \ 2 \ -1 \ \ 0 \ \dots \ 0]\,\frac{1}{h}$

$\boxed{\text{* Example 2:}}$

• Consider the advection-diffusion eqn in 1D:

$\partial_t v + a\,\partial_x v - \mu\,\partial_{xx} v = 0$ with $\partial_x = 0$ on the boundary:

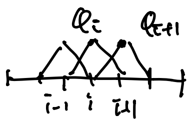
The discrete prob. is:

$$\sum_j \dot{v}_j \underbrace{\int_\Omega \varphi_i \varphi_j \, dx}_{=M_{ij}} + a \sum_j v_j \underbrace{\int_\Omega \frac{\partial \varphi_j}{\partial x} \varphi_i \, dx}_{=T_{ij}} + \mu \underbrace{\int_\Omega \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} dx}_{=S_{ij}} = 0$$

• From the previous example, we have M_{ij} and S_{ij}

What is T_{ij}?

- consider $T_{i\,i+1} = \int_\Omega \frac{\partial \varphi_{i+1}}{\partial x} \varphi_i \, dx = \int_{x_i}^{x_{i+1}} (\dots)\, dx = \int_0^1 (1-x)\, dx$



$$= \left(x - \frac{x^2}{2}\right)\Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$T_{i\,i-1} = \int_\Omega \frac{\partial \varphi_{i-1}}{\partial x} \varphi_i \, dx = \int_{x_{i-1}}^{x_i} (\dots)\, dx = \int_0^1 (-1)\, x\, dx = -\frac{x^2}{2} = -\frac{1}{2}$$

$$T_{ii} = \int_\Omega \frac{\partial \varphi_i}{\partial x} \varphi_i \, dx = \int_{x_{i-1}}^{x_i} (\dots)\, dx + \int_{x_i}^{x_{i+1}} (\dots)\, dx = \int_0^1 x\, dx - \int_0^1 (1-x)$$

$$= \frac{x^2}{2}\Big|_0^1 + \frac{(1-x)^2}{2}\Big|_0^1 = \frac{1}{2} - \frac{1}{2} = 0$$

- Then we have $(T)_i = [0 \dots 0 \; -1/2 \; 0 \; 1/2 \; 0 \dots 0]$

- Recall the discretization:

$$\sum_j \dot{v}_j M_{ij} + \sum_j v_j T_{ij} + \mu \sum_j v_j b_{ij} = 0$$

- For the i-th equation we have:

$$\boxed{\frac{h}{6}\left[\dot{v}_{i-1} + 4\dot{v}_i + \dot{v}_{i+1}\right] + a\left\{\frac{v_{i+1} - v_i}{2h}\right\} - \mu\left\{\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2}\right\} = 0}$$

* Example of quad. via Gauss-Legendre:

$$\partial_t v + a\partial_x v - \mu\partial_{xx} v = 0 \Rightarrow \sum_j \dot{v}_j \int_\Omega \varphi_i \varphi_j dx + \sum_j v_j \left\{ a\int_\Omega \frac{\partial\varphi_j}{\partial x}\varphi_i dx + \int_\Omega \mu \frac{\partial \varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x} dx \right\} = 0$$

- If $p=1$ then the largest degree polynomial in the integrands is: $\varphi_i\varphi_j \in \mathbb{P}^2(K)$
- How many quad points do we need?

$$2N_q - 1 \geq 2 \Rightarrow N_q \geq 3/2 \Rightarrow N_q = 2$$

- With $N_q = 2$, I should be able to integrate exactly polynomials of degree up to $2(2)-1=3$.

Let's verify this.

- Consider $p(x) = ax^3 + bx^2 + cx + d \Rightarrow I := \int_{-1}^{1} p(x) = \frac{b}{3} + 2d$

and
$$\sum_{q=1}^{2} w_q p(x_q) = 1\cdot p(-1/\sqrt{3}) + 1\, p(1/\sqrt{3})$$
$$= \cancel{a\left(-\frac{1}{\sqrt{3}}\right)^3} + b\left(-\frac{1}{\sqrt{3}}\right)^2 + \cancel{c\left(\frac{-1}{\sqrt{3}}\right)} + d + \cancel{a\left(\frac{1}{\sqrt{3}}\right)^3} + b\left(\frac{1}{\sqrt{3}}\right)^2 + \cancel{c\left(1/\sqrt{3}\right)} + d = 2b/3 + 2d$$