

# Solving the Navier-Stokes equations via a projection scheme

Consider the incompressible Navier-Stokes equations in non-conservative form

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \mu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad (1a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (1b)$$

where  $\rho$  and  $\mu$  are the density and viscosity of the fluid,  $\mathbf{u} = [u, v, w]^T$  is the velocity,  $p$  is the pressure and  $\mathbf{f}$  is an external force. Our objective is to solve a two-dimensional version of (1) via the second-order projection scheme by [Germond and Salgado 2016 (JCP)]. The algorithm is given as follows:

$$\rho \left( \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} \right) + \rho (\mathbf{u}^* \cdot \nabla \mathbf{u}^{n+1}) - \mu \Delta \mathbf{u}^{n+1} + \nabla p^* = \mathbf{f}^{n+1}, \quad \mathbf{u}^{n+1}|_{\partial\Omega} = 0,$$

where

$$\mathbf{u}^* = 2\mathbf{u}^n - \mathbf{u}^{n-1}, \quad p^* = p^n + \frac{4}{3}\phi^n - \frac{1}{3}\phi^{n-1}.$$

Here  $\phi$  is an auxiliary variable that represents an increment in pressure and is given by

$$\Delta \phi^{n+1} = \frac{3\rho}{2\Delta t} \nabla \cdot \mathbf{u}^{n+1}, \quad \nabla \phi^{n+1} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

Finally, the pressure is updated as follows:

$$p^{n+1} = p^n + \phi^{n+1}.$$

## 1 First step: solving the heat equation via BDF2

Let us start by solving

$$\rho \left( \frac{\partial u}{\partial t} \right) - \mu \Delta u = f(\mathbf{x}),$$

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega,$$

via second-order backward differences. That is, we consider

$$\rho \left( \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} \right) - \mu \Delta u^{n+1} = f^{n+1}, \quad u^{n+1}|_{\partial\Omega} = 0.$$

We can use our implementation for the heat equation.

## Weak formulation

The discrete weak formulation is given by

$$\rho \int_{\Omega} \left( \frac{3u_h^{n+1} - 4u_h^n + u_h^{n-1}}{2\Delta t} \right) \varphi d\mathbf{x} + \mu \int_{\Omega} \nabla u_h^{n+1} \cdot \nabla \varphi d\mathbf{x} = \int_{\Omega} f(\mathbf{x}, t^{n+1}) \varphi d\mathbf{x}, \quad \forall \varphi \in V_h. \quad (2)$$

Using the fact that  $u_h(\mathbf{x}) = \sum_j U_j \varphi(\mathbf{x})$ , we get

$$\sum_j U_j^{n+1} \underbrace{\int_{\Omega} \left[ \varphi_i \varphi_j + \frac{2\Delta t \mu}{3\rho} (\nabla \varphi_i \cdot \nabla \varphi_j) \right] d\mathbf{x}}_{= M_{ij} + \frac{2\Delta t \mu}{3\rho} S_{ij}} = \underbrace{\int_{\Omega} \left( \frac{4}{3} u_h^n - \frac{1}{3} u_h^{n-1} \right) \varphi d\mathbf{x} + \frac{2\Delta t}{3\rho} \int_{\Omega} f(\mathbf{x}, t^{n+1}) \varphi_i d\mathbf{x}}_{= r_i^{n+1}}$$

Therefore, we get

$$\left( M + \frac{2\Delta t \mu}{3\rho} S \right) U^{n+1} = R^{n+1}.$$

## Numerical experiment

Consider  $\rho = 1$ ,  $\mu = 1$  and

$$\begin{aligned} u(\mathbf{x}, 0) &= u_0(\mathbf{x}) = \sin(2\pi x) \sin(2\pi y), \\ f(\mathbf{x}, t) &= 2\pi [4\pi \cos(2\pi t) - \sin(2\pi t)] \sin(2\pi x) \sin(2\pi y), \end{aligned}$$

and solve the equation up to  $t = 1$ . The exact solution is given by

$$u = \cos(2\pi t) \sin(2\pi x) \sin(2\pi y).$$

The results of a convergence test are shown in Table 1.

