Prof. Dr. Stefan Hofmann Winter term 2017/18

Exercises on General Relativity TVI TMP-TC1

Problem set 11, due January 21th

Exercise 1 – Parallel transport

A useful formula to find the components of a vector field V parallel transported along a curve is

$$U^\alpha \nabla_\alpha V^\beta = 0 \tag{1}$$

with $U^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}$ and x^α the parametrization of the curve. This expression and its derivation will be discussed in the tutorials.

(i) Show explicitly that parallel transport preserves the length of the vector.

(ii) Perform a parallel transport of a vector field V along a full circle (R, φ) with a fixed radius R in $\mathbb{R}^2$ by solving (1) and find the angle δ between $V(\varphi = 0)$ and $V(\varphi = 2\pi)$:

$$\cos\delta = \frac{V^\alpha(0)V_\alpha(2\pi)}{V^\alpha(0)V_\alpha(0)} \tag{2}$$

Convince yourself of the result with a picture.

Now consider a parallel transport on a 2-sphere. The metric g of a 2-sphere of radius R is given as

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = R^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \tag{3}$$

with $\mu, \nu \in I(2)$.

(iii) Draw a picture of the parallel transport of an arbitrary tangent vector along the following closed curve: $A \to B \to C \to A$ connected via geodesics with $A = (\theta = \pi/2, \varphi = 0)$, $B = (\theta = \pi/2, \varphi = \pi/2)$ and $C = (\theta = 0, \varphi = 0)$.

(iv) Now take as the curve a circle on the sphere with fixed angle $\theta = \theta_c$ parameterized as $x^\alpha = (\theta_c, \varphi)$. Convince yourself that the unit tangent of this curve is

$$U^\alpha = \frac{1}{R\sin\theta_c}(0, 1) \; . \tag{4}$$

Solve (1) and take as the starting point $\varphi = 0$ for which the vector field is given by $V^\alpha(0) = (V_0^\theta, V_0^\varphi)$ and as the end point an arbitrary angle φ.
The solutions are

$$V^\theta(\varphi) = V_0^\theta \cos\bar\varphi + V_0^\varphi \sin\theta_c \sin\bar\varphi \tag{5}$$

$$V^\varphi(\varphi) = V_0^\varphi \cos\bar\varphi - \frac{V_0^\theta}{\sin\theta_c}\sin\bar\varphi \tag{6}$$

with $\bar\varphi = \varphi\cos\theta_c$. Find the angle δ between $V(0)$ and $V(2\pi)$.
What is the behavior of δ at the equator and near the poles? What does this imply for analogue parallel transports on a cylinder?

Exercise 2 – Torsion

Let M be a smooth manifold and ∇ a connection on M. We define a tensor by

$$T(X,Y) = \nabla_X(Y) - \nabla_Y(X) - [X,Y]\ . \tag{7}$$

Prove that T is skew-symmetric and $C^\infty(M)$-bilinear.

Exercise – 3 Lie Derivatives

The Lie derivative of a vector field v along the vector field u was defined as $\mathcal{L}_u v = [u, v]$.

(i) Given that the action on a one form, σ, is defined by requiring the relation $\langle \mathcal{L}_u \sigma, v\rangle = u\langle \sigma, v\rangle - \langle \sigma, [u, v]\rangle$ to hold for all v, determine $\mathcal{L}_u \sigma$ in a coordinate basis.

(ii) Write down $\mathcal{L}_u T$ in a coordinate basis, if T is a $(0, 2)$ tensor.

(iii) For functions f, g show $\mathcal{L}_u fg = f\mathcal{L}_u g + g\mathcal{L}_u f$.

General information

The lecture takes place on Monday at 14:00-16:00 and on Friday at 10:00 - 12:00 in A348 (Theresienstraße 37).

Presentation of solutions:
Monday at 16:00 - 18:00 in B 138

There are six tutorials:
Monday at 12:00 - 14:00 in A 249
Thursday at 16:00 - 18:00 in A 449
Friday at 14:00 - 16:00 in C 113 and A 249
Friday at 16:00 - 18:00 in A 249

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/tvi_tc1_gr/index.html`