IMPROVED ALGORITHMS OF ILLINOIS-TYPE FOR THE NUMERICAL SOLUTION OF NONLINEAR EQUATIONS

J.A. FORD*

Department of Computer Science, University of Essex, Wivenhoe Park,
Colchester, Essex, United Kingdom

ABSTRACT

An approach is described for the construction of a class of derivative-free methods for the solution of a single nonlinear equation in one variable and several new methods are obtained. The prototype for the class is the so-called "Illinois" Method (Dowell and Jarratt [1]), which itself is a variant of the classical method of Regula Falsi. These methods deal with the problem of "end-point retention" in Regula Falsi and the consequent failure to achieve superlinear convergence by modifying (with a suitably-chosen scaling factor) one of the function-values used in the linear interpolation. The results of numerical experiments on new and existing algorithms in the class are reported, indicating that the performance of some of the new methods obtained here is very promising. The local convergence of these methods is analysed and the asymptotic orders of convergence and patterns of behaviour are determined and compared with those of existing methods.

*e-mail: fordj@essex.ac.uk

1 INTRODUCTION

This paper will be concerned with a class of methods, which we shall call Illinois-type methods, for the solution of the single nonlinear equation

$$f(x) = 0 \tag{1}$$

in the single real variable x. The need to solve such equations arises frequently in many areas of numerical computation. A typical situation occurs when a nonlinear equation must be solved as a sub-problem within an iterative method (say, to determine a parameter required later in the iteration; see, for example, Ford and Ghandhari [2] or Ford and Moghrabi [3]), when the need for efficiency and speed is evident. The name for the class of methods is derived from the first such method, described by Dowell and Jarratt [1]. Each member of the class may be viewed as a modification of the classical Regula Falsi method and, as such, is derivative-free and guaranteed to converge to a solution, on condition that f is continuous and given a bracket $[a, b]$ for which $f(a).f(b) < 0$. Other known members of the class are the Pegasus method (Dowell and Jarratt [4]) and the method of Anderson and Björck [5]. We shall describe the construction of several other members of the class and analyse their asymptotic behaviour. The results of numerical tests will also be presented and these will demonstrate the superiority, from a practical point of view, of some of the new methods over the known methods. We observe, here, that all of these methods (both old and new) are easily programmed and that existing code for any of the known methods may readily be modified in order to implement the new methods.

We shall assume that the root x^* of interest in equation (1) is simple. Given two estimates (r and s, say) of x^* such that x^* lies within the bracket $[r, s]$, Regula Falsi computes a new estimate, t, by means of linear interpolation:

$$t = s - f(s)/f[s,r] \quad (2)$$

$$= r - f(r)/f[s,r]$$

$$= \{sf(r) - rf(s)\}/\{f(r) - f(s)\}, \quad (3)$$

where $f[s,r]$ denotes the standard divided difference

$$f[s,r] = \{f(s) - f(r)\}/\{s - r\}.$$

The next iterationis then carried out with a bracket constituted by the points t and either s (if $f(s).f(t) < 0$) or r. In this way, the root always lies within the current bracket and convergence is guaranteed. However, it is well-known that, for many functions, this process results ultimately in one end-point being permanently retained and, thus, only linear convergence. Illinois-type methods seek to eliminate this "end-point retention" in the following manner:

given two estimates x_{i-1} and x_i of x^* with $f_{i-1}f_i < 0$, where

$$f_j \stackrel{\text{def}}{=} f(x_j)\ \forall j, \quad (4)$$

apply (2)/(3) with $r = x_{i-1}$ and $s = x_i$, and call the result x_{i+1}. If $f_{i+1}f_i < 0$, then $r := x_i$, $s := x_{i+1}$, and apply (2)/(3) again. Otherwise, $f_{i+1}f_i > 0$, so $s := x_{i+1}$ but, in place of (2)/(3), the modified formula

$$t = \{s[\gamma f(r)] - rf(s)\}/\{[\gamma f(r)] - f(s)\} \quad (5)$$

is applied, where r has retained the value x_{i-1}. In other words, in the situation where $[x_i, x_{i+1}]$ does not constitute a bracket for x^*, the value f_{i-1} is scaled by the factor γ.

The various methods of this type correspond to differing choices of the parameter γ :

Method	Value of γ
Illinois	0.5
Pegasus	$f_i/(f_i + f_{i+1})$
Anderson & Björck	$f[x_{i+1}, x_i]/f[x_i, x_{i-1}]$

A step utilising formula (3) is known as an *unmodified* step and is denoted by the letter **U**, whereas a step employing (5) is said to be *modified* and will be denoted here by **I**, **P** or **A**, according to which of the three methods described above is in use. It is possible for the expression for γ in Anderson and Björck's method to yield a negative value, in which case $\gamma = 0.5$ is used and the step is denoted by **M**. In this method, however, $\gamma \sim 1$ asymptotically, so that **M**-steps are only to be expected far from the root, if at all. It is worth pointing out, at this juncture, that it is still possible (in any of these methods) for an end-point to be retained for several iterations (though not permanently), so that a particular function-value f_{i-1} may, on such occasions, be scaled by several "γ - factors". Nevertheless, it can be proved (see the discussion below) that, asymptotically, each method exhibits its own definite pattern of behaviour, whereby a modified step (or pair of steps, in some cases) is always followed by unmodified steps.

