AMCS 394E: Contemp. Topics in Computational Science.
Computing with the finite element method

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Strong and weak forms of a PDE

Given a PDE, for example

$$-\frac{d^2u(x)}{dx^2} = f(x), \quad \forall x \in \Omega = (0, 1),$$

how can we find an approximation of its solution?

Strong and weak forms of a PDE

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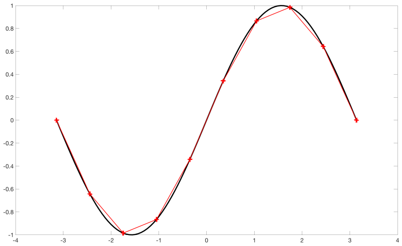

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* We expand the solution $u_h(x)$ as follows:

$$u_h(x) = \sum_j U_j \varphi_j(x),$$

where (for this interpolatory basis) $U_j = u_h(x_j)$.

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NOTE:
In order to represent the solution $u_h(x)$, we only need the (finitely many) coordinates U_j. These coordinates are indeed the solution of our linear algebraic systems.

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Problem:
This is too weak. For instance,

$$1, \quad 2x, \quad 3x^2, \quad \ldots, \quad (n+1)x^n$$

have the same integral over $\Omega := (0, 1)$.

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We can define that $f(x) = g(x)$ if

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Question:
For a given (vector) space V, what is a good set of functionals to weakly define that $f(x) = g(x)$?

Strong and weak forms of a PDE

Given a vector space V with some inner product $\langle \cdot, \cdot \rangle$,
$f, g \in V$ are weakly equal if

$$\langle f - g, v \rangle = 0, \quad \forall v \in V$$

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Example
If $V = L^2(\Omega)$, f and g are weakly equal if

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If $V = L^2(\Omega)$, f and g are weakly equal if

$$\int_\Omega (f - g) v \, dx = 0, \qquad \forall v \in V$$

Note

If $B = \{\varphi\}_i$ is a basis for V, then $f = g$ if

$$\int_\Omega (f - g) \varphi_i \, dx = 0, \qquad \forall \varphi_i \in B$$

Strong and weak forms of a PDE

The **strong form** of a PDE is the differential form; e.g.,

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$$-\int_0^1 \frac{d^2u(x)}{dx^2}\, v dx = \int_0^1 f(x)\, v dx, \quad \forall v \in V$$

or, integrating by parts:

$$\int_0^1 \frac{du}{dx}\, \frac{dv}{dx} dx + BCs = \int_0^1 f(x)\, v dx, \quad \forall v \in V$$

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Strong and weak forms of a PDE

Why and when should we integrate by parts?

* If we expect the solution to be not “sufficiently” smooth.
* If we need to impose boundary conditions via a numerical flux.
* We need to consider the finite dimensional space we aim to use.

Strong and weak forms of a PDE

Examples of strong to weak forms

Finite dimensional approximation of the weak form

Strong formulation:

Consider the following PDE in strong form

$$-\frac{d^2u}{dx} = f(x), \qquad\qquad \forall x \in \Omega = (0,1),$$

with

$$du/dx|_{x=1} = g_r, \qquad \text{and} \qquad du/dx|_{x=0} = g_l,$$

as boundary conditions.

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Weak formulation:

* Get the weak form of the PDE and apply the boundary conditions.

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Weak formulation:

* Get the weak form of the PDE and apply the boundary conditions.
* What is an appropriate space for the solution?

Finite dimensional approximation of the weak form

Weak formulation:

$$\int_0^1 \frac{du}{dx}\frac{d\varphi}{dx}\,dx - [g_r\varphi(1) - g_l\varphi(0)] = \int_0^1 f(x)\varphi\,dx, \qquad \forall\varphi \in V$$

where $V = H^1(\Omega)$.

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* The weak solution $u \in V$ is an infinite dimensional solution.
* We need to approximate u via a finite dimensional space.
* We can use the space space of continuous piecewise polynomials

$$V_h = \{\varphi \in C^0(\Omega) \mid \varphi_K \in \mathbb{P}^p(K)\}$$

Finite dimensional approximation of the weak form

Finite dimensional weak formulation:

Find $u_h \in V_h$ such that

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$$\underbrace{\int_0^1 \frac{du_h}{dx}\frac{d\varphi}{dx}\,dx - [g_r\varphi(1) - g_l\varphi(0)]}_{=:a(u_h,\varphi)} = \underbrace{\int_0^1 f(x)\varphi\,dx}_{=:F(\varphi)}, \qquad \forall\varphi \in V_h$$

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The finite element problem is then to find $u_h \in V_h$ such that

$$a(u_h,\varphi) = F(\varphi), \qquad \forall\varphi \in V_h$$

Galerkin orthogonality

Let $u \in V$ and $u_h, \varphi \in V_h$ and consider

$$a(u, \varphi) - a(u_h, \varphi) = F(\phi) - F(\phi) = 0$$

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Then, we get

$$a(\underbrace{u - u_h}_{=e_h}, \varphi) = 0, \qquad \forall \varphi \in V_h,$$

which means that the error e_h is orthogonal to V_h.

Finite element projection

Given a general function $f(x)$. We want to represent $f(x)$ in V_h. That is, find

$$f_h(x) \in V_h \qquad \text{s.t.} \qquad f_h(x) = f(x)$$

in a weak sense in V_h.

From finite dimensional weak form to a linear algebra problem

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iii. Plug $u_h = \sum_j U_j \phi_j$ into the finite dim. weak form.

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Note:
Always test again ϕ_i and always expand u_h using ϕ_j, then i and j will represent rows and columns, respectively.

