Prof. Dr. Karsten Urban
M. Sc. Lewin Ernst

Winter term 2024/2025
Sheet 14, 26 Points

Numerical Optimization
exercise sheet

review on 05.02.2024 during the exercise class

1. *(Projected Subgradient Method)*
For nonsmooth optimization we are forced to generalize the concept of classical derivatives of a function f at a point x_0. For this, we have introduced for convex functions f defined on a convex set the convex subdifferential ∂f, compare Definition 5.3.3.

   a) Calculate the following sub-differentials.

      i) $\partial f(0)$, for $f : \mathbb{R} \to \mathbb{R}$, $\quad f(x) := \begin{cases} x^2, & x < 0, \\ x, & x \geq 0. \end{cases}$

      ii) $\partial f(0)$, for $f : \mathbb{R}^n \to \mathbb{R}$, $\quad f(x) := \|x\|_2 = \sqrt{x^T x}$.

   b) The Projected Subgradient Method is a method for non-smooth optimization, where (for example) the cost function f is not differentiable in the classical sense. In contrast to the general gradient method, the Projected Subgradient Method uses on the one hand an element of the convex sub-differential instead of it's classical gradient and on the other hand a projection of $x^{(k+1)}$ onto the feasible domain $\mathcal{F}$ (compare Algorithm 5.3.1). However, the choice of a reasonable step-size is not that trivial due to the missing smoothness of f.

      Consider now the unconstrained optimization problem

$$\min_{x \in \mathbb{R}} f(x), \tag{1}$$

      for $f(x) = |x|$ with its solution $x^* = 0$. Use the Projected Subgradient Method with $P_{\mathcal{F}} = id$ (because of $\mathcal{F} = \mathbb{R}^n$) for solving the optimization problem (1).

      Choose the step sizes

      i) $\sigma^{(k)} = \frac{1}{k+1}$, as well as

      ii) $\sigma^{(k)} = \frac{f(x^{(k)}) - f(x^*)}{|s^{(k)}|}$,

      and the initial vector $x^{(0)} = 2$ for computing $x^{(1)}, \ldots, x^{(5)}$ (until $s^{(k)} = 0$).

      Hint: You can use without a proof that the convex sub-differential of f in $x_0 = 0$ is given by $\partial f(0) = [-1, 1]$.

   c) Consider the optimization problem (1) with $f(x) = (|x| + 1)^2$ and its solution $x^* = 0$. Prove that for the Projected Subgradient Method with step size $\sigma^{(k)} = (f(x^{(k)}) - f(x^*))/|s^{(k)}|$, it holds

$$|x^{(k+1)} - x^*| \leq \frac{1}{2}|x^{(k)} - x^*|, \qquad k = 1, 2, \ldots.$$

(4 + 4 + 6 = 14 Points)

2. *(Projected Subgradient Method)*

a) Apply the routine `projected_subgradient_method.m` to the Wolfe function

$$f^{\text{Wolfe}}(x,y) := \begin{cases} 5\sqrt{9x^2+16y^2}, & x \geq |y|, \\ 9x+16|y|, & 0 < x < |y|, \\ 9x+16|y|-x^9, & x \leq 0 \end{cases}$$

using different step sizes $\sigma^{(k)} = n/(k+1)$, e.g. $n = 1, 2, 3$. Plot the iteration path together with the solution $x^* = (1,0)^T$ and the contour lines of f^{Wolfe}.

b) Write a MATLAB routine

```
x = Projection_Parabel(x, a, b),
```

which performs a projection on the convex set

$$\mathcal{F} = \{(x,y)^\top \in \mathbb{R}^2 : y \geq (x-a)^2 + b\},$$

where $a, b \in \mathbb{R}$ are two parameters. Solve the constrained optimization problem

$$\min_{(x,y)^T \in \mathcal{F}} f^{\text{Wolfe}}(x,y)$$

for different parameter a and b by using the Projected Subgradient Method. Plot again the iteration path together with the set $\mathcal{F}$ and the contour lines of f^{Wolfe}.

(6 + 6 = 12 Points)