Before listing some of the properties of these methods, we introduce some standard notation. Let

$$c_k = f^{(k)}(x^*)/k!,\ k = 1, 2, \ldots,$$

$$\beta = c_2/c_1$$

and

$$K = \beta^2 - c_3/c_1.$$

(Note that we may infer the condition

$$c_1 \neq 0,$$

from the assumption made above that the root x^* is simple.) Furthermore, we define the errors ε_i by

$$\varepsilon_i = x_i - x^*$$

and recall (Dowell and Jarratt [1], for example) that, for an unmodified step, the successive errors are related by (ignoring the higher-order terms)

$$\varepsilon_{i+1} \sim \beta \varepsilon_i \varepsilon_{i-1}. \tag{6}$$

The following results concerning asymptotic behaviour patterns and speed of convergence are known for the Illinois-type methods described above (see Dowell and Jarratt [1, 4]; Anderson and Björck [5]) :

Method	**Asymptotic Step Pattern**	**Order over Pattern**	**Efficiency Index**
Illinois	**IUU**	3	1.44225
Pegasus	**PPUU**	7.27492	1.64232
A & B ($K > 0$)	**AUU**	5	1.70998
A & B ($K < 0$)	**AAUU**	8	1.68179

(The "Order over Pattern" column is obtained by relating the errors at the beginning and end of a complete pattern of steps. For example, in the Illinois method, it may be shown that, over one complete pattern **IUU** of steps,

$$\varepsilon_{i+3} \sim \beta^2 \varepsilon_i^3.$$

Hence, the order is three over the stated pattern and the efficiency index is (since *three* steps / function evaluations were required) $3^{1/3} \approx 1.44225$. The efficiency index is that defined by Traub [6].)

2 NEW METHODS

In order to develop new methods of the "Illinois" type, it is instructive to examine a typical instance in which a modified step is required. We therefore consider the situation illustrated in Figure 1, where the standard linear interpolation between the points (x_{i-1}, f_{i-1}) and (x_i, f_i) has yielded an estimate x_{i+1} on the *same side* of the root x^* as x_i, necessitating (in the terminology of Section 1) a modified step. It may readily be seen from the figure that, ideally, we would wish to choose the factor γ so that the chord joining the points $(x_{i-1}, \gamma f_{i-1})$ and (x_{i+1}, f_{i+1}) passes through the desired point $(x^*, 0)$. Examination of the slope of the straight line passing through this point and some elementary co-ordinate geometry then yields the relation

$$\gamma f_{i-1}/(x_{i-1} - x^*) = f_{i+1}/(x_{i+1} - x^*),$$

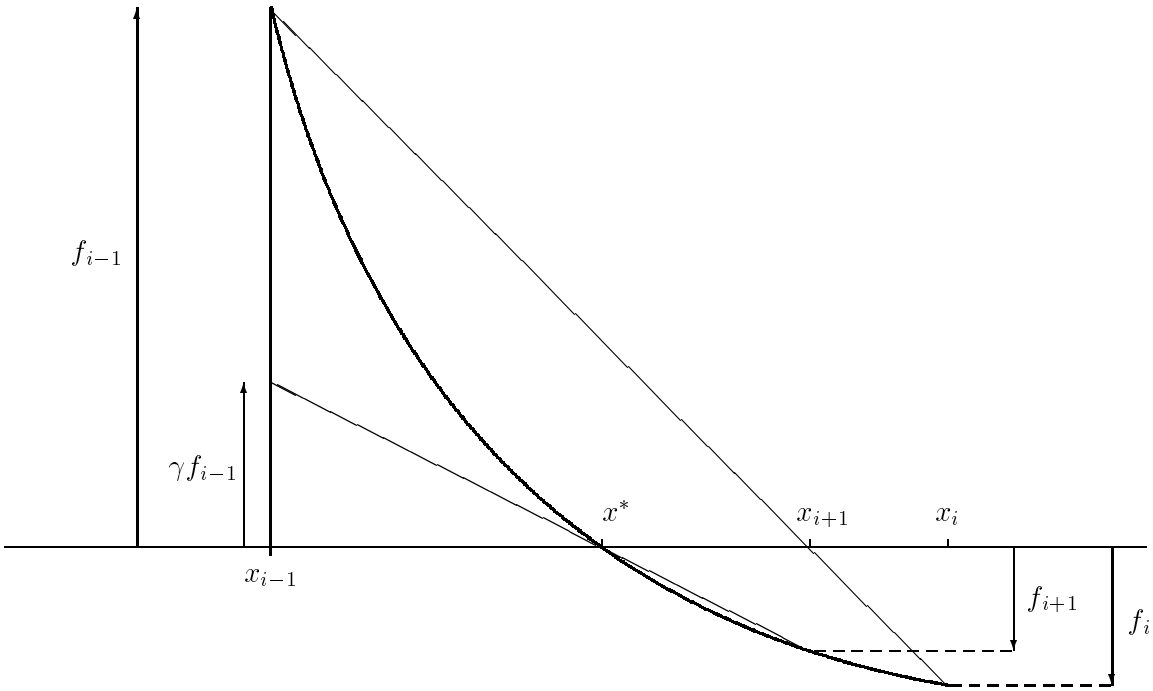
so that

$$\gamma = \{f_{i+1}/(x_{i+1} - x^*)\}.\{(x_{i-1} - x^*)/f_{i-1}\}. \tag{7}$$

Evidently, this expression for γ is not computable but, by approximating the constituents of the expression with other, known, quantities, we are able to obtain new (and, in some cases, better) methods. Recalling that x^* is a zero of f, we write (7) in the form

