

AMCS 394E: Contemp. Topics in Computational Science.

Computing with the finite element method

David I. Ketcheson and Manuel Quezada de Luna



Solution of PDEs

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For example:

The solution $u(x)$ of

$$-\Delta u = f(x), \quad \forall x \in (0, 1)$$

describes some distribution (e.g., of heat) at every point between $(0, 1)$.

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- * Use finite information to approximate the solution.

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- * how do I obtain the solution?

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Global approximations:

For example via Taylor series:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \\ &\approx \sum_{n=0}^N \alpha_n (x - x_0)^n \end{aligned}$$

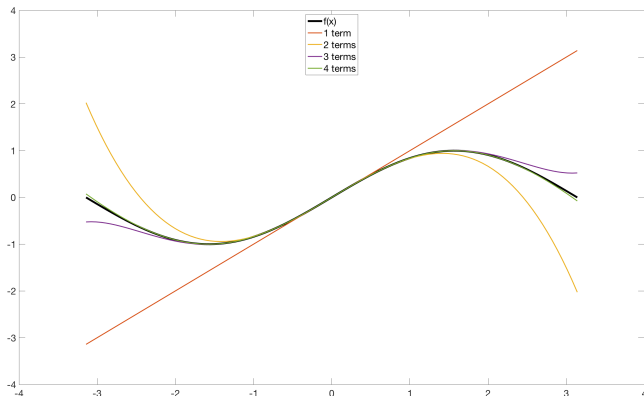
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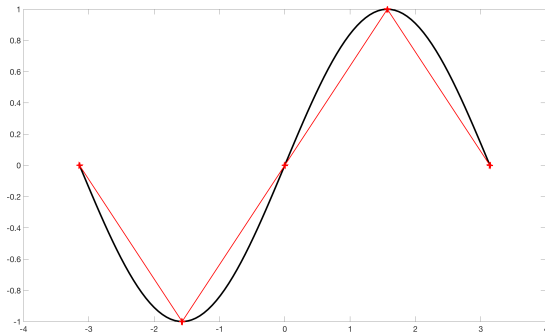


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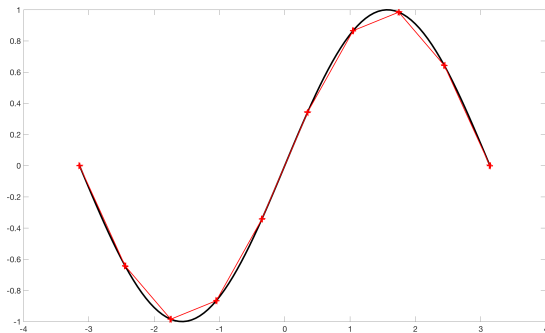


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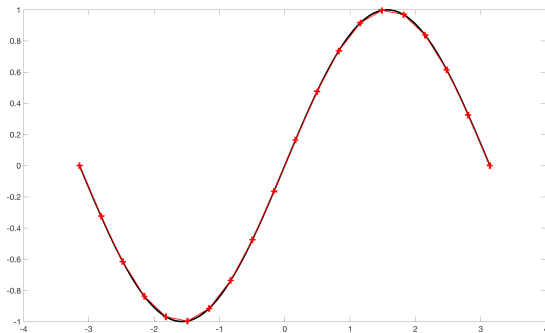


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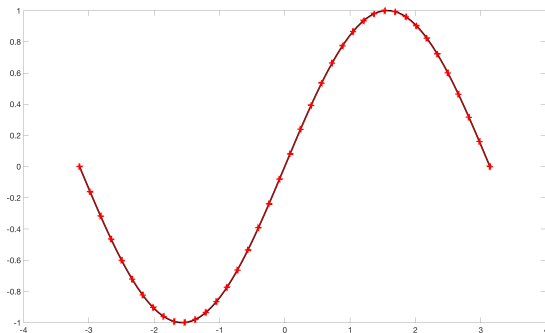


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Some definitions

Vector space

Is a set of objects that can be added together and multiplied by scalars (of a given field) and yield a new object also in the set.

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Important concepts of a vector space:

Basis and dimension.

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Requirement:

The elements of the basis B must be linearly independent.

Dimension of a vector space

The dimension of the vector space is the number of elements in the basis. Note that any basis of V has the same number of elements.

Examples of vector spaces and basis

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- * Dimension of the space is 3.

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- * Dimension of the space is $p + 1$.

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Dense set

A set S is dense in V if every element in V is either in S or is a limit point of S .

Limit point of a set S

The limit point of S is any point that can be approx. by points in S .

Example:

The rationals are dense in the real numbers.

