Prof. Dr. Stefan Hofmann Winter term 2017/18

Exercises on General Relativity TVI TMP-TC1

Problem set 7, due December 11th

Exercise 1 – Maxwell theory in spherical coordinates

(i) Consider the Maxwell action functional

$$S[A] = -\tfrac{1}{4}\int_{M_4} F^2(A)\ . \tag{1}$$

Write down this action and the corresponding equations of motion explicitly in spherical spatial coordinates.

(ii) A two-sphere of fixed radius R in three-dimensional Euclidean space is considered using polar coordinates $(\theta,\varphi)\in[0,\pi]\times[0,2\pi[$:

$$\begin{aligned} x^1(\theta,\varphi) &= R\sin\theta\cos\varphi \\ x^2(\theta,\varphi) &= R\sin\theta\sin\varphi \\ x^3(\theta,\varphi) &= R\cos\theta\ . \end{aligned}$$

Calculate the area of the two-sphere with

$$A = \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, \sqrt{\det(M)} \tag{2}$$

using the results of the previous exercise and

$$M = \begin{pmatrix} \frac{\partial \boldsymbol{x}}{\partial\theta}\cdot\frac{\partial \boldsymbol{x}}{\partial\theta} & \frac{\partial \boldsymbol{x}}{\partial\theta}\cdot\frac{\partial \boldsymbol{x}}{\partial\phi} \\ \frac{\partial \boldsymbol{x}}{\partial\phi}\cdot\frac{\partial \boldsymbol{x}}{\partial\theta} & \frac{\partial \boldsymbol{x}}{\partial\phi}\cdot\frac{\partial \boldsymbol{x}}{\partial\phi} \end{pmatrix}\ . \tag{3}$$

Exercise 2 – Stereographic projections of a n-sphere

Consider the n-sphere:

$$\mathbb{S}^n = \{x\in\mathbb{R}^{n+1}\ \big|\ \|x\|^2 = 1\}\,. \tag{4}$$

Let $N := (0,\dots,0,1)$ be the north pole and $S := (0,\dots,0,-1)$ the south pole of the n-sphere and define

$$U^+ := \mathbb{S}^n\setminus\{N\} \tag{5}$$

$$U^- := \mathbb{S}^n\setminus\{S\}\ . \tag{6}$$

We construct a map

$$\phi_+ : U^+ \longrightarrow \mathbb{R}^n\times\{0\}\subset\mathbb{R}^{n+1}\ , \tag{7}$$

by defining the image $\phi_+(p)$ of a point $p\in U^+$ to be the intersection point of the line through p and N with the plane $\mathbb{R}^n\times\{0\}$. The map ϕ_+ is called stereographic projection through N. Similarly, a map $\phi_- : U^-\longrightarrow\mathbb{R}^n\times\{0\}\subset\mathbb{R}^{n+1}$ is given by using lines passing through S instead of N. Show that $\{U^\pm,\phi_\pm\}$ defines an atlas on $\mathbb{S}^n$.

Exercise 3 – Diffeomorphism invariance of the measure

It was explicitly shown how the measure of the action of electrodynamics changes when a coordinate transformation is performed. Show that

$$\int \mathrm{d}\mu := \int \mathrm{d}^4x\sqrt{g} \tag{8}$$

with $g = -\det(g_{\mu\nu})$ is diffeomorphism invariant by using that the metric transforms as

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}\ . \tag{9}$$

General information

The lecture takes place on Monday at 14:00-16:00 and on Friday at 10:00 - 12:00 in A348 (Theresienstraße 37).

Presentation of solutions:
Monday at 16:00 - 18:00 in B 138

There are six tutorials:
Monday at 12:00 - 14:00 in A 249
Thursday at 16:00 - 18:00 in A 449
Friday at 14:00 - 16:00 in C 113 and A 249
Friday at 16:00 - 18:00 in A 249

The webpage for the lecture and exercises can be found at

`www.physik.uni-muenchen.de/lehre/vorlesungen/wise_17_18/tvi_tc1_gr/index.html`