$$\begin{aligned} \gamma &= \{(f_{i+1} - f(x^*))/(x_{i+1} - x^*)\}/\{(f_{i-1} - f(x^*))/(x_{i-1} - x^*)\} \\ &= f[x_{i+1}, x^*]/f[x_{i-1}, x^*]. \end{aligned} \tag{8}$$

Figure 1: The ideal value of γ



Thus, it is apparent that the ideal value for γ is the ratio of the slopes of the chords joining, respectively, (x_{i-1}, f_{i-1}) with $(x^*, f(x^*))$ and (x_{i+1}, f_{i+1}) with $(x^*, f(x^*))$. It is then possible to appreciate why the method of Anderson and Björck is effective, because it amounts to approximating x^* in (8) with x_i:

$$\gamma_{AB} = f[x_{i+1}, x_i]/f[x_{i-1}, x_i].$$

(It should be remarked, however, that Anderson and Björck derived their formula by other means, using a derivative-free variant of Newton's Method. Their method is also closely connected to hyperbolic interpolation.)

Once it is recognized that the essential problem is one of obtaining two slopes, it is possible to derive many new methods by employing and combining various estimates. A valuable tool in this respect is the expression for the derivative of the quadratic polynomial interpolating

f at the three points x_{i-1}, x_i and x_{i+1}. Traub [6] shows that, if $\{j,k,l\}$ is some permutation of $\{i-1,i,i+1\}$, then the derivative of the interpolating quadratic at x_j is given by the expression

$$f[x_j,x_k]+f[x_j,x_l]-f[x_k,x_l].$$

As an example of the application of this expression, it can be seen that one possibility for estimating the numerator in (8) is to replace x^* by the most recent approximation, x_{i+1}, and then use the derivative of the interpolating quadratic (q, say):

$$\begin{aligned} f[x_{i+1},x^*] &\approx f[x_{i+1},x_{i+1}] \\ &= f'(x_{i+1}) \text{ (formally)} \\ &\approx q'(x_{i+1}) \\ &= f[x_{i+1},x_i]+f[x_{i+1},x_{i-1}]-f[x_i,x_{i-1}]. \end{aligned}$$

We therefore define the following four new methods:

Method	**Approximation to** $f[x_{i+1},x^*]$	**Approximation to** $f[x_{i-1},x^*]$
1	$f[x_{i+1},x_i]+f[x_{i+1},x_{i-1}]-f[x_i,x_{i-1}]$	$f[x_{i+1},x_{i-1}]+f[x_i,x_{i-1}]-f[x_{i+1},x_i]$
2	$f[x_{i+1},x_i]$	$f[x_{i+1},x_{i-1}]$
3	$f[x_{i+1},x_i]+f[x_{i+1},x_{i-1}]-f[x_i,x_{i-1}]$	$f[x_{i+1},x_{i-1}]$
4	$f[x_{i+1},x_i]+f[x_{i+1},x_{i-1}]-f[x_i,x_{i-1}]$	$f[x_i,x_{i-1}]$

In each case, γ is defined to be the ratio of the two stated approximations. A fifth method, defined by the expression

$$\gamma = f[x_{i+1},x_i]/\{f[x_{i+1},x_{i-1}]+f[x_i,x_{i-1}]-f[x_{i+1},x_i]\}$$

was discarded after initial numerical tests showed its performance to be no better than the method of Anderson and Björck and, often, significantly worse.

Given that the step which produced the iterate x_{i+1} was unmodified, it is possible to re-write the expressions used in the various methods. We first obtain (using (2)) the following relationships between the divided differences:

$$\begin{aligned} f[x_{i+1}, x_i] &= (1 - \varphi_i) f[x_i, x_{i-1}], \\ f[x_{i+1}, x_{i-1}] &= (1 - \varphi_{i-1}) f[x_i, x_{i-1}], \end{aligned}$$

where we define

$$\varphi_j \stackrel{\text{def}}{=} \frac{f_{i+1}}{f_j}. \tag{9}$$

Then, we can write down the following alternative expressions for the values of γ:

Table 1: Alternative expressions for γ

Method	**Value of γ**
Anderson & Björck	$1 - \varphi_i$
Method 1	$(1 - \varphi_i - \varphi_{i-1})/(1 + \varphi_i - \varphi_{i-1})$
Method 2	$(1 - \varphi_i)/(1 - \varphi_{i-1})$
Method 3	$1 - [\varphi_i/(1 - \varphi_{i-1})]$
Method 4	$1 - \varphi_i - \varphi_{i-1}$
Method 5	$(1 - \varphi_i)/(1 + \varphi_i - \varphi_{i-1})$

It is clear that the methods outlined above by no means exhaust the list of possible choices for γ and we might well consider, for example, suitably-chosen convex combinations of appropriate terms. We shall restrict ourselves here, however, to consideration of the methods we have numbered 1 to 4.

