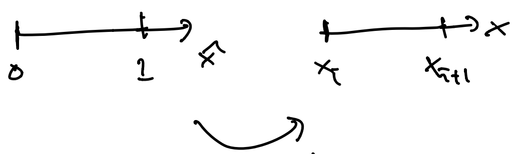
WEEK 6

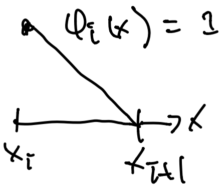
* Some clarifications;



$$T(\hat{x}) = x(\hat{x}) = \hat{x}(x_{i+1} - x_i) + x_i$$

$$\Rightarrow \frac{\partial x(\hat{x})}{\partial \hat{x}} = x_{i+1} - x_i$$

- Now consider:



$$\varphi_i(x) = 1 - \frac{(x - x_i)}{h} = 1 - \frac{1}{h}\Big\{ \underbrace{\hat{x}\overbrace{(x_{i+1} - x_i)}^{=h} + \cancel{x_i}}_{=x(\hat{x})} - \cancel{x_i} \Big\}$$

$$= \boxed{1 - \hat{x}} \quad \text{This is } \varphi_0(\hat{x}) \text{ in } \hat{K}$$

$$\Rightarrow \int_{x_i}^{x_{i+1}} \varphi_i(x)\,dx = \int_0^1 \varphi_i(x(\hat{x}))\,|J|\,d\hat{x} \quad \Big\} \text{ No need to call } \varphi_i \text{ as } \hat{\varphi}_i$$

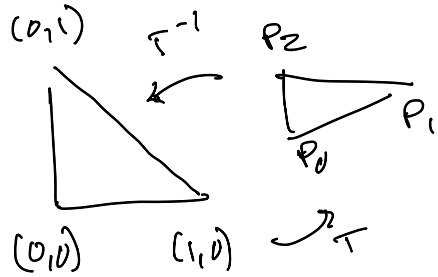
and
$$\frac{\partial \varphi_i(x(\hat{x}))}{\partial \hat{x}} = \frac{\partial \varphi_i(x(\hat{x}))}{\partial x} \cdot \underbrace{\frac{\partial x(\hat{x})}{\partial \hat{x}}}_{=J}$$

$$\Rightarrow \boxed{\frac{\partial \varphi_i(x(\hat{x}))}{\partial x} = J^{-1} \frac{\partial \varphi_i(x(\hat{x}))}{\partial \hat{x}}}$$

- Also consider

$$\int_{x_i}^{x_{i+1}} f(x(\hat{x}))\,dx = \int_0^1 f\Big(\underbrace{\hat{x}(x_{i+1}-x_i)-x_i}_{=T(\hat{x})=x(\hat{x})}\Big)|J|\,d\hat{x}$$

clarification about transformation using triangles:



$$\varphi_0(\hat{x},\hat{y}) = 1-\hat{x}-\hat{y}$$

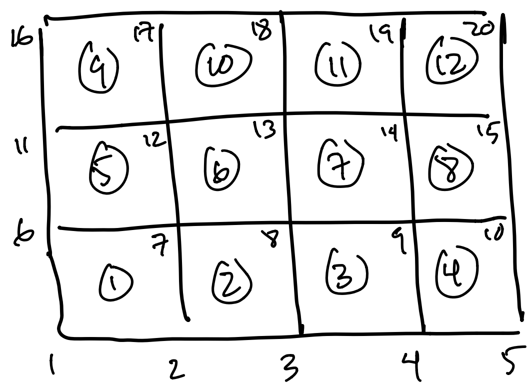
$$\varphi_1(\hat{x},\hat{y}) = \hat{x}$$

$$\varphi_2(\hat{x},\hat{y}) = \hat{y}$$

$$\Rightarrow \vec{x} = \vec{x}_0\underbrace{(1-\hat{x}-\hat{y})}_{=\varphi_0} + \vec{x}_1\underbrace{\hat{x}}_{\varphi_1} + \vec{x}_2\underbrace{\hat{y}}_{\varphi_2} = \sum_{j=0}^{2}\vec{x}_j\,\varphi_j(\hat{x},\hat{y})$$

