Prof. Dr. Stefan Hofmann Winter term 2017/18

Exercises on General Relativity TVI TMP-TC1

Problem set 5, due November 27th

Exercise 1 – Gravitational waves

Due to the non-linearity of gravity the gravitational wave itself serves as a source. Therefore we consider a weak wave in order to neglect any non-linearities and work with Fierz-Pauli without self-interaction.
In this theory the equations of motion are as discussed before

$$\Box \bar{h}_{\mu\nu} = 0 \; . \tag{1}$$

We are familiar with this differential equation from electrodynamics, therefore we take as an ansatz the plane-wave expansion:

$$\bar{h}_{\mu\nu}(x) = \int \mathrm{d}\omega_k \sum_s \epsilon^s_{\mu\nu}(k) a^s(k) \exp(i k_\alpha x^\alpha) + \text{c.c.} \tag{2}$$

where $\epsilon^s_{\mu\nu}(k)$ is a momentum-dependent symmetric rank-2 tensor, $a^s(k)$ is a complex function and $\mathrm{d}\omega_k$ is the abbreviation of $\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega}$ with $\omega := k^0$. The sum runs over all linear independent components of the tensor.

(i) Using this ansatz and plugging it into (1) what properties of the wave can you deduce?

(ii) Since no further conditions on ϵ were imposed this ansatz possesses 10 degrees of freedom and therefore has to be gauged. Show that using de-Donder gauge introduces 4 conditions:

$$k_\mu \epsilon^{\mu\nu}_s = 0 \; . \tag{3}$$

As we discussed in exercise 3.2 there are 4 degrees of freedom left which are removed by a coordinate transformation with an arbitrary vector field e^μ satisfying $\Box e^\mu = 0$. It can be shown that this freedom of choosing e^μ and of choosing a Lorentz frame allows us to make ϵ^s traceless and $\epsilon^s_{0\nu} = 0$. This choice is called the transverse traceless gauge.

(iii) Considering a monochromatic wave-mode propagating in z-direction with $k_\alpha = (k_0, 0, 0, k_3)$, determine the components of ϵ_s in terms of the remaining two degrees of freedom. How can these degrees of freedom be interpreted physically?

(iv) How does a ring with radius R lying in the (x, y) plane is distorted during the propagation of the gravitational wave through the ring? Sketch your results and discuss how you could build a detector of gravitational waves.

Exercise 2 – Sources of gravitational waves

The goal of this exercise is to derive the amplitude of a gravitational wave generated by a specific source which could be measured on earth.
Analogously to electrodynamics we find the solutions for an arbitrary source with the Green's function method. The defining equation for the Green's function G is given by

$$\Box\, G(x^\alpha - y^\alpha) = \delta^{(4)}(x^\alpha - y^\alpha)\ . \tag{4}$$

Then the general solution to (1) for a source given by the energy-momentum tensor T is

$$\bar{h}_{\mu\nu}(x^\alpha) = -16\pi G_N \int \mathrm{d}^4 y\, G(x^\alpha - y^\alpha) T_{\mu\nu}(y^\alpha)\ . \tag{5}$$

Convince yourself with exercise 3.2 that the retarded Green's function for the d'Alembert operator is given by

$$G(x^\alpha - y^\alpha) = -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta\left(|\boldsymbol{x}-\boldsymbol{y}| - (x^0 - y^0)\right) H(x^0 - y^0) \tag{6}$$

with H the Heaviside step function.

(i) Plug the retarded Green's function into the general solution and perform the y^0 integration. Further go into frequency space using the Fourier transformation:

$$\hat{\bar{h}}_{\mu\nu}(\omega, \boldsymbol{x}) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x^0\, \mathrm{e}^{i\omega x^0}\, \bar{h}_{\mu\nu}(x^0, \boldsymbol{x})\ . \tag{7}$$

(ii) Assuming the distance r between the source and the earth is huge compared to the extend of the source a, we can approximate $|\boldsymbol{x}-\boldsymbol{y}|$ as the constant r. Additionally, using the de-Donder gauge in frequency space should give you as an result

$$\hat{\bar{h}}_{ij}(\omega, \boldsymbol{x}) = \frac{4G}{r}\mathrm{e}^{i\omega r} \int \mathrm{d}^3 y\, \hat{T}_{ij}(\omega, \boldsymbol{y})\ . \tag{8}$$

(iii) Show that the integral can be rewritten as

$$\int \mathrm{d}^3 y\, \hat{T}_{ij}(\omega, \boldsymbol{y}) = -\frac{\omega^2}{2}\hat{Q}_{ij}(\omega) \tag{9}$$

with the quadrupole tensor

$$\hat{Q}^{ij}(\omega) = \int \mathrm{d}^3 y\, y^i y^j \hat{T}^{00}(\omega, \boldsymbol{y})\ , \tag{10}$$

by performing the following steps:

- since the source we want to describe is compact, boundary terms vanish; use partial integration
- exploit the conservation of the source $\partial_\mu T^{\mu\nu} = 0$ in frequency space
- consider the vanishing boundary term $\partial_k \left(y^i y^j \hat{T}^{0k}\right)$

(iv) Show that using the result of c), restoring the speed of light c and transforming back equation (8) the metric perturbation becomes

$$\bar{h}_{ij}(t, \boldsymbol{x}) = \frac{2G_N}{rc^4}\frac{\mathrm{d}}{\mathrm{d}t^2}Q_{ij}(t-r) \tag{11}$$

with $t := x^0$. Why is the first contribution to the generation of gravitational waves given by the quadrupole moment and not by the dipole moment as in electrodynamics?

(v) As an example consider a binary system with two point masses circling around each other in the (x,y) plane with masses $M_1 = 2M_\odot$, $M_2 = M_\odot$ and a distance between them of $a = 100\,\mathrm{km}$.
Compute the $\bar{h}_{11}$ component of the gravitational wave and estimate the amplitude as it could be measured on earth with the distance between the source and the detector $r = 40\,\mathrm{Mpc}$.
Optional: Research whether there are actual experiments which can detect such waves. Over 20 years ago gravitational waves generated by such a binary system were detected indirectly, can you imagine how?

General information

The lecture takes place on Monday at 14:00-16:00 and on Friday at 10:00 - 12:00 in A348 (Theresienstraße 37).

Presentation of solutions:
Monday at 16:00 - 18:00 in B 138

There are six tutorials:
Monday at 12:00 - 14:00 in A 249
Thursday at 16:00 - 18:00 in A 449
Friday at 14:00 - 16:00 in C 113 and A 249
Friday at 16:00 - 18:00 in A 249

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/tvi_tc1_gr/index.html`