Cells	DoFs	$E_2$	rate
256	289	6.76e-03	—
1024	1089	1.66e-03	2.03
4096	4225	4.07e-04	2.03
16384	16641	1.01e-04	2.01
65536	66049	2.53e-05	2.00

Table 1: Convergence test for the heat equation.

## 2 Second step: solving two heat equations via BDF2

Now let us solve

$$\begin{aligned} \rho \left( \frac{\partial u}{\partial t} \right) - \mu \Delta u &= f(\mathbf{x}, t), & u(\mathbf{x}, t) &= u_B(\mathbf{x}, t), & \mathbf{x} &\in \partial\Omega, \\ \rho \left( \frac{\partial v}{\partial t} \right) - \mu \Delta v &= g(\mathbf{x}, t), & v(\mathbf{x}, t) &= v_B(\mathbf{x}, t), & \mathbf{x} &\in \partial\Omega, \end{aligned}$$

via second-order backward differences. That is, we consider

$$\begin{aligned}\rho \left( \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} \right) - \mu \Delta u^{n+1} &= f^{n+1}, & u^{n+1}|_{\partial\Omega} &= u_B(\mathbf{x}, t^{n+1}), \\ \rho \left( \frac{3v^{n+1} - 4v^n + v^{n-1}}{2\Delta t} \right) - \mu \Delta v^{n+1} &= g^{n+1}, & v^{n+1}|_{\partial\Omega} &= v_B(\mathbf{x}, t^{n+1}).\end{aligned}$$

We need to duplicate the implementation to solve for each variable  $u$  and  $v$ . Later, we will couple these two equations via the nonlinear term in the Navier-Stokes equations.

### Numerical experiment

Test the implementation and verify we get second-order convergence for each component. Consider  $\rho = 1$ ,  $\mu = 1$ ,

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) = \sin(x) \sin(y), \quad v(\mathbf{x}, 0) = v_0(\mathbf{x}) = \cos(x) \cos(y),$$

and

$$\begin{aligned}f(\mathbf{x}, t) &= \underbrace{\cos(t + y) \sin(x)}_{\text{time derivative}} + \underbrace{2\mu \sin(x) \sin(t + y)}_{\text{viscosity}}, \\ g(\mathbf{x}, t) &= \underbrace{-\cos(x) \sin(t + y)}_{\text{time derivative}} + \underbrace{2\mu \cos(x) \cos(t + y)}_{\text{viscosity}}\end{aligned}$$

Solve the equation up to  $t = 1$ . The exact solution is given by

$$u = \sin(x) \sin(y + t), \quad v = \cos(x) \cos(y + t).$$

The results of a convergence test are shown in Table 2.

Cells	DoFs	$E_2(u)$	rate	$E_2(v)$	rate
256	289	3.16E-04	—	1.57E-04	—
1024	1089	7.90E-05	2.00	3.93E-05	2.00
4096	4225	1.97E-05	2.00	9.88E-06	1.99
16384	16641	4.93E-06	2.00	2.47E-06	2.00
65536	66049	1.23E-06	2.00	6.17E-07	2.00

Table 2: Convergence test for the heat equation.

## 3 Third step: include the nonlinear term

By considering the nonlinear term

$$\rho \mathbf{u}^* \cdot \nabla \mathbf{u}^{n+1} = \rho \begin{bmatrix} u^* \\ v^* \end{bmatrix} \cdot \begin{bmatrix} u_x^{n+1} & u_y^{n+1} \\ v_x^{n+1} & v_y^{n+1} \end{bmatrix} = \rho \begin{bmatrix} u^* u_x^{n+1} + v^* u_y^{n+1} \\ u^* v_x^{n+1} + v^* v_y^{n+1} \end{bmatrix},$$

we get

$$\begin{aligned} \rho \left( \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} \right) + \rho (u^* u_x^{n+1} + v^* u_y^{n+1}) - \mu \Delta u^{n+1} &= f^{n+1}, & u^{n+1}|_{\partial\Omega} &= u_B(\mathbf{x}, t^{n+1}), \\ \rho \left( \frac{3v^{n+1} - 4v^n + v^{n-1}}{2\Delta t} \right) + \rho (u^* v_x^{n+1} + v^* v_y^{n+1}) - \mu \Delta v^{n+1} &= g^{n+1}, & v^{n+1}|_{\partial\Omega} &= v_B(\mathbf{x}, t^{n+1}). \end{aligned}$$

