Q: A self-consistent completion ?

A: By Deser (1970)

Idea: 1st order formalism

$$L(q, \dot{q}) = \langle \dot{q} \, p \rangle - H(q, p) \;\wedge\; p = f(\dot{q})$$

Here,

$$q \mathrel{\hat{=}} h \in LT^{(2)}_{0}(M_4) \,, \quad \mathrm{weight}(h) = 1 \,,$$

$$[h] = 1$$

$$p \mathrel{\hat{=}} \Gamma \in LT^{1}_{(2)}(M_4) \,, \quad [\Gamma] = 1$$

$$\dot{q} \mathrel{\hat{=}} \langle \partial h \rangle$$

Then,

$$\langle \dot{q} \, p \rangle \mathrel{\hat{=}} \mu \int_{M_4} \langle \partial h \, \Gamma \rangle \doteq -\mu \int_{M_4} \langle h \, \partial \Gamma \rangle$$

$$\stackrel{\text{sym}}{=} -\mu \int_{M_4} h \, \langle \partial \Gamma \rangle$$

$$\stackrel{!}{\doteq} -M_p^2 \int_{M_4} M_p^{-1} \, h \, \langle \partial \Gamma \rangle$$

Q: What about $\mathcal{H}$ (Hamilton density)?

A: For a free theory, $H = \langle p p \rangle$. So,

$$\mathcal{H}_{\text{free}}(\Gamma) = \mu^2 \eta \langle \Gamma \Gamma \rangle$$
$$\stackrel{!}{=} M_P^2 \, \eta \langle \Gamma \Gamma \rangle$$

The symbolic action becomes (1st. order formalism!)

$$S^{\mathrm{FP}}[h, \Gamma] = -M_P^2 \int_{M_4} \left\{ M_P^{-1} h \langle \partial \Gamma \rangle - \eta \langle \Gamma \Gamma \rangle \right\}$$

It will turn out: This is just a 1st. order reformulation of Fierz-Pauli theory!

<u>Note:</u>

$$S^{\mathrm{FP}}[h, \Gamma] = -M_P^2 \int_{M_4} \left\{ M_P^{-1} h^I \langle \partial \Gamma \rangle_I - \eta^I \langle \Gamma \Gamma \rangle_I \right\}$$

$$I = (i_1, i_2), \quad i_j \in I_0(3) \text{ for } j \in \{1, 2\}.$$

The structural symbolism is quite useful:

$$h^I \langle \partial\Gamma \rangle_I = c_{11} h^I \partial_a \Gamma_I{}^a + \\ + c_{12} h^I \partial_{i_1} \Gamma_{i_2 a}{}^a$$

$$\eta^I \langle \Gamma\Gamma \rangle_I = c_{21} \eta^I \Gamma_I{}^a \Gamma_{ab}{}^b + \\ + c_{22} \eta^I \Gamma_{i_1 b}{}^a \Gamma_{i_2 a}{}^b$$

with unknown real coefficients c.

Variation:

1) $$\frac{\delta S^{FP}}{\delta h^I} = -M_p \langle \partial\Gamma \rangle_I \overset{!}{=} 0 \qquad (\dot{p} = \dots)$$

$$\rightsquigarrow \langle \partial\Gamma \rangle_I = 0 \rightsquigarrow$$

$$c_{11} \partial_a \Gamma_I{}^a + c_{12} \frac{1}{2} \partial_{(i_1} \Gamma_{i_2)a}{}^a = 0$$

2) $$\frac{\delta_1 S^{FP}}{\delta \Gamma_I{}^a} = - M_p \int_{M_4} h^J \frac{\delta}{\delta \Gamma_I{}^a} \langle \partial \Gamma \rangle_J + $$

$$+ M_p^2 \int_{M_4} \eta^J \frac{\delta}{\delta \Gamma_I{}^a} \langle \Gamma \Gamma \rangle_J$$

$$= + M_p \langle \partial h \rangle^I{}_a + M_p^2 \langle \eta \Gamma \rangle^I{}_a \underset{(\dot{q} \sim P + \dots)}{\overset{!}{=}} 0$$

