AMCS 394E: Contemp. Topics in Computational Science.

Computing with the finite element method

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Solution of PDEs

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For example:

The solution $u(x)$ of

$$-\Delta u = f(x), \qquad \forall x \in (0, 1)$$

describes some distribution (e.g., of heat) at every point between $(0, 1)$.

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* Use finite information to approximate the solution.

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* how do I obtain the solution?

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In general we can consider local or global approximations.

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Global approximations:
For example via Taylor series:

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\
&\approx \sum_{n=0}^{N} \alpha_n (x - x_0)^n
\end{aligned}
$$

Exercise:
Consider $f(x) = sin(x)$ and approximate it via Taylor series.

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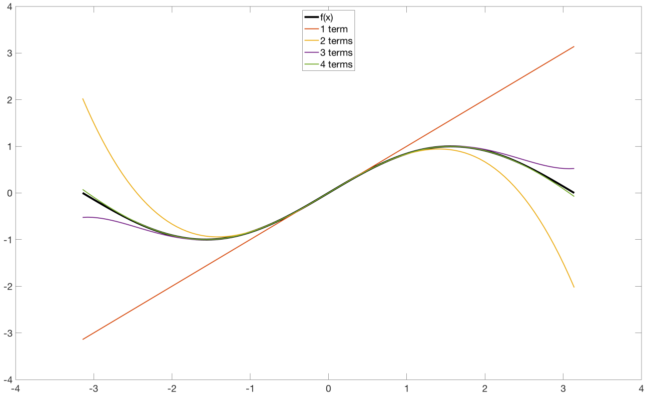

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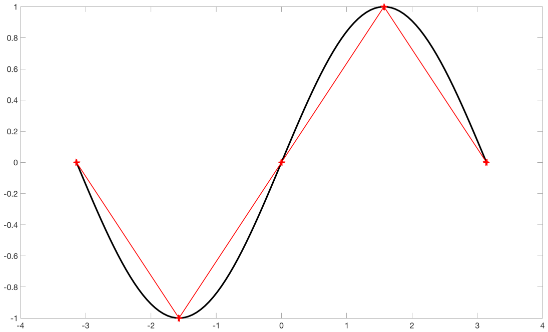
For example, via local (linear) polynomial approximations.



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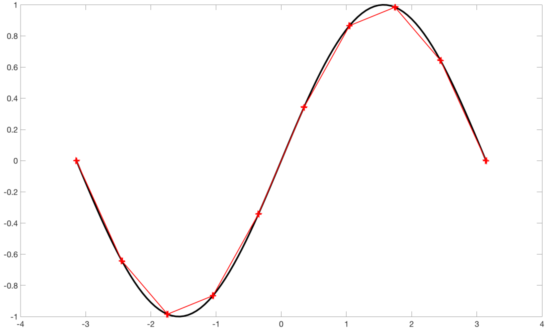

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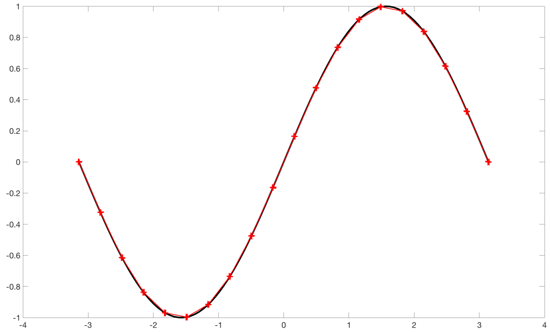

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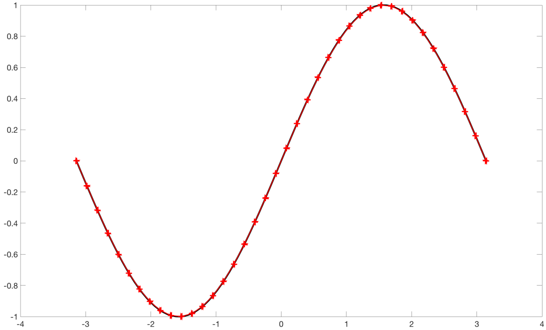

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Questions:

* Why can we consider polynomials?
* Should we use global or local approximations?

Some definitions

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Important concepts of a vector space:

Basis and dimension.

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Requirement:

The elements of the basis B must be linearly independent.

Dimension of a vector space

The dimension of the vector space is the number of elements in the basis. Note that any basis of V has the same number of elements.

Examples of vector spaces and basis

Example 1:

Three dimensional euclidean space.

* Elements in the space are all vectors in $\mathbb{R}^3$.
* Example of a basis is the set of orthonormal vectors $(\hat{i}, \hat{j}, \hat{k})$.
* Dimension of the space is 3.

Examples of vector spaces and basis

Example 2:

Polynomial space of degree p.

* Elements in the space are all polynomials with degree at most p.
* Example of a basis is the set of monomials $\{1, x, x^2, ..., x^p\}$.
* Dimension of the space is $p + 1$.

Examples of vector spaces and basis

Example 3:

Space of infinitely smooth functions in 1D: $C^\infty(\mathbb{R})$.

* Elements in the space are all 1D infinitely smooth functions.
* Examples of basis functions are:
    - set of monomials $\{1, x, x^2, ...\}$.
    - trigonometric functions $\{e^{2\pi inx}\}_{n=-\infty}^{\infty}$.
* Dimension of the space is infinity.

