Exercises on General Relativity TVI TMP-TC1

Problem set 1, due October 30th

Exercise 1 – Noether currents and charges

Consider a d-dimensional action for a scalar field of the form $\alpha \in \{0, 1, \cdots, d\}$

$$\mathcal{S} = \int d^d x \mathcal{L}\left(\phi, \partial_\alpha \phi\right). \tag{1}$$

(i) Using the equations of motion, show that if $\mathcal{L}$ is invariant under the following transformation:

$$\phi(x) \to \phi(x) + \delta\phi(x), \tag{2}$$

there is a conserved Noether current of the form

$$j^\alpha(x) := \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} \delta\phi(x) \tag{3}$$

(ii) Show that the Noether charge

$$Q = \int dx^1 \cdots dx^d j^0(x) \tag{4}$$

is conserved in time.

(iii) Consider now a Lagrange function $L\left(x^\mu(t), \dot{x}^\mu(t)\right)$.
Find the Noether charge corresponding to small Lorentz transformations close to the identity. What is their physical interpretation?

Hint: Expand the Lorentz transformations Λ to first order, to obtain $\delta_s x^\mu(t) = \omega^\mu_\nu\, x^\nu(t)$ where ω^μ_ν is an antisymmetric matrix.

Exercise 2 – Energy momentum tensor of the free scalar field

Given the following Lagrange density of a free massive scalar field with an external scource $J(\boldsymbol{x})$:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + J\phi \tag{5}$$

(i) Determine the equations of motion for the field ϕ.

(ii) Use the condition that $\mathcal{L}$ does not depend explicitly on the space-time coordinates (it is invariant under space-time translations) to derive the energy momentum tensor $T^{\mu\nu}$, for $J = 0$ show that it is conserved i.e. $\partial_\mu T^{\mu\nu} = 0$.
Hint: The result should be: $T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi + \frac{1}{2}\eta^{\mu\nu}\left[-\partial_\alpha \phi \partial^\alpha \phi - m^2\phi^2 + 2J\phi\right]$

(iii) Prove that $T^{00} \geq 0$ and give its physical interpretation.

Exercise 3 – Conserved current in classical electrodynamics

The Lagrangian of classical electrodynamics (massless vector field) is the following:

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6}$$

Use the condition that $\mathcal{L}$ does not depend explicitly on the space-time coordinates (it is invariant under space-time translations) to derive the following equation

$$\partial_\mu \theta^{\mu\nu} = 0$$

where $\theta^{\mu\nu}$ is the corresponding Noether current.
In order to make $\theta^{\mu\nu}$ symmetric and gauge invariant add a suitable conserved quantity. Denote the new quantity as $T^{\mu\nu}$ and write it down under the assumption that the positive definiteness of T^{00} should be fulfilled. Spell out the component T^{00} recalling from the classical electrodynamics course.
Take this tensor and check whether a vertex density v, which couples electromagnetic fields to a scalar gravity field Φ, is possible as discussed in the lecture:

$$v = \Phi \, \mathrm{tr}_\eta(T) \,. \tag{7}$$

Recall why this rules out the scalar model for gravity.

General information

The lecture takes place on Monday at 14:00-16:00 and on Friday at 10:00 - 12:00 in A348 (Theresienstraße 37).

Presentation of solutions:
Monday at 16:00 - 18:00 in B 138

There are six tutorials:
Monday at 12:00 - 14:00 in A 249
Thursday at 16:00 - 18:00 in A 449
Friday at 14:00 - 16:00 in B 139, C 113 and A 249
Friday at 16:00 - 18:00 in A 249

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/tvi_tc1_gr/index.html`