Construction of Uniform Cubic B-Splines

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A group of local control points are used to determine the geometry of curve segments which form a B-spline. A curve segment does not necessarily have to pass through a control point but its desired at the two end points of the B-spline. Cubic B-splines are popular because of their continuity characteristics which make the segment joints smooth [1]. Figure 1 shows a cubic curve constructed from a series of curve segments $\mathbf{S_0}, \mathbf{S_1}, \mathbf{S_2}, ..., \mathbf{S_{m-3}}$, using $m+1$ control points $\mathbf{P}, \mathbf{P_1}, \mathbf{P_2}, ..., \mathbf{P_m}$.

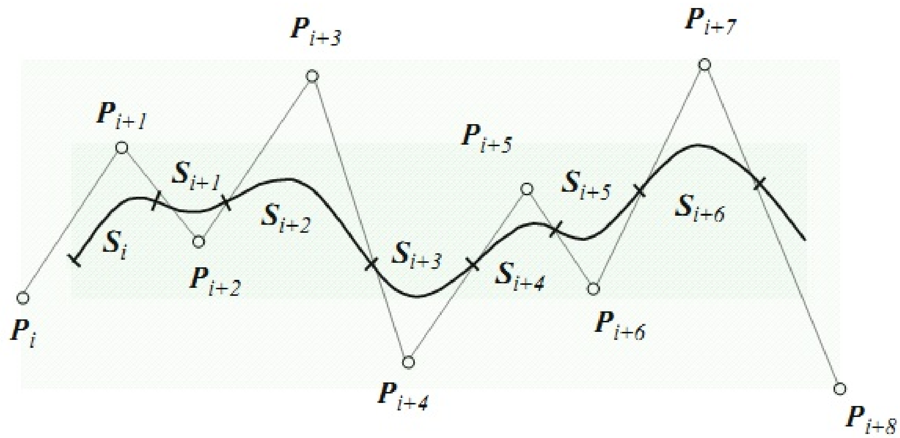

Figure 1: Construction of a uniform non-rational B-spline curve. [1]

Any single segment $\mathbf{S}_i(t)$ of a B-spline curve is defined by

$$\mathbf{S}_i(t) = \sum_{r=0}^{3} \mathbf{P}_{i+r} B_r(t) \tag{1}$$

where t $(0 \leq t \leq 1)$ is a parameter and the B-spline basis functions are as below [1]

$$B_0(t) = \frac{-t^3 + 3t^2 - 3t + 1}{6} = \frac{(1-t)^3}{6} \tag{2}$$

$$B_1(t) = \frac{3t^3 - 6t^2 + 4}{6} \tag{3}$$

$$B_2(t) = \frac{-3t^3 + 3t^2 + 3t + 1}{6} \tag{4}$$

$$B_3(t) = \frac{t^3}{6} \tag{5}$$

The B-spline curve, $Q_1(t)$ is represented in matrix form [1] by

$$\mathbf{Q_1}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \cdot \frac{1}{6} \cdot \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_i \\ P_{i+1} \\ P_{i+2} \\ P_{i+3} \end{bmatrix} \tag{6}$$

where $\mathbf{P_i}, \mathbf{P_{i+1}}, \mathbf{P_{i+2}}$ and $\mathbf{P_{i+3}}$ are the 4 consecutive control points of the total number of control points. So, if there are 5 control points then the first curve segment is made up of the first 4 control points (1, 2, 3, 4) and the second curve segment is made up of the second 4 consecutive control points (2, 3, 4, 5).

Continuity

The B-spline curve consists of many curve segments and it is essential to match the slopes of the curves to obtain a smooth continuous resultant spline curve. It is also necessary to match the rate of change of the slopes at the join since the slopes of the curve segments are changing constantly. The basis curves are defined such that first, second and third level of curve continuity is achieved as explained by the author in [1] .

- The first level of continuity, $\mathbf{C^0}$, ensures that the physical end of one basis curve corresponds with the next one [1].

$$S_i(1) = S_{i+1}(0) \tag{7}$$

- The second level of continuity, $\mathbf{C^1}$, ensures that the slope at the end of one basis curve matches that of the following curve [1].
- The third level of continuity, $\mathbf{C^2}$, ensures that the rate of change of slope at the end of one basis curve matches the following curve [1].

The continuity properties of cubic B-splines are shown in figure 2 where the slopes and the derivatives of slopes are calculated for the joins between B_3, B_2, B_1, B_0 at $t = 0$ and $t = 1$.

	t			t			t	
$\mathbf{C}^0$	**0**	**1**	$\mathbf{C}^1$	**0**	**1**	$\mathbf{C}^2$	**0**	**1**
$B_3(t)$	0	1/6	$B_3'(t)$	0	0.5	$B_3''(t)$	0	1
$B_2(t)$	1/6	2/3	$B_2'(t)$	0.5	0	$B_2''(t)$	1	−2
$B_1(t)$	2/3	1/6	$B_1'(t)$	0	−0.5	$B_1''(t)$	−2	1
$B_0(t)$	1/6	0	$B_0'(t)$	−0.5	0	$B_0''(t)$	1	0

Figure 2: Continuity properties of cubic B-splines. [1]

End Points as Collocation Points

The one draw back of this method is that the spline curve does not pass through the two end control points. This property is necessary and is achieved as explained here for n control points. An extra control point $\mathbf{P_0}$ is defined such that, $\mathbf{P_1}$ is at an equal distance from $\mathbf{P_0}$ and $\mathbf{P_2}$. Similarly, $\mathbf{P_{n+1}}$ is defined such that, $\mathbf{P_n}$ is at an equal distance from $\mathbf{P_{n+1}}$ and $\mathbf{P_{n-1}}$. This makes the curve segment to pass through $\mathbf{P_1}$and $\mathbf{P_n}$ which now are the tangent points to the curve due to the extra control points. The points $\mathbf{P_0}$ and $\mathbf{P_{n+1}}$ are not new input points but are defined internally using $\mathbf{P_1}, \mathbf{P_2}$ and $\mathbf{P_{n-1}}, \mathbf{P_n}$ as given below:

$$P_0 = 2P_1 - P_2 \tag{8}$$

$$P_{n+1} = 2P_n - P_{n-1} \tag{9}$$

References

[1] John Vince. *Mathematics for Computer Graphics 2nd Edition*. Springer, New Jersey, 2006.