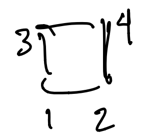
$$= \vec{x}_0 + \begin{bmatrix}\vec{x}_1-\vec{x}_0 & \vec{x}_2-\vec{x}_0\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} = \begin{bmatrix}x_1-x_0 & x_2-x_0\\ y_1-y_0 & y_2-y_0\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix} + \begin{bmatrix}x_0\\ y_0\end{bmatrix}$$

* For connectivity:



	1	2	3	4
1	1	2	6	7
2	2	3	7	8

Locally:



* Project some function $u(x)$ onto V_h

* find $u_h(x) \in V_h$ s.t. $u(x) \approx u_h(x)$ in a weak sense
* Weak formulation:

$$\int_\Omega u_h(x)\,\varphi\,dx = \int_\Omega u(x)\,\varphi\,dx \quad \forall \varphi \in V_h$$

* Lin. algebra:

$$\sum_j v_j \underbrace{\int_\Omega \varphi_i \varphi_j\,dx}_{=m_{ij}} = \int_\Omega u(x)\varphi_i\,dx =: F_i$$

* See the code in Matlab

$\boxed{\text{* Solve the Poisson equation}}$

$$
\begin{aligned}
-\Delta v &= f(\vec{x}) \quad \forall \vec{x} \in \Omega = [0,1]^2 \\
v &= 0 \quad \forall \vec{x} \in \partial\Omega
\end{aligned}
$$

* Discrete weak form:

$$
\int_\Omega \nabla v_h \cdot \nabla \varphi_i \, dx - \cancelto{0}{\int_{\partial\Omega} (\nabla v_h \cdot \vec{n}) \underbrace{\varphi_i}_{=0} ds} = \int_\Omega f(\vec{x}) \varphi_i \, dx
$$

* Linear Algebra problem:

$$
\sum_j v_j \underbrace{\int_\Omega \nabla \varphi_i \cdot \nabla \varphi_j \, dx}_{= S_{ij}} = \underbrace{\int_\Omega f(\vec{x}) \varphi_i \, dx}_{=: F_i}
$$

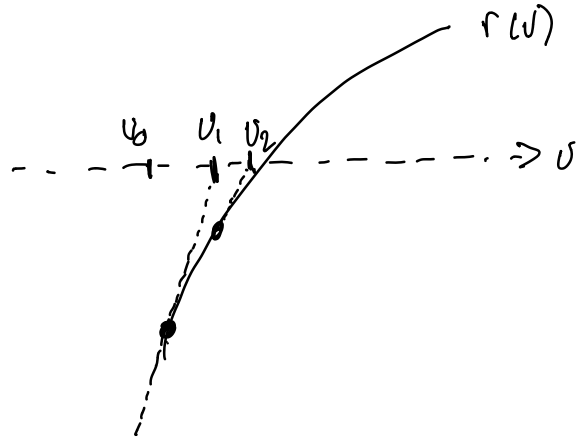
* to impose the B.C.s, say the i-th node is at the Boundary. Then

$$
\begin{array}{c}
S_i = [0 \dots 0 \; 1 \; 0 \dots 0] \\
\uparrow \\
\text{i-th column}
\end{array}
$$

$F_i = 0 \leftarrow$ Boundary value at i-th node.

