AMCS 394E: Contemp. Topics in Computational Science.
Computing with the finite element method

David I. Ketcheson and Manuel Quezada de Luna

King Abdullah University of Science and Technology

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* We are assuming Ω and the basis functions don't change in time.

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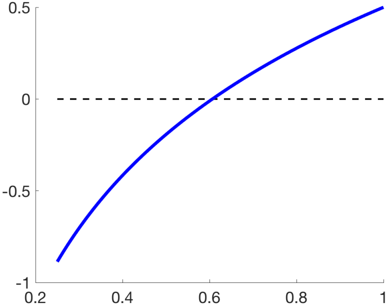
In this class we will see how to use Newton's method to solve a nonlinear elliptic equation.

Newton's method for the algebraic scalar equation $r(u)$



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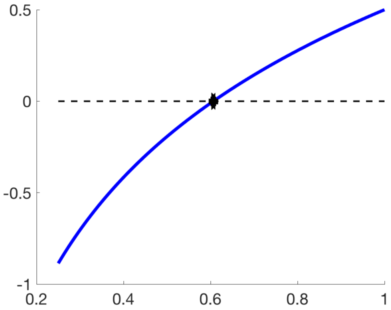
* We want to find u_* such that $r(u_*) = 0$.



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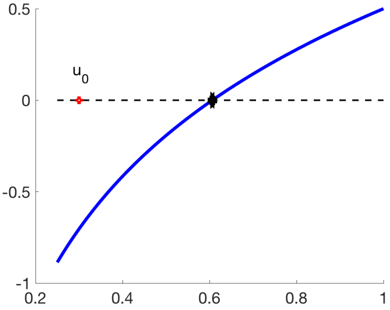
* Consider an initial guess u_0.



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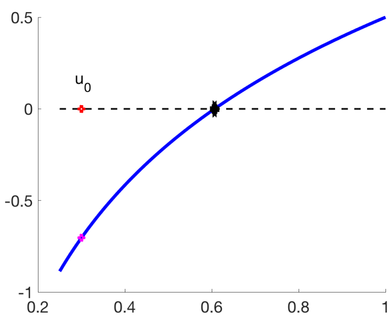
* Perform a linear approximation of $r(u)$ around u_0.



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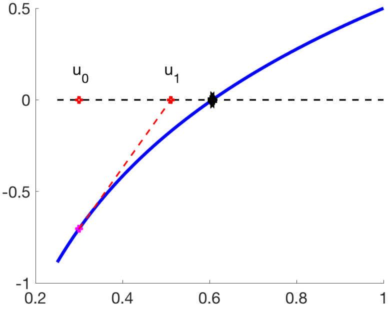
* Find the root of the linear approximation and call that u_1.



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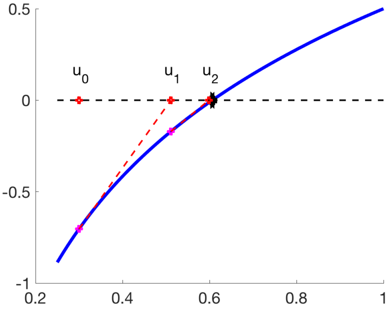
* Repeat the process until 'convergence is achieved'.



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$$\alpha m_i^{\Gamma(\tilde{u}_h)}(U_i - \tilde{U}_i) + \int_\Omega S_\epsilon(\tilde{u}_h)\left[|\nabla u_h| - 1\right]\phi d\mathbf{x} + c\int_\Omega h(\mathbf{x})\nabla u_h \cdot \nabla\phi d\mathbf{x} = 0$$

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$$\underbrace{\alpha m_i^{\Gamma(\tilde{u}_h)}(U_i - \tilde{U}_i)}_{\text{penalization}} + \int_\Omega S_\epsilon(\tilde{u}_h)\left[|\nabla u_h| - 1\right]\phi d\mathbf{x} + c\int_\Omega h(\mathbf{x})\nabla u_h \cdot \nabla\phi d\mathbf{x} = 0$$

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Here $\alpha = 1 \times 10^{10}$, and

$$\begin{aligned}\int_{\Gamma(\tilde{u})}(u_h - \tilde{u}_h)\phi_i d\mathbf{x} &\approx \int_\Omega \delta_\epsilon(\tilde{u})(u_h - \tilde{u}_h)\phi_i d\mathbf{x} \\ &= \sum_j (U_j - \tilde{U}_j)\int_\Omega \delta_\epsilon(\tilde{u})\phi_i\phi_j d\mathbf{x} \\ &\approx (U_i - \tilde{U}_i)\int_\Omega \delta_\epsilon(\tilde{u})\phi_i d\mathbf{x} = m_i^{\Gamma(\tilde{u})}(U_i - \tilde{U}_i)\end{aligned}$$

is a penalization term that forces the solution u_h to be close to $\tilde{u}_h$

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The residual (within Newton's method) is

$$R(U) = \alpha m_i^{\Gamma(\tilde{u}_h)}(U_i - \tilde{U}_i) + \underbrace{\int_\Omega S_\epsilon(\tilde{u}_h)\left[|\nabla u_h| - 1\right]\phi d\mathbf{x}}_{=:E(U)} + S^h U$$

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Let us work out the details to get

$$\frac{\partial E(U)}{\partial U} = \dots$$

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