## Weak formulation

The discrete weak formulation changes in the definition of the matrices. Now we have

$$\begin{aligned} \sum_j U_j^{n+1} \int_{\Omega} \left[ \varphi_i \varphi_j + \frac{2\Delta t}{3} (u_h^* \partial_x \varphi_j + v_h^* \partial_y \varphi_j) \varphi_i + \frac{2\Delta t \mu}{3\rho} (\nabla \varphi_i \cdot \nabla \varphi_j) \right] d\mathbf{x} &= r_i^{(u),n+1}, \\ \sum_j V_j^{n+1} \int_{\Omega} \left[ \varphi_i \varphi_j + \frac{2\Delta t}{3} (u_h^* \partial_x \varphi_j + v_h^* \partial_y \varphi_j) \varphi_i + \frac{2\Delta t \mu}{3\rho} (\nabla \varphi_i \cdot \nabla \varphi_j) \right] d\mathbf{x} &= r_i^{(v),n+1}. \end{aligned}$$

## Numerical experiment

Test the implementation and verify we get second-order convergence for each component. Consider  $\rho = 1$ ,  $\mu = 1$ ,

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) = \sin(x) \sin(y), \quad v(\mathbf{x}, 0) = v_0(\mathbf{x}) = \cos(x) \cos(y),$$

and

$$\begin{aligned} f(\mathbf{x}, t) &= \underbrace{\cos(t+y) \sin(x)}_{\text{time derivative}} + \underbrace{2\mu \sin(x) \sin(t+y)}_{\text{viscosity}} + \underbrace{\cos(x) \sin(x)}_{\text{nonlinearity}}, \\ g(\mathbf{x}, t) &= \underbrace{-\cos(x) \sin(t+y)}_{\text{time derivative}} + \underbrace{2\mu \cos(x) \cos(t+y)}_{\text{viscosity}} - \underbrace{\frac{1}{2} \sin(2(t+y))}_{\text{nonlinearity}}, \end{aligned}$$

Solve the equation up to  $t = 1$ . The exact solution is given by

$$u = \sin(x) \sin(y + t), \quad v = \cos(x) \cos(y + t).$$

The results of a convergence test are shown in Table 3.

Cells	DoFs	$E_2(u)$	rate	$E_2(v)$	rate
256	289	3.00E-04	—	1.69E-04	—
1024	1089	7.50E-05	2.00	4.22E-05	2.00
4096	4225	1.87E-05	2.00	1.06E-05	2.00
16384	16641	4.68E-06	2.00	2.64E-06	2.00

Table 3: Convergence test including the nonlinear terms.

## 4 Fourth step: solve the Poisson equation in the algorithm

At each time step, after solving the momentum equation, we need to solve

$$-\Delta\phi^{n+1} = -\frac{3\rho}{2\Delta t}\nabla \cdot \mathbf{u}^{n+1}, \quad \nabla\phi^{n+1} \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

### Weak formulation

The discrete weak formulation is given as follows:

$$\int_{\Omega} \nabla\phi_h^{n+1} \cdot \nabla\varphi d\mathbf{x} - \underbrace{\int_{\partial\Omega} (\nabla\phi_h^{n+1} \cdot \mathbf{n})\varphi d\mathbf{x}}_{=0} = -\frac{3\rho}{2\Delta t} \int_{\Omega} (\nabla \cdot \mathbf{u}_h^{n+1})\varphi d\mathbf{x}.$$

**Remark:** we will use a different space for the pressure; therefore, in the implementation, we must create a new space, DoFHandler, index sets for locally relevant and locally owned DoFs, etc.

