Homework 1

Using the method of lines

Consider the one-dimensional advection diffusion equation

$$u_t + u_x + \mu u_{xx} = 0, \qquad \Omega = [0,1], \tag{1}$$

where $\mu > 0$ is a coefficient. Consider periodic boundary conditions and the following initial condition:

$$u(x,0) = \sin(2\pi x).$$

What do we expect the exact solution to do? Due to the advective part, the initial condition travels at constant speed to the right. At the same time, due to the diffusive term, the initial condition is dissipated at a rate that depends on μ.

Discretization based on finite differences

Consider the following discretization. Use second-order central finite differences to approximate u_x and u_{xx}. Use forward and backward Euler to obtain full discretizations (write down the schemes). Consider a fixed mesh with $\Delta x = 1 \times 10^{-2}$.

Exercise 1

Consider a final time of $t = 1$ and $\mu = 0.01$. For each full discretization proceed as follows:

- Experiment using the following time step sizes: $\Delta t = 1 \times 10^{-4}, 1 \times 10^{-3}, 1 \times 10^{-2}$ and 1×10^{-1}.
- How do the explicit and implicit methods behave for these time steps?

Exercise 2

Consider $\mu = 0$ and solve (1) using the explicit and the implicit methods. Use $\Delta t = 1 \times 10^{-4}$ and solve the problem for the following final times: $t = 1, 5, 10, 15$ and 20. Comment on the behavior of each full discretization as the final time increases.

Approximation of functions

Consider the function

$$f(x) = \sin^4(2\pi x), \qquad \forall x \in \Omega = [0, 1].$$

Our goal is to find multiple global and local approximations of $f(x)$. Let $f_h(x)$ be such an approximation for a given grid. We consider the following errors:

$$E_1 := \int_0^1 |f(x) - f_h(x)| dx, \qquad E_2 := \int_0^1 (f(x) - f_h(x))^2 dx.$$

Exercise 3: global approximations

Consider

a) the Taylor series (around $x = 0.5$) with N terms,

b) the Fourier series with N terms, and

c) a global polynomial interpolation given by

$$f_h(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{N-1} x^{N-1}$$

with $f_h(x_i) = f(x_i)$ for an evenly spaced set of N points between (and including) $[0, 1]$.

Consider $N = 4, 5, 6 \ldots, 10$ and for each approximation compute and report E_1 and E_2.

Exercise 4: local approximations

Split the domain Ω into N cells. For each cell K, compute linear and quadratic approximations $f_K(x)$ with $f_K(x_i) = f(x_i)$ where x_i are evenly spaced grid points (including the boundaries of the cell) within cell K. Compute and report the errors E_1 and E_2 for different number of cells; e.g., $N = 4, 5, 6, \ldots, 10$.