* Newton's method for Nonlinear Scalar equation

* Given $r = r(u)$, find u^* s.t. $r(u^*) = 0$



* The $(k+1)$-th iteration is:

$$u^{(k+1)} = u^{(k)} - \left[r'(u^{(k)}) \right]^{-1} r(u^{(k)})$$

* Newton's method for system of equations

* Given $R(u)$, find $u^* \in \mathbb{R}^N$ s.t. $R(u^*) = 0$
* The $(k+1)$-th iteration is:

$$\boxed{u^{(k+1)} = u^{(k)} - \left(\frac{\partial R(u^{(k)})}{\partial u} \right)^{-1} R(u^{(k)})}$$

* **Example:** Given

$$R_i(v) = \alpha\, m_i^{\Gamma(\tilde{v})} (v_i - \tilde{v}_i) + \int_\Omega S_\epsilon(\tilde{v}_h)\left[|\nabla v_h| - 1\right]\varphi\, dx$$

$$+ ch \int_\Omega \nabla v_h \cdot \nabla \varphi\, ds$$

where $\alpha = 10^{10}$, $\tilde{v}_h = \sum_j \tilde{v}_j \varphi_j$ is given

$c = 1/2$

$\epsilon = 0.1h$

and $$m_i^{\Gamma(\tilde{v})} = \int_\Omega \delta_\epsilon(\tilde{v})\, \varphi_i\, dx$$

and $$S_\epsilon(\tilde{v}) = 2H_\epsilon(\tilde{v}) - 1$$

$$\delta_\epsilon(\tilde{v}) = H_\epsilon'(\tilde{v})$$

$$H_\epsilon'(\tilde{v}) = \begin{cases} 0 & \text{if } \tilde{v} \le -\epsilon \\ \frac{1}{2}\left[1 + \frac{\tilde{v}}{\epsilon} + \frac{1}{\pi}\sin\left(\pi\frac{\tilde{v}}{\epsilon}\right)\right] & \text{if } -\epsilon \le \tilde{v} \le \epsilon \\ 1 & \text{if } \tilde{v} \ge \epsilon \end{cases}$$

* Find $v^* \in \mathbb{R}^N$ s.t. $R(v^*) = 0$

* What is the Jacobian ?

$$J = \frac{\partial R(v)}{\partial v} = \alpha m_i^{r(\tilde{v})} + \frac{\partial E(v)}{\partial v} + chS$$

where $E(v)$ is a vector whose i-th entry is:

$$E_i(v) = \int_\Omega S_\epsilon(\tilde{v}_n)\left[|\nabla v_n| - 1\right]\varphi_i \, dx$$

* Note that $\partial E(v)/\partial v$ is a Matrix

* What is the (i,j)-th entry of $\frac{\partial E(v)}{\partial v}$?

* $$\left(\frac{\partial E(v)}{\partial v}\right)_{ij} = \frac{\partial E_i(v)}{\partial v_j} = \int_\Omega S_\epsilon(\tilde{v}_n)\,\frac{\partial |\nabla v_n|}{\partial v_j}\,\varphi_i \, dx$$

where

$$\frac{\partial}{\partial v_j}|\nabla v_n| = \frac{\partial(\nabla v_n \cdot \nabla v_n)^{1/2}}{\partial v_j} = \cancel{\tfrac{1}{2}}(\nabla v_n \cdot \nabla v_n)^{-1/2}\,\cancel{2}\nabla v_n \cdot \underbrace{\frac{\partial}{\partial v_j}\sum_\ell v_\ell \nabla\varphi_\ell}_{= \nabla\varphi_j}$$

* Then,

$$\frac{\partial E_i(v)}{\partial v_j} = \int_{\Omega} S_{\varepsilon}(\hat{v}_h) \frac{(\nabla v_h \cdot \nabla \varphi_j)\,\varphi_i}{|\nabla v_h|}\, dx$$

* Finally,

$$\boxed{J_{ij}^{(k)} = \alpha m_i^{F(v)} + \int_{\Omega} S_{\varepsilon}(\hat{v}_h) \frac{(\nabla v_h^{(k)} \cdot \nabla \varphi_j)\,\varphi_i}{|\nabla v_h^{(k)}|}\, dx + ch \int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j \, dx}$$

* We need to compute this matrix. (See the code in Matlab)