Examples of vector spaces and basis

Example 4:

square integrable functions in 1D: $L^2(\mathbb{R})$.

* Elements in the space are all 1D functions s.t.: $\int_{\mathbb{R}} f^2(x)dx < \infty$
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Some definitions

Dense set
A set S is dense in V if every element in V is either in S or is a limit point of S.

Limit point of a set S
The limit point of S is any point that can be approx. by points in S.

Example:
The rationals are dense in the real numbers.

Why do we even consider polynomials in the first place?

Weierstrass approximation theorem:

Let $f(x)$ be a real valued and continuous function in an interval $[a, b]$. Then for any $\epsilon > 0$, there is a polynomial $p(x)$ in $[a, b]$ s.t.

$$|f(x) - p(x)| < \epsilon, \quad \forall x \in [a, b]$$

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  (uniformly means at every point)

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* Computational cost.

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Relevance within finite elements:

Since $L^2(\Omega)$ is more general than $C^0(\Omega)$, solutions with discontinuities can be represented by the L^2 space.

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Properties:

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In simple words, ...

we can approximate functions in $L^2(\Omega)$ via (sequences of) continuous functions.

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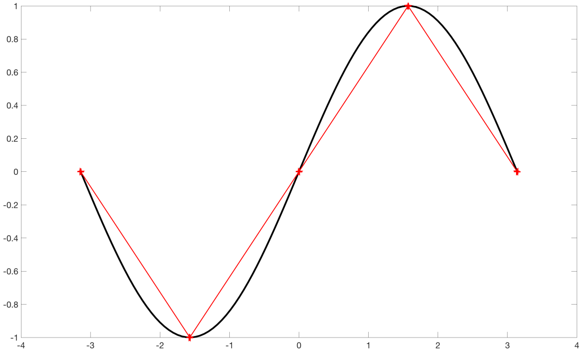

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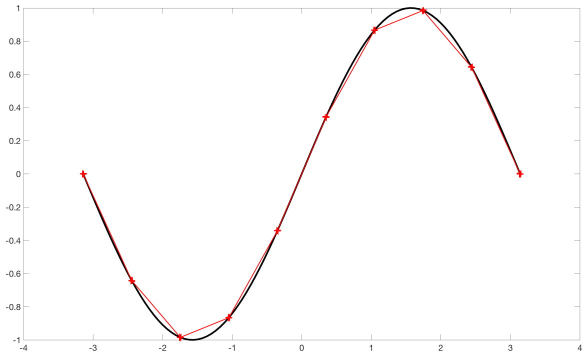

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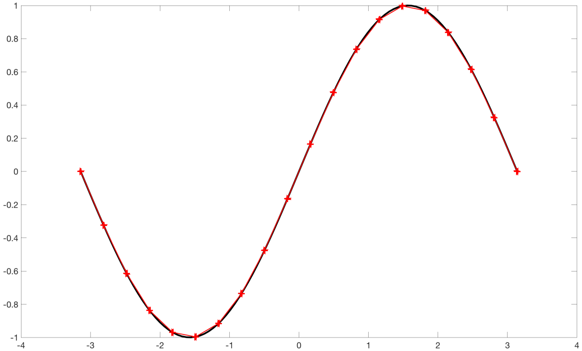

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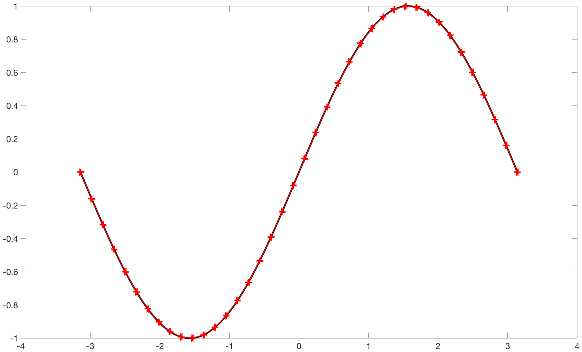

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Then,

$$||f(x) - f_h(x)||_{L^2([a,b])} \leq C(b-a)^{p+1},$$

where p is the degree of the polynomial and C is a constant independent of u and $|b-a|$.

Common piecewise polynomial spaces used in finite elements

The most common (finite dimensional) space in finite elements is

$$\underbrace{V_h := \{v \in C^0(\Omega) \mid v|_K \in \mathbb{P}^p(K) + \text{BCs}\}}_{\text{(finite dim.) space of the finite element soln.}} \subset \underbrace{V := \{v \in L^2(\Omega) + \text{BCs}\}}_{\text{(infinite dim.) space of the exact soln.}}$$

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* Interpolatory versus non-interpolatory basis functions.

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Finite element solution

Given the basis $\{\varphi_i,\ j = 1, ..., \dim(V_h)\}$, the finite element solution is

$$u_h(x) = \sum_{n=1}^{\dim(V_h)} a_i \varphi_i(x),$$

where a_i are called degrees of freedom (DoFs) or control points.

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Note: if the basis is interpolatory; i.e., if $\varphi_i(x_j) = \delta_{ij}$, then

$$a_i = u_h(x_i)$$