This equation is a constraint for Γ.
In greater detail,

$$\langle \partial h \rangle^I{}_a = c_{11} \, \partial_a h^I +$$

$$+ c_{12} \, \frac{1}{2} \, \eta^{(i_1}{}_a \, \partial_c h^{i_2)c}$$

<u>Note</u>: The expression must not be symmetric in $(a \, i_j)$.

$$\langle \eta \Gamma \rangle^I{}_a = c_{121} \, \eta^I \Gamma_{ac}{}^c + c_{121} \frac{1}{2} \eta_a{}^{(i_1} \Gamma_c{}^{c \, i_2)} +$$

$$+ c_{122} \, \eta^{c \, (i_1} \Gamma_{ac}{}^{i_2)}$$

Solving the constraint equation is cumbersome.
It allows for a solution of the form

$$\Gamma = \langle \partial \bar{h} \rangle \in LT_{(2)1}(M_4)$$

where $\bar{h} = h - \frac{1}{2} \operatorname{tr}(h)\, \eta$.
Inserting this into the variation with respect to h
($\hat{=}$ eom for Γ) we find

$$\langle \partial \Gamma \rangle \Big|_{\Gamma = \langle \partial \bar{h} \rangle} = 0$$

which is equivalent to the source-free Fierz-Pauli
equation for h.
So far, we found that

$$S^{1FP}[h, \Gamma] = -M_p^2 \int_{M_4} \left\{ M_p^{-1} h \langle \partial \Gamma \rangle - \eta \langle \Gamma \Gamma \rangle \right\}$$

is the appropriate 1st order formulation of free
Fierz-Pauli theory.

Q: What about a source term?

A: Try minimal coupling,

$$v = \overset{M_p^{-1}}{\vphantom{h}} h \left(T^{FP} \right) \quad , \quad T^{FP} = -M_p^2 \underset{\sim p^2}{\langle \Gamma\Gamma \rangle}$$

with $T^{FP} \in LT_{(2)}(M_4)$.

The action becomes

$$S[h,\Gamma] = S^{FP} + \int_{M_4} v(h,\Gamma) .$$

In greater detail,

$$S[h,\Gamma] = M_p^2 \int_{M_4} \left\{ M_p^{-1}(-h) \langle \partial\Gamma \rangle + \right.$$
$$\left. + \left(\eta - M_p^{-1} h \right) \langle \Gamma\Gamma \rangle \right\}$$

In a global inertial coordinate system,

$$S[h,\Gamma] = M_p^2 \int_{M_4} \left\{ \left(\eta - M_p^{-1} h \right) \langle \partial\Gamma \rangle + \right.$$
$$\left. + \left(\eta - M_p^{-1} h \right) \langle \Gamma\Gamma \rangle \right\}$$

This suggests a shift

$$h =: M\eta - M\mathfrak{g}\,.$$

In global inertial coordinates, this is a constant shift! Therefore, we may consider $\mathfrak{g}$ to carry the dynamical degrees of freedom (instead of h). Then,

$$S[\mathfrak{g},\Gamma] = M_P^2 \int_{M_4} \mathfrak{g}\left\{\langle\partial\Gamma\rangle + \langle\Gamma\Gamma\rangle\right\}$$

$$\equiv M_P^2 \int_{M_4} \mathfrak{g}^I \, Ric_I(\Gamma)$$

<u>Note:</u> weight $(\mathfrak{g}) = 1$ and we are still using the first-order formalism!

Variations:

1) $\frac{\delta S}{\delta g^I} \stackrel{!}{=} 0 \rightsquigarrow \mathrm{Ric}_I(\Gamma) = 0$

2) $\frac{\delta S}{\delta \Gamma_I{}^a} \stackrel{!}{=} 0$ constraint for Γ

The constraint equation is very cumbersome to solve. Its solution can be written as

$$\Gamma_I{}^a = \langle g^{-1} \partial g \rangle_I{}^a ,$$

where $g = \det^{1/2}(g)\, g^{-1}$.
Using the solution of the constraint for Γ in the equation of motion, we find

$$\mathrm{Ric}(g) = 0 . \quad \lrcorner$$