Why do we even consider polynomials in the first place?

Weierstrass approximation theorem:

Let $f(x)$ be a real valued and continuous function in an interval $[a, b]$.
Then for any $\epsilon > 0$, there is a polynomial $p(x)$ in $[a, b]$ s.t.

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Relevance within finite elements:

Since $L^2(\Omega)$ is more general than $C^0(\Omega)$, solutions with discontinuities can be represented by the L^2 space.

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In simple words, ...

we can approximate functions in $L^2(\Omega)$ via
(sequences of) continuous functions.

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- * Sobolev spaces with $p = 2$.

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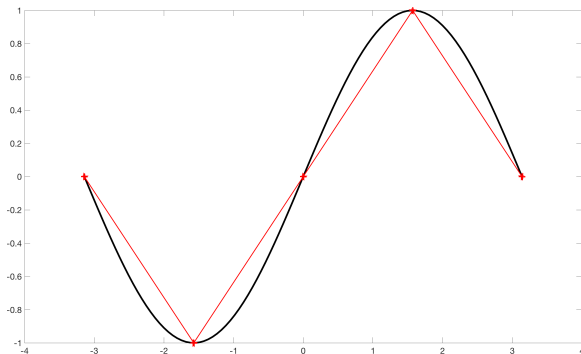
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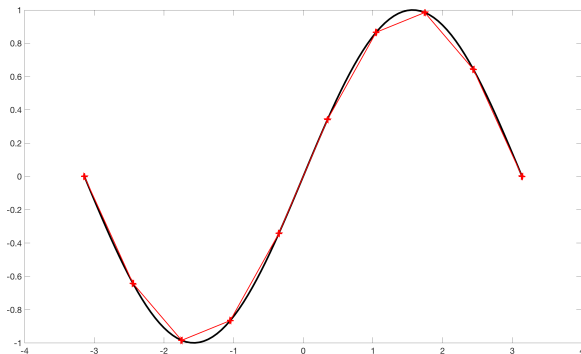
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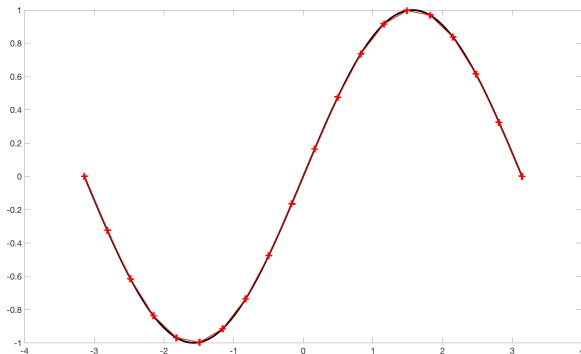
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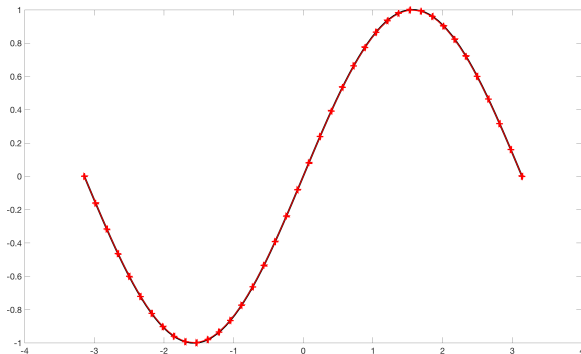
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Then,

$$\|f(x) - f_h(x)\|_{L^2([a,b])} \leq C(b-a)^{p+1},$$

where p is the degree of the polynomial and C is a constant independent of u and $|b-a|$.

Common piecewise polynomial spaces used in finite elements

The most common (finite dimensional) space in finite elements is

$$\underbrace{V_h := \{v \in C^0(\Omega) \mid v|_K \in \mathbb{P}^p(K) + \text{BCs}\}}_{\text{(finite dim.) space of the finite element soln.}} \subset \underbrace{V := \{v \in L^2(\Omega) + \text{BCs}\}}_{\text{(infinite dim.) space of the exact soln.}}$$

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Basis functions for V_h .

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- * Consider the ‘hat’ functions.
- * Interpolatory versus non-interpolatory basis functions.

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Finite element solution

Given the basis $\{\varphi_j, j = 1, \dots, \dim(V_h)\}$, the finite element solution is

$$u_h(x) = \sum_{n=1}^{\dim(V_h)} a_n \varphi_n(x),$$

where a_i are called degrees of freedom (DoFs) or control points.

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Note: if the basis is interpolatory; i.e., if $\varphi_i(x_j) = \delta_{ij}$, then

$$a_i = u_h(x_i)$$