Prof. Dr. Stefan Hofmann Winter term 2017/18

Exercises on General Relativity TVI TMP-TC1

Problem set 10, due January 14th

Exercise 1 – Connection

Let $U \subset \mathbb{R}^n$ be an open set. Consider the vector fields X, Y on U given by

$$X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j} \,. \tag{1}$$

Show that ∇ defined by

$$\nabla_X(Y) := \sum_{i,j=1}^{n} f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} \tag{2}$$

defines a connection on U.

Exercise 2 – Christoffel symbols

(i) Consider the Levi-Civita connection and find the transformation property of the Christoffel symbols Γ under arbitrary coordinate transformations:

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\mu}_{\rho\sigma} \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial \tilde{x}^{\beta}} \frac{\partial x^{\sigma}}{\partial \tilde{x}^{\gamma}} + \frac{\partial^2 x^{\sigma}}{\partial \tilde{x}^{\beta} \partial \tilde{x}^{\gamma}} \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\sigma}} \,, \tag{3}$$

where Γ is the Christoffel symbol in coordinates x and $\tilde{\Gamma}$ in coordinates $\tilde{x}$. Find a subset of the general coordinate transformations under which the Christoffel symbol transforms like a tensor. What does this tell you about the connection in arbitrary Cartesian coordinates if all Christoffel symbols vanish in a given Cartesian coordinate system?

(ii) As an example find the Christoffel symbols of $\mathbb{S}^2$ in the coordinates $x^{\mu} = (\theta, \phi)$ of problem 9.2.

(iii) Show that in these coordinates the Christoffel symbols fail to describe geodesics going through $\theta = \{0, \pi\}$. Find coordinates in which you can circumvent this problem. One cannot cover $\mathbb{S}^2$ globally by Euclidean coordinates, though, can you think of a way how this can be introduced in the neighborhood of an arbitrary point on the sphere?

(iv) Now consider two sets of connection coefficients Γ and $\hat{\Gamma}$. Take their difference

$$\Gamma^{\alpha}_{\beta\gamma} - \hat{\Gamma}^{\alpha}_{\beta\gamma} \tag{4}$$

and find whether it transforms as a tensor under arbitrary coordinate transformations. What are the consequences?

General information

The lecture takes place on Monday at 14:00-16:00 and on Friday at 10:00 - 12:00 in A348 (Theresienstraße 37).

Presentation of solutions:
Monday at 16:00 - 18:00 in B 138

There are six tutorials:
Monday at 12:00 - 14:00 in A 249
Thursday at 16:00 - 18:00 in A 449
Friday at 14:00 - 16:00 in C 113 and A 249
Friday at 16:00 - 18:00 in A 249

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/tvi_tc1_gr/index.html`