Exercises on General Relativity TVI TMP-TC1

Problem set 3, due November 13th

Exercise 1 – The Fierz-Pauli action functional

We introduced in the lecture the general functionals of spin-2 field h:

$$S_1[h] = \int_{M_4} h\big(P_\perp(h)\big)\ , \quad S_2[h] = \int_{M_4} h\big(P_\|(h)\big)\ , \tag{1}$$

$$S_3[h] = \int_{M_4} \mathrm{tr}_\eta(h)\,\mathrm{tr}_\eta(P_\perp(h))\ , \quad S_4[h] = \int_{M_4} \mathrm{tr}_\eta(h)\,\mathrm{tr}_\eta(P_\|(h))\ , \tag{2}$$

where $P_\| := \partial \otimes \partial$ and $P_\perp := \eta\,\Box - P_\|$.
Find the transformations of the above functionals under $\delta_\epsilon h_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)}$ with ϵ a smooth vector field.
In addition, show explicitely that the following expression,

$$S^k[h] = (8\bar{G}_N)^{-1}\big((S_1 - S_2)[h] - (S_3 - S_4)[h]\big)\ , \tag{3}$$

is invariant under this transformation. This is the so-called Fierz-Pauli action.

Exercise 2 – Coupling of the spin-2 field to a massive scalar field

(i) Use the Fierz-Pauli action $S^k[h]$ from equation (3) and vary it to derive the equations of motion, namely

$$E_{\mu\nu} := \Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial^\alpha\partial_{(\mu}h_{\nu)\alpha} - \eta_{\mu\nu}\left(\Box h - \partial_\alpha\partial_\beta h^{\alpha\beta}\right) = 0\ , \tag{4}$$

where we used the shorthand notation $h := h_\mu{}^\mu$.

(ii) In order to couple this field to a massive scalar, we introduce a source term $S_I[h]$ to the Fierz-Pauli action,

$$S_I[h] = \int_{M_4} h_{\mu\nu}T^{\mu\nu}\ , \tag{5}$$

where T is the energy-momentum tensor (EMT), changing the equations of motion to $E'_{\mu\nu} = E_{\mu\nu} + E^I_{\mu\nu} = 0$. Write down the new expression for the equation of motion explicitly. Do you find any similarities to the source theory of classical electrodynamics?

(iii) We now consider the example of a scalar scource field (at rest) with mass $M \in \mathbb{R}^+$. The corresponding EMT is given by

$$T^{\mu\nu}(\boldsymbol{x}) = M\eta_0^\mu\eta_0^\nu\delta^{(3)}(\boldsymbol{x})\ . \tag{6}$$

Argue, why this is a reasonable EMT for the system we want to describe.

(iv) Introduce for convenience the variable $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and eliminate the non-physical degrees of freedom of the massless spin-2 field using harmonic gauge $\partial^\mu\bar{h}_{\mu\nu} = 0$ at the level of the equations of motion.

(v) Count the remaining (physical) degrees of freedom that propagate, under the assumption that $h_{\mu\nu}$ is a totally symmetric tensor and solve the equations of motion $E'_{\mu\nu} = 0$ for $\bar{h}_{\mu\nu}$.

(vi) Finally, express the components of the field, h_{00} and h_{ij}, using the definition $\Phi(\boldsymbol{x}) = -\frac{\bar{G}_N M}{4\pi\,|\boldsymbol{x}|}$ and give a physical interpretation for h_{00}.

General information

The lecture takes place on Monday at 14:00-16:00 and on Friday at 10:00 - 12:00 in A348 (Theresienstraße 37).

Presentation of solutions:
Monday at 16:00 - 18:00 in B 138

There are six tutorials:
Monday at 12:00 - 14:00 in A 249
Thursday at 16:00 - 18:00 in A 449
Friday at 14:00 - 16:00 in B 139, C 113 and A 249
Friday at 16:00 - 18:00 in A 249

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/tvi_tc1_gr/index.html`