AMCS 394E: Contemp. Topics in Computational Science.
Computing with the finite element method

David I. Ketcheson and Manuel Quezada de Luna

King Abdullah University of Science and Technology

What have we seen so far?

Approximation theory:

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* Why do we go from a strong to a weak formulation?

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* Why do we go from a strong to a weak formulation?
* How do we get a finite dimensional approx. of the weak form?

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Approximation of PDEs:

* Why do we go from a strong to a weak formulation?
* How do we get a finite dimensional approx. of the weak form?
* How do we go from a finite dimensional approx. of the weak form to a linear algebra problem?

What have we seen so far?

Example:

Consider the following equation:

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\begin{aligned}
u - \Delta u &= 0, && \forall x \in \Omega, \\
u &= u_D, && \forall x \in \Gamma_D, \\
\partial_{\mathbf{n}} u &= b(x), && \forall x \in \Gamma_N.
\end{aligned}
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What is the weak formulation?

$$\int_\Omega u\varphi dx + \int_\Omega \nabla u \cdot \nabla \varphi dx - \int_{\partial\Omega} (\nabla u \cdot \mathbf{n})\varphi ds = 0, \quad \forall \varphi \in V$$

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where

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ if } x \in \Gamma_D\}.$$

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What is the discrete weak formulation?

$$\int_{\Omega} u_h \varphi dx + \int_{\Omega} \nabla u_h \cdot \nabla \varphi dx = \int_{\partial\Omega} b \varphi ds, \quad \forall \varphi \in V_h$$

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How do we get a linear algebra problem?

Consider and plug $\phi = \phi_i$ and $u_h = \sum_j U_j \varphi_j$ into the equation.

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We want to solve

$$\sum_j U_j \underbrace{\int_\Omega \varphi_i\varphi_j + \nabla\varphi_i \cdot \nabla\varphi_j dx}_{=A_{ij}} = \underbrace{\int_{\partial\Omega} b\varphi_i ds}_{=F(\phi_i)}, \quad i = 1, \ldots, \dim(V_h).$$

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Recall the space V_h and that $\phi_i \in V_h$

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* Let's consider a discretization of Ω.
* Let's compute the matrices and RHS.

Linear algebra problem from a FE discretization

Second example:

Let's consider the 1D equation (from the first week):

$$\partial_t u + a\partial_x u - \mu\partial_{xx} u = r(x), \quad \forall x \in \Omega = (0, 1),$$

with $\mu > 0$.

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* Consider piecewise linear continuous finite elements.
* What is the discrete weak formulation?
* Compute the mass, transport and stiffness matrices.
* Express the equation for the i-th DoF.
* Compare this versus the finite difference formulation.

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* We performed the integration based on a reference element.

Quadratures

We approximate the integrals via quadratures. That is (for 1D),

$$\int_{-1}^{1} f(x)dx \approx \sum_{q=1}^{N_q} w_q f(x_q),$$

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* Gauss-Legendre quadrature is exact for polynomials of degree up to $2N_q - 1$.

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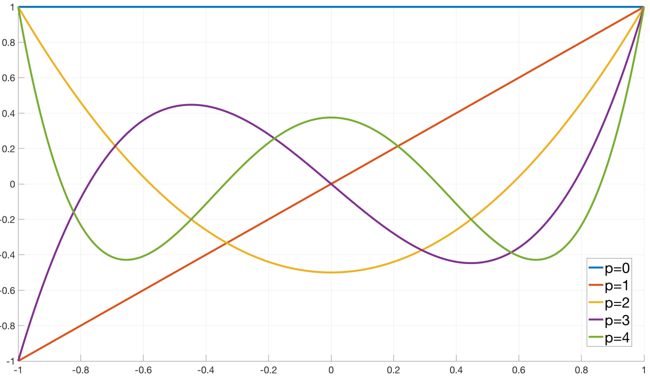

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Example:

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* What is the highest degree of polynomials that I need to integrate?
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Example:

Consider piecewise linear poly. to solve the advection-diffusion eqn.

- What is the highest degree of polynomials that I need to integrate?
- The highest degree is given by the mass matrix.
- The integrand of the mass matrix is quadratic.

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Consider piecewise linear poly. to solve the advection-diffusion eqn.

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* The highest degree is given by the mass matrix.
* The integrand of the mass matrix is quadratic.
* Then I need $2N_q - 1 \geq 2$. Therefore, $N_q \geq 3/2 \implies N_q = 2$.

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* With $N_q = 2$, I can integrate exactly polynomials up to degree 3.

Verify that I can integrate third order polynomials with $N_q = 2$.

The finite element (FE) loop

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* Combine (or assemble) the element integrals to global integrals.

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How to compute the element based integrals?

For each element, proceed as follows:

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For each element, proceed as follows:

* Compute all quantities with respect to the reference element.
* Loop on quadrature points. We do this since we approximate the integrals via quadratures.

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For each element, proceed as follows:

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* Loop on i-DoFs.
* Loop on j-DoFs and more indices if needed.

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The FE loop looks as follows:

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for $K = 1$ to N_{el} **do**

end for

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The FE loop looks as follows:

for $K = 1$ to N_{el} **do**
Compute quantities wrt reference element.

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The FE loop looks as follows:

for $K = 1$ to N_{el} **do**
  Compute quantities wrt reference element.
  for $q = 1$ to N_q **do**

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The FE loop looks as follows:

for $K = 1$ to N_{el} **do**
  Compute quantities wrt reference element.
  for $q = 1$ to N_q **do**
    for $i = 1$ to DoFs per element **do**

    end for

  end for
end for

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for $K = 1$ to N_{el} **do**
  Compute quantities wrt reference element.
  for $q = 1$ to N_q **do**
    for $i = 1$ to DoFs per element **do**
      Compute element based vectors.
    end for
  end for
end for

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   Compute quantities wrt reference element.
   for q = 1 to N_q do
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         for j = 1 to DoFs per element do

         end for
      end for

   end for
end for
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    for q = 1 to N_q do
        for i = 1 to DoFs per element do
            Compute element based vectors.
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                Compute element based matrices.
            end for
        end for

    end for
end for
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  Compute quantities wrt reference element.
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    for $i = 1$ to DoFs per element **do**
      Compute element based vectors.
      for $j = 1$ to DoFs per element **do**
        Compute element based matrices.
      end for
    end for
    Assemble from local to global operators.
  end for
end for