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$$\int_0^1 \frac{du_h}{dx}\frac{d\varphi}{dx}\,dx = \int_0^1 f(x)\varphi\,dx + [g_r\varphi(1) - g_l\varphi_l(0)] =: F(\phi)$$

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Given a basis $B = \{\varphi_i\}_{i=0}^{\dim(V_h)}$ for V_h. Choose φ to be φ_i.
Since $u_h \in V_h$, we have

$$\int_0^1 \frac{d}{dx}\left(\sum_j U_j\varphi_j\right)\frac{d\varphi_i}{dx}\,dx = \sum_j U_j \int_0^1 \frac{d\varphi_j}{dx}\frac{d\varphi_i}{dx}\,dx = F(\varphi_i)$$

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From finite dimensional weak form to a linear algebra problem

More examples:

Strong, weak and finite dimensional problems

Consider the following PDE in strong form:

$$
\begin{aligned}
u_t + \nabla \cdot (\mathbf{a}u) - \nabla \cdot \mu \nabla u &= 0, \\
u &= u_B, & \forall x \in \Gamma_D, \\
\partial_{\mathbf{n}} u &= f(x), & \forall x \in \Gamma_N, \\
u(\mathbf{a} \cdot \mathbf{n}) - \mu \partial_{\mathbf{n}} u &= g(x), & \forall x \in \Gamma_R
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The corresponding weak form is given by

$$\partial_t \int_\Omega u\varphi dx + \int_\Omega \nabla \cdot (\mathbf{a}u)\varphi dx - \int_\Omega \nabla \cdot (\mu \nabla u)\varphi dx = 0, \quad \forall \varphi \in V$$

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$$
\int_\Omega \nabla \cdot (\mathbf{a} u) \varphi dx = - \int_\Omega u (\mathbf{a} \cdot \nabla \varphi) dx + \int_{\partial \Omega} u (\mathbf{a} \cdot \mathbf{n}) \varphi ds
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-\int_\Omega \nabla \cdot (\mu \nabla u)\varphi dx = \int_\Omega \mu \nabla u \cdot \nabla \varphi dx - \int_{\partial\Omega} \mu (\nabla u \cdot \mathbf{n})\varphi ds
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Strong, weak and finite dimensional problems

The weak form is given by

$$\partial_t \int_\Omega u\varphi dx - \int_\Omega u(\mathbf{a} \cdot \nabla\varphi) dx + \int_\Omega \mu \nabla u \cdot \nabla\varphi dx + B(u,\varphi) = 0, \quad \forall \varphi \in V$$

where $B(u,\varphi)$ is an operator that accounts for the boundary conditions.

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Strong, weak and finite dimensional problems

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What is the space V?

$$V = \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ in } \Gamma_D\}.$$

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where $B(u,\varphi)$ is an operator that accounts for the boundary conditions.

The boundary conditions are imposed via

$$B(u,\varphi) = \int_{\partial\Omega} \left[u(\mathbf{a} \cdot \mathbf{n}) - \mu(\nabla u \cdot \mathbf{n})\right] \varphi ds$$

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The boundary conditions are imposed via

$$\begin{aligned} B(u,\varphi) &= \int_{\partial\Omega} \left[u(\mathbf{a}\cdot\mathbf{n}) - \mu(\nabla u \cdot \mathbf{n})\right] \varphi ds \\ &= \int_{\Gamma_N} \left[u(\mathbf{a}\cdot\mathbf{n}) - \mu f\right] \varphi ds + \int_{\Gamma_R} g\varphi ds \end{aligned}$$

Note that the integral in Γ_D vanishes since $\varphi = 0$ in Γ_D.

Strong, weak and finite dimensional problems

The **discrete** weak form is given by

$$\partial_t \int_\Omega u_h \varphi dx - \int_\Omega u_h (\mathbf{a} \cdot \nabla \varphi) dx + \int_\Omega \mu \nabla u_h \cdot \nabla \varphi dx + B(u_h, \varphi) = 0, \quad \forall \varphi \in V_h$$

Strong, weak and finite dimensional problems

To get the **linear algebra** problem, choose $\phi = \phi_i \in B$. Then we get

$$\partial_t \int_\Omega u_h \varphi_i dx - \int_\Omega u_h (\mathbf{a} \cdot \nabla \varphi_i) dx + \int_\Omega \mu \nabla u_h \cdot \nabla \varphi_i dx + B(u_h, \varphi_i) = 0$$

Strong, weak and finite dimensional problems

To get the **linear algebra** problem, choose $\phi = \phi_i \in B$. Then we get

$$\partial_t \int_\Omega u_h \varphi_i dx - \int_\Omega u_h (\mathbf{a} \cdot \nabla \varphi_i) dx + \int_\Omega \mu \nabla u_h \cdot \nabla \varphi_i dx + B(u_h, \varphi_i) = 0$$

Or in matrix form

$$\sum_j \dot{U}_j M_{ij} - \sum_j U_j K_{ij} + \sum_j U_j S_{ij} + BCs = 0,$$

where M_{ij}, K_{ij} and S_{ij} are the entries of the mass, transport and stiffness matrix.

Strong, weak and finite dimensional problems

During the implementation we need to consider the time integration method.

Integration by parts (IBP) in multiple dimensions

Divergence theorem:

$$\int_{\Omega} \nabla \cdot \mathbf{a} dx = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} dx$$

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IBP based on the divergence:

$$\int_\Omega (\nabla \cdot \mathbf{a})\varphi dx = -\int_\Omega \mathbf{a} \cdot \nabla\varphi dx + \int_{\partial\Omega} (\mathbf{a} \cdot \mathbf{n})\varphi dx$$