3 NUMERICAL TESTS

Before embarking upon an examination of the local convergence properties of the new methods, we present the results of numerical experiments. Tests have been carried out on a wide variety of functions: a representative sample of the results obtained is presented here. Some of the functions used here were drawn from the literature on the subject of solving nonlinear equations: function (3) is taken from Anderson and Björck [5], function (5) is studied by Hopgood and McKee [7], function (8) comes from the paper by Shacham and Kehat [8] and function (9) is considered by Ortega and Poole [9]. Other test functions were specifically designed to test the capabilities of the methods under examination on different types of function. In order to reduce the possibility of freak results, a selection of different initial brackets was employed for each function. The tests were carried out in double precision (about 16 decimal places). A tolerance, τ, provided by the user is converted to a "program tolerance", ε, by means of the relation

$$\varepsilon = \tau + 2^{-53} max(\mid a \mid, \mid b \mid, 1),$$

where a and b are the end-points of the initial bracket containing the desired root. Convergence was assumed when a point was obtained where the function-value was less than ε in modulus or when the length of the current bracket was less than 0.95ε. In all the experiments recorded here, τ was 10^{-14} and a limit of 200 iterations was imposed.

The test functions

1.

$$\begin{aligned} f(x) &\equiv 4\cos(x) - \exp(x); \\ x^* &= 0.90478821787302. \end{aligned}$$

2.

$$\begin{aligned} f(x) &\equiv \sum_{i=1}^{10}\{\exp(xt_i) - \exp(5t_i)\} \text{ (where } t_i = 0.1i); \\ x^* &= 5.0. \end{aligned}$$

3.

$$\begin{aligned} f(x) &\equiv 2x\exp(-20) + 1 - 2\exp(-20x); \\ x^* &= 0.034657358821882. \end{aligned}$$

4.

$$\begin{aligned} f(x) &\equiv \exp(x^{-1} - 25) - 1; \\ x^* &= 0.04. \end{aligned}$$

5.

$$\begin{aligned} f(x) &\equiv 10^{-8}(x-1)\prod_{i=1}^{10}(x^2 + x + i); \\ x^* &= 1.0. \end{aligned}$$

6.

$$\begin{aligned} f(x) &\equiv 10^{10}(x^{1/x}) - 1; \\ x^* &= 0.1. \end{aligned}$$

7.

$$\begin{aligned} f(x) &\equiv x^{20} - 1; \\ x^* &= 1.0. \end{aligned}$$

8.

$$\begin{aligned} f(x) &\equiv \exp(21,000/x)/(1.11 \times 10^{11}x^2) - 1; \\ x^* &= 551.77382493033. \end{aligned}$$

9.

$$\begin{aligned} f(x) &\equiv 1/x + \ln(x) - 100; \\ x^* &= 0.0095556044375379. \end{aligned}$$

10.

$$\begin{aligned} f(x) &\equiv \exp[\exp(x)] - \exp[\exp(1.0)]; \\ x^* &= 1.0. \end{aligned}$$

11.

$$\begin{aligned} f(x) &\equiv \sin(0.01/x) - 0.01; \\ x^* &= 0.99998333286109. \end{aligned}$$

In the following tables (Tables 2 and 3) of results, we give the number of iterations required by each method to locate the root to within the specified tolerance. The notation "200+" indicates that the method required more than 200 iterations (in many cases, substantially more than 200) in order to converge.

Table 2: Comparison of old and new methods

Function (root)	**Initial Bracket**	**Illinois**	**Pegasus**	**Anderson & Björck**	**Method 1**	**Method 2**	**Method 3**	**Method 4**
1	[0, 1.5]	8	7	6	8	7	6	7
(0.904788)	[-1, 3]	11	10	10	13	11	10	10
	[-1.5, 6]	20	20	17	19	18	17	17
2	[4, 6.5]	11	8	9	9	8	7	8
(5.0)	[2, 8]	12	11	12	11	13	11	10
	[0, 15]	24	21	23	15	26	13	14
	[-5, 25]	40	39	125	19	128	16	15
3	[0, 1]	9	10	11	10	14	9	10
(0.034657)	[-0.1, 1.5]	15	14	43	11	46	12	12
	[-0.5, 2]	33	31	200+	16	200+	14	13
	[-1, 4]	47	44	200+	20	200+	16	16
4	[0.035, 0.05]	14	14	18	9	21	12	10
(0.04)	[0.03, 0.09]	27	26	200+	15	200+	7	13
	[0.025, 0.5]	49	46	200+	19	200+	17	15
	[0.02, 1]	58	55	200+	19	200+	18	18
5	[0.9, 1.1]	8	7	7	8	8	7	7
(1.0)	[0.5, 1.5]	15	14	12	20	12	17	17
	[-5, 5]	54	70	58	49	46	46	42
	[-5, 10]	70	103	45	65	51	65	59
6	[0.095, 1.0]	35	33	12	17	14	12	13
(0.1)	[0.075, 0.15]	23	23	200+	17	200+	16	16
	[0.08, 0.5]	38	35	200+	31	200+	22	21
	[0.05, 0.2]	36	34	200+	21	200+	15	18

Table 3: Comparison of old and new methods (continued)