## 5 Fifth step: combine momentum and Poisson equations

Up to this point, we are solving a nonlinear advection-diffusion-reaction equation to obtain  $u$  and  $v$  and a Poisson equation to obtain  $\phi$ . Now, we need to combine these equations to solve the Navier-Stokes equations. We get

$$\begin{aligned} \rho \left( \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} \right) + \rho (u^* u_x^{n+1} + v^* u_y^{n+1}) + p_x^n + \frac{4}{3}\phi_x^n - \frac{1}{3}\phi_x^{n-1} - \mu\Delta u^{n+1} &= f^{n+1}, \\ \rho \left( \frac{3v^{n+1} - 4v^n + v^{n-1}}{2\Delta t} \right) + \rho (u^* v_x^{n+1} + v^* v_y^{n+1}) + p_y^n + \frac{4}{3}\phi_y^n - \frac{1}{3}\phi_y^{n-1} - \mu\Delta v^{n+1} &= g^{n+1}, \end{aligned}$$

### Weak formulation

The discrete weak formulation changes in the right hand side. Now we have

$$\begin{aligned} \sum_j U_j^{n+1} \int_{\Omega} [\dots] d\mathbf{x} &= r_i^{(u),n+1} = \dots - \frac{2\Delta t}{3\rho} \int_{\Omega} \left( p_x^n + \frac{4}{3}\phi_x^n - \frac{1}{3}\phi_x^{n-1} \right) \varphi d\mathbf{x}, \\ \sum_j V_j^{n+1} \int_{\Omega} [\dots] d\mathbf{x} &= r_i^{(v),n+1} = \dots - \frac{2\Delta t}{3\rho} \int_{\Omega} \left( p_y^n + \frac{4}{3}\phi_y^n - \frac{1}{3}\phi_y^{n-1} \right) \varphi d\mathbf{x}, \end{aligned}$$

**Remark:** for stability reasons we will consider the so called Taylor-Hood finite elements, which consist on using  $\mathbb{Q}^{k+1}$  spaces for the velocity and a  $\mathbb{Q}^k$  space for the pressure. In particular, let's consider  $k = 1$ .

## Numerical experiment

Test the implementation and verify we get second-order convergence for each component. Consider  $\rho = 1$ ,  $\mu = 1$ ,

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) = \sin(x) \sin(y), \quad v(\mathbf{x}, 0) = v_0(\mathbf{x}) = \cos(x) \cos(y),$$

and

$$\begin{aligned} f(\mathbf{x}, t) &= \underbrace{\cos(t+y) \sin(x)}_{\text{time derivative}} + \underbrace{2\mu \sin(x) \sin(t+y)}_{\text{viscosity}} + \underbrace{\cos(x) \sin(x)}_{\text{nonlinearity}} - \underbrace{\sin(x) \sin(t+y)}_{\text{pressure}}, \\ g(\mathbf{x}, t) &= \underbrace{-\cos(x) \sin(t+y)}_{\text{time derivative}} + \underbrace{2\mu \cos(x) \cos(t+y)}_{\text{viscosity}} - \underbrace{\frac{1}{2} \sin(2(t+y))}_{\text{nonlinearity}} + \underbrace{\cos(x) \cos(t+y)}_{\text{pressure}} \end{aligned}$$

Solve the equation up to  $t = 1$ . The exact solution is given by

$$u = \sin(x) \sin(y+t), \quad v = \cos(x) \cos(y+t), \quad p = \cos(x) \sin(y+t).$$

The results of a convergence test are shown in Table 4.

Cells	DoFs	$E_2(u)$	rate	$E_2(v)$	rate	$E_2(p)$	rate
256	1089	9.54E-06	—	4.80e-05	—	8.90e-04	—
1024	4225	2.35E-06	2.02	1.19e-05	2.01	3.42e-04	1.38
4096	16641	5.58E-07	2.07	2.85e-06	2.07	1.44e-04	1.25
16384	66049	1.39E-07	2.00	7.11e-07	2.00	6.73e-05	1.10

Table 4: Convergence test for the projection scheme.