Function (root)	**Initial Bracket**	**Illinois**	**Pegasus**	**Anderson & Björck**	**Method 1**	**Method 2**	**Method 3**	**Method 4**
7	[0.9, 1.05]	8	8	9	9	10	7	8
(1.0)	[0.7, 1.2]	15	14	23	13	24	11	11
	[0, 2.5]	42	42	200+	19	200+	16	16
	[-0.5, 5]	60	59	200+	21	200+	19	19
8	[550, 560]	7	5	5	6	6	5	6
(551.774)	[525, 590]	11	9	9	9	10	9	9
	[400, 600]	29	27	18	15	20	12	12
	[350, 850]	44	42	200+	15	200+	15	19
9	[0.005, 0.02]	10	7	7	7	8	9	8
(0.009556)	[0.001, 0.05]	13	13	7	12	9	11	10
	[0.0001, 0.1]	17	16	8	11	11	12	12
	[0.001, 100]	21	18	17	13	23	17	22
10	[0, 2]	13	14	12	11	13	11	11
(1.0)	[-4, 2]	19	17	32	14	36	10	12
	[-10, 3]	43	42	200+	16	200+	14	14
	[0.5, 3.5]	51	48	11	13	12	10	12
11	[0.5, 2]	10	6	5	5	8	7	8
(0.999983)	[0.2, 6]	12	10	5	10	9	9	10
	[0.05, 20]	15	11	5	12	9	10	11
	[0.004, 200]	21	18	7	13	10	13	15

4 DISCUSSION OF NUMERICAL RESULTS

In our discussion of the results presented in Tables 2 and 3, it should first be emphasized that, for "simple" functions such as (1), the behaviour of all the methods may be expected to be broadly similar, as the figures indeed indicate. On functions of greater complexity or difficulty, however, it does appear to be possible to draw some firm conclusions about the relative performance of the seven methods we have considered. We observe, first, that the behaviour of Method 2 is very similar to that of Anderson and Björck's method. Only on rare occasions is there a significant difference in the figures returned by the two methods and we shall therefore omit further discussion of Method 2.

Perhaps the most noticeable feature of the results is the frequent inability of the method of Anderson and Björck to converge within the specified limit of 200 iterations, whereas none of the other methods (except Method 2) experienced this difficulty at all. It appears to be the case that Anderson and Björck's method may have this difficulty when the initial bracket is (relatively) large, so that a and b are far from the desired root. Typically, the method will almost immediately commence the asymptotic "**AUU**" pattern of behaviour, but make extremely slow progress towards the root.

Notwithstanding this difficulty, Anderson and Björck's method is noticeably better than all the other methods, on occasion; function (11) probably provides the best demonstration of this. (We note, recalling the observation of Section 2 that this method is related to hyperbolic interpolation, that this function is almost hyperbolic in its behaviour near the root.) Conversely, there are also functions for which Anderson and Björck's method is clearly worse (for example, function (2)).

With regard to the new methods, it would appear to be the case that, overall, Method 1 is not quite as effective as Methods 3 and 4, although it may out-perform them on occasion

(see, for instance, function (9)). Methods 3 and 4 are rather more difficult to separate; more extensive testing has indicated that Method 3 may have a slight overall advantage. It is, however, certainly fair to conclude from the results presented here that all three of the new methods we are now considering have performed better, in general, than the older methods. The Illinois and Pegasus methods are reliable but sometimes comparatively slow (for example, functions (7) and (10)), while the method of Anderson and Björck suffers from the disadvantages discussed above.

5 ANALYSIS OF CONVERGENCE

We shall examine the local convergence of Methods 1, 3 and 4 by studying the application of one or more modified steps in each case, on the assumption that the preceding step (that is, the step producing x_{i+1}) was unmodified. It then follows that the latest error, ε_{i+2}, is related to the prevous errors by (compare equations (2) and (3)):

$$\begin{aligned} \varepsilon_{i+2} &= \{\gamma f_{i-1}\varepsilon_{i+1} - f_{i+1}\varepsilon_{i-1}\}/\{\gamma f_{i-1} - f_{i+1}\} \\ &= \frac{\varepsilon_{i-1}\varepsilon_{i+1}}{\gamma f_{i-1} - f_{i+1}}\left(\gamma\frac{f_{i-1}}{\varepsilon_{i-1}} - \frac{f_{i+1}}{\varepsilon_{i+1}}\right). \end{aligned} \tag{10}$$

In all three of the new methods, it is evident (using the relations (6) and (9)) that $\gamma \sim 1$ asymptotically (see Table 1). Hence, we choose to express γ in the form

$$\gamma = 1 + \nu/\eta, \tag{11}$$

where we assume (and will demonstrate in each case) that

$$\nu = O(\varepsilon);\ \eta = O(1), \tag{12}$$

where

$$\varepsilon = \max(|\ \varepsilon_{i-1}\ |, |\ \varepsilon_i\ |). \tag{13}$$

It follows that the bracketed term (which we denote by ψ) in (10) may be expressed in the form

$$\psi = c_2(\varepsilon_{i-1} - \varepsilon_{i+1}) + c_3\varepsilon_{i-1}^2 + \frac{\nu}{\eta}(c_1 + c_2\varepsilon_{i-1}) + O(\varepsilon^3), \tag{14}$$

since

$$\varepsilon_{i+1} = \beta\varepsilon_{i-1}\varepsilon_i + O(\varepsilon^3) = O(\varepsilon^2), \tag{15}$$

from (6). Relation (15) implies that, asymptotically,

$$\mathrm{sgn}(\varepsilon_{i+1}) = -\mathrm{sgn}(\beta) \tag{16}$$

and, since a modified step was necessary to produce ε_{i+2}, it is evident that

$$\mathrm{sgn}(\varepsilon_i) = -\mathrm{sgn}(\beta) \tag{17}$$

and

$$\mathrm{sgn}(\varepsilon_{i-1}) = \mathrm{sgn}(\beta). \tag{18}$$

5.1 Method 1

It is not difficult to show from the expression for γ in this method that

$$\begin{aligned} \nu &= 2\{c_2(\varepsilon_{i+1} - \varepsilon_{i-1}) - c_3\varepsilon_{i-1}(\varepsilon_i + \varepsilon_{i-1}) + O(\varepsilon^3)\}; \\ \eta &= c_1 + 2c_2\varepsilon_{i-1} + c_3\varepsilon_{i-1}(2\varepsilon_{i-1} + \varepsilon_i) + O(\varepsilon^3). \end{aligned}$$

Thus, from (14),

$$\psi = -c_2\varepsilon_{i-1} + O(\varepsilon^2)$$

and, substituting in (10), we obtain

$$\varepsilon_{i+2} = -\beta\varepsilon_{i-1}\varepsilon_{i+1} + O(\varepsilon^4). \tag{19}$$

From (16) and (18), we deduce that, in the neighbourhood of the root,

$$\mathrm{sgn}(\varepsilon_{i+2}) = -\mathrm{sgn}(\varepsilon_{i+1}),$$

so that the next step will be unmodified. Consequently, if we denote a modified step for this method by **B**, the asymptotic step pattern will be "**BUU**". To compute the order of the method over the complete composite step **BUU**, we may derive the relation

$$\varepsilon_{i+4} \sim \beta^2 \varepsilon_{i+1}^4 \varepsilon_{i-2}^{-1}.$$

Thus, the order is the dominant zero, $2+\sqrt{3} \approx 3.73205$, of the quadratic $\rho^2 - 4\rho + 1$ and the efficiency index is

$$(2+\sqrt{3})^{1/3} \approx 1.55113. \tag{20}$$

5.2 Method 3

Returning to equation (14), we can show that the quantities ν and η (equation (11)) for this method are given by

$$\begin{aligned} \nu &= c_2(\varepsilon_{i+1} - \varepsilon_{i-1}) - c_3\varepsilon_{i-1}(\varepsilon_i + \varepsilon_{i-1}) + O(\varepsilon^3); \\ \eta &= c_1 + c_2(\varepsilon_{i+1} + \varepsilon_{i-1}) + c_3\varepsilon_{i-1}^2 + O(\varepsilon^3). \end{aligned}$$

It follows that

$$\psi = -c_3\varepsilon_{i-1}\varepsilon_i + O(\varepsilon^3).$$

Thus, from (10), we obtain

$$\varepsilon_{i+2} = -(c_3/c_1)\varepsilon_{i-1}\varepsilon_i\varepsilon_{i+1} + O(\varepsilon^5). \tag{21}$$

(It is interesting to note that the same result, to $O(\varepsilon^5)$, is obtained when the Müller-Traub method is used; see Anderson and Björck [5], for instance.) It is apparent that the properties of Method 3 will depend upon the sign of (c_3/c_1). If it is negative, then equations (16), (17) and (18) imply that

$$\mathrm{sgn}(\varepsilon_{i+2}) = \mathrm{sgn}(\beta) = -\mathrm{sgn}(\varepsilon_{i+1})$$

and the next step will be unmodified. Denoting a modified step for this method by **E**, we conclude that the asymptotic step sequence, in the case when $(c_3/c_1) < 0$, will be "**EUU**" and the order over this sequence may be derived from the relation

$$\varepsilon_{i+4} \sim (c_3/c_1)^2 \varepsilon_{i+1}^5.$$

Hence, the order is 5 and the efficiency index is

$$5^{1/3} \approx 1.70998. \tag{22}$$

Turning to the case when $(c_3/c_1) > 0$, it is clear that (21) then implies that

$$\mathrm{sgn}(\varepsilon_{i+2}) = -\mathrm{sgn}(\beta) = \mathrm{sgn}(\varepsilon_{i+1}), \tag{23}$$

so that a further modified step will be required. In order to analyse the effect of a second such step, we will require some relations (valid whatever the actual value of γ used) derived by Anderson and Björck [5]:-

$$\begin{aligned}
\frac{\tilde{f}_{i-1} - f_{i+1}}{x_{i-1} - x_{i+1}} &= \tilde{f}[x_{i-1}, x_{i+1}] \\
&= \frac{f_{i+1}}{(\varepsilon_{i+1} - \varepsilon_{i+2})}; && (24) \\
\gamma f_{i-1} &= \tilde{f}_{i-1} \\
&= \{(\varepsilon_{i-1} - \varepsilon_{i+2})/(\varepsilon_{i+1} - \varepsilon_{i+2})\} f_{i+1} && (25) \\
&= \theta f_{i+1}, \text{say.} && (26)
\end{aligned}$$

The "γ - factor" required for scaling $\tilde{f}_{i-1}$ for the second modified step will therefore be given by

$$\begin{aligned}
\hat{\gamma} &= (f[x_{i+2}, x_{i+1}] + \tilde{f}[x_{i+2}, x_{i-1}] - \tilde{f}[x_{i+1}, x_{i-1}])/\tilde{f}[x_{i+2}, x_{i-1}] \\
&= \{f_{i+1} - (1 + \theta^{-1}) f_{i+2}\}/\{f_{i+1} - \theta^{-1} f_{i+2}\}.
\end{aligned}$$

Thus, the scaled function-value to be used in the new step is

$$\begin{aligned}\hat{f}_{i-1} &= \hat{\gamma}\tilde{f}_{i-1} \\ &= \theta\{f_{i+1} - (1+\theta^{-1})f_{i+2}\}f_{i+1}/\{f_{i+1} - \theta^{-1}f_{i+2}\}. \qquad (27)\end{aligned}$$

This step will yield an error ε_{i+3} defined by

$$\begin{aligned}\varepsilon_{i+3} &= (\hat{f}_{i-1}\varepsilon_{i+2} - f_{i+2}\varepsilon_{i-1})/(\hat{f}_{i-1} - f_{i+2}) \\ &= \phi_1/\phi_2, \text{ say,} \qquad (28)\end{aligned}$$

where, using (27),

$$\begin{aligned}\phi_1 &= \theta\varepsilon_{i+2}f_{i+1}\{f_{i+1} - (1+\theta^{-1})f_{i+2}\} - \varepsilon_{i-1}f_{i+2}\{f_{i+1} - \theta^{-1}f_{i+2}\} \\ \text{and} \quad \phi_2 &= \theta f_{i+1}\{f_{i+1} - (1+\theta^{-1})f_{i+2}\} - f_{i+2}\{f_{i+1} - \theta^{-1}f_{i+2}\}.\end{aligned}$$

Observing, from (25) and (26), that $\theta = O(\varepsilon^{-1})$, we may then show that

$$\begin{aligned}\phi_1 &= c_2\varepsilon_{i-1}\varepsilon_{i+1}^2\varepsilon_{i+2}f_{i+1}/(\varepsilon_{i+1} - \varepsilon_{i+2}) + O(\varepsilon^{10}) \\ \text{and} \quad \phi_2 &= \theta f_{i+1}^2 + O(\varepsilon^5).\end{aligned}$$

Thus, in (28),

$$\varepsilon_{i+3} = \beta\varepsilon_{i+1}\varepsilon_{i+2} + O(\varepsilon^7). \qquad (29)$$

We deduce, by means of (23), that

$$\text{sgn}(\varepsilon_{i+3}) = \text{sgn}(\beta) = -\text{sgn}(\varepsilon_{i+2}),$$

so that the next step will be unmodified. We have therefore shown that, asymptotically, the pattern of steps in the case when $(c_3/c_1) > 0$ will be "**EEUU**" with corresponding order 8, derived from the relation

$$\varepsilon_{i+5} \sim -\beta(c_3/c_1)^3\varepsilon_{i+1}^8.$$

Finally, the efficiency index is

$$8^{1/4} \approx 1.68179. \qquad (30)$$

5.3 Method 4

Returning to equation (14) again, it may first be shown, for Method 4, that

$$\begin{aligned} \nu &= c_2(2\varepsilon_{i+1} - \varepsilon_{i-1} - \varepsilon_i) - c_3(\varepsilon_{i-1}^2 + \varepsilon_{i-1}\varepsilon_i + \varepsilon_i^2) + O(\varepsilon^3); \\ \eta &= c_1 + c_2(\varepsilon_{i-1} + \varepsilon_i) + c_3(\varepsilon_{i-1}^2 + \varepsilon_{i-1}\varepsilon_i + \varepsilon_i^2) + O(\varepsilon^3). \end{aligned}$$

Thus,

$$\psi = -c_2\varepsilon_i + O(\varepsilon^2)$$

and, by (10),

$$\varepsilon_{i+2} = -\beta\varepsilon_i\varepsilon_{i+1} + O(\varepsilon^4). \tag{31}$$

Hence, (16) and (17) show that, as the root is approached,

$$\mathrm{sgn}(\varepsilon_{i+2}) = -\mathrm{sgn}(\beta) = \mathrm{sgn}(\varepsilon_{i+1}), \tag{32}$$

so that a further modified step will be necessary. To analyse the effect of such a step, we note, first, that equations (24), (25) and (26) still hold and that, therefore, the new "γ - factor" will be

$$\begin{aligned} \hat{\gamma} &= (f[x_{i+2}, x_{i+1}] + \tilde{f}[x_{i+2}, x_{i-1}] - \tilde{f}[x_{i+1}, x_{i-1}])/\tilde{f}[x_{i+1}, x_{i-1}] \\ &= \{f_{i+1} - (1 + \theta^{-1})f_{i+2}\}/f_{i+1}, \end{aligned}$$

which implies that the scaled version of $\tilde{f}_{i-1}$ is given by

$$\begin{aligned} \hat{f}_{i-1} &= \hat{\gamma}\tilde{f}_{i-1} \\ &= \theta\{f_{i+1} - (1 + \theta^{-1})f_{i+2}\}, \end{aligned} \tag{33}$$

by virtue of (26). The second modified step will thus yield an error determined by

$$\begin{aligned} \varepsilon_{i+3} &= (\hat{f}_{i-1}\varepsilon_{i+2} - f_{i+2}\varepsilon_{i-1})/(\hat{f}_{i-1} - f_{i+2}) \\ &= \phi_1/\phi_2, \text{ say,} \end{aligned}$$

where, from (33),

$$\begin{aligned} \phi_1 &= \theta\varepsilon_{i+2}\{f_{i+1} - (1+\theta^{-1})f_{i+2}\} - \varepsilon_{i-1}f_{i+2} \\ \text{and} \quad \phi_2 &= \theta\{f_{i+1} - (1+\theta^{-1})f_{i+2}\} - f_{i+2}. \end{aligned}$$

Since $\theta = O(\varepsilon^{-1})$ again, it follows, using (31), that

$$\begin{aligned} \phi_1 &= c_2\varepsilon_{i+1}^2\varepsilon_{i+2}(\varepsilon_{i-1} + 2\varepsilon_i/(\varepsilon_{i+1} - \varepsilon_{i+2}) + O(\varepsilon^7); \\ \phi_2 &= \theta f_{i+1} + O(\varepsilon^3), \end{aligned}$$

and hence that

$$\varepsilon_{i+3} = \beta\varepsilon_{i+1}\varepsilon_{i+2}(1 + 2\varepsilon_i/\varepsilon_{i-1}) + O(\varepsilon^6).$$

Therefore, as long as $\varepsilon_i/\varepsilon_{i-1}$ remains less than $1/2$ in magnitude (in practice, we would anticipate that $\varepsilon_i/\varepsilon_{i-1} \to 0$), we may deduce that, asymptotically,

$$\varepsilon_{i+3} \sim \alpha\varepsilon_{i+1}\varepsilon_{i+2}, \tag{34}$$

for some constant α with the same sign as β. From (32), we infer that ε_{i+3} and ε_{i+2} will have opposite sign, so that the next step will be unmodified. The sequence of iterations as the root is approached may therefore be expected to be "**FFUU**" (where **F** denotes a modified step for this method) and the order may be determined from the relation

$$\varepsilon_{i+9} \sim -\alpha\beta^4\varepsilon_{i+5}^7\varepsilon_{i+1}^{-1}.$$

The order is thus the dominant zero of the quadratic $\rho^2 - 7\rho + 1$, which is $\tau^4 \approx 6.85410$ (where $\tau = (1+\sqrt{5})/2$, the Golden Section), while the efficiency index is therefore

$$\tau \approx 1.61803.$$

For the sake of completeness, we may state here that Method 2 exhibits the asymptotic pattern "**CUU**" (where **C** denotes a modified step for this method), has order $\tau^3 \approx 4.23607$ over this pattern and an efficiency index of τ.

To facilitate comparisons, we collect together the results on local convergence for all the methods (existing and new) considered in this paper:

Method	**Asymptotic Step Pattern**	**Order over Pattern**	**Efficiency Index**
Illinois	**IUU**	3	1.44225
Pegasus	**PPUU**	7.27492	1.64232
A & B ($K > 0$)	**AUU**	5	1.70998
A & B ($K < 0$)	**AAUU**	8	1.68179
Method 1	**BUU**	3.73205	1.55113
Method 2	**CUU**	4.23607	1.61803
Method 3 ($c_3/c_1 < 0$)	**EUU**	5	1.70998
Method 3 ($c_3/c_1 > 0$)	**EEUU**	8	1.68179
Method 4	**FFUU**	6.85410	1.61803

We remark, finally, that the numerical experiments reported in Section 3 confirm the conclusions of this section regarding the patterns of steps as the root is approached. In particular, Method 3 exhibited, asymptotically, the behaviour pattern "**EEUU**" for all the test functions, except function (1), for which the observed pattern was "**EUU**". (It is easily verified that $c_3/c_1 < 0$ for this function.)

6 SUMMARY AND CONCLUSIONS

A technique has been developed for the construction of methods of Illinois-type for the solution of single-variable nonlinear equations. Three of the new methods thus constructed have been shown, empirically, to exhibit improved numerical performance and their asymptotic properties have been determined.

Our analysis of the local convergence of the new methods has shown that only Method 3 is competitive with Anderson and Björck's method insofar as efficiency indices are concerned. In that practical experience hitherto appears to indicate that (c_3/c_1) and K are both more likely to be positive than negative, it may be reckoned that, locally, Method 3 is at a very slight disadvantage when compared with Anderson and Björck's method. We have already observed, however, that, globally, the picture is very different, with Methods 1, 3 and 4 often out-performing Anderson and Björck's method by a considerable margin. The behaviour of the new methods demonstrates clearly that such measures as order and efficiency index by no means tell the whole story and that other factors must be taken into account in assessing the relative merits of methods. Such behaviour may also be taken as a strong indication that attempts to develop better methods in this class should devote at least as much attention to the issue of global behaviour as to improving the order of convergence.

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