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Winter term 2024/2025
Sheet 01, 30 Points

# Numerical Optimization
# exercise sheet

review on 23.10.2024 during the exercise class

**Modus Operandi of the exercise class:**

- You choose a group (four members at max) for the exercises on Moodle. There will be a poll, where you can choose your group. If you have no group by 18th of October you will get automatically assign to one.
- There will be an exercise sheet, uploaded every Wednesday before the exercise class starts. It usually consists of three up to four tasks, covering theoretical aspects or numerical experiments (with matlab).
- You have one week to work on the sheet with your group.
- Before the subsequent exercise class you and your group can pick on moodle the tasks for which you have a solution. You have to upload the solutions on moodle. Your group will already get 50 percent of the points for those tasks. For the submitted tasks, you must pick also at least one expert of your group, who is willing to present the solution. The experts will get 100 percent of the points for those tasks.
- During the exercise class I will pick for each task an expert, who presents then his/her solution to the class.
- In the end you should have at least 70 percent of all points to pass the preliminaries (a.k.a. Vorleistung).

1. *(Derivation of Gauß-Newton Method)*
   Derive the Gauß-Newton method by using Taylor's expansion of the residual function $F : \mathbb{R}^n \to \mathbb{R}^m$ with a second order remainder term, i.e. expand $F(x)$ at the current iteration $x^{(k)} \in \mathbb{R}^n$ to approximate $F(x^{(k)} + s^{(k)})$ with an affine linear function.

   (10 Points)

2. *(Convergence of Gauß–Newton)*
   Consider for a parameter $\lambda \in \mathbb{R}$ the parametrized function

$$F_\lambda : \mathbb{R} \to \mathbb{R}^2, \qquad F_\lambda(x) = \begin{pmatrix} x+1 \\ \lambda x^2 + x - 1 \end{pmatrix}$$

   and the nonlinear least squares problem

$$g(x) := \frac{1}{2}\|F_\lambda(x)\|_2^2 \to \min . \tag{1}$$

   a) Prove that the function $g$ has a local minimum at $x^* = 0$, if $\lambda < 1$ and it's the only local minimum, if $\lambda < 7/16$.
   b) Prove that $x^* = 0$ is a repulsive fixpoint[1] of the Gauß-Newton-Method if $\lambda < -1$, i.e. there exists a $\delta > 0$, such that

$$|x_{k+1} - 0| > |x_k - 0| \qquad \text{for all } x_k \text{ with } 0 < |x_k - 0| < \delta.$$

   $(4 + 4 = 8$ Points)

[1] You can ask ChatGPT what a repulsive fixpoint is.

3. *(Gauß-Newton method,* MATLAB*)*

a) Write a MATLAB function

```
x = GaussNewton(F, dF, x0, maxIt, tol),
```

which computes the solution $\mathtt{x} \in \mathbb{R}^n$ of the nonlinear least squares problem

$$\mathtt{x} = \arg\min \|F(x)\|_2^2,$$

by using the Gauß-Newton-Method. You should be able to explain what happens in the code!
**Hint:** You are allowed to use ChatGPT (and you actually should do this).

b) Write a MATLAB function

```
x = Newton(F, J, x0, maxIter, tol),
```

which computes the solution $\mathtt{x} \in \mathbb{R}^n$ of the nonlinear root finding problem

$$\text{find } \mathtt{x} \in \mathbb{R}^n : F(x) = 0,$$

by using the Newton-Method, where `J` is the Jacobian of `F`.
**Hint:** You are allowed to use ChatGPT (and you actually should do this).

c) Extend the functions `GaussNewton.m` and `Newton.m`, such that all iterates `x` are returned. Further, adjust `GaussNewton.m` in a way, that for all iterations the relative error of the Hessian approximation of $g$ is calculated and returned. This means calculate for all iterations $k = 1, 2, ...$

$$\frac{\|(\nabla^2 g)(x^{(k)}) - \left(F'(x^{(k)})\right)^T F'(x^{(k)})\|_2}{\|(\nabla^2 g)(x^{(k)})\|_2}$$

and return it.

d) Write a script in which you apply your MATLAB-functions `GaussNewton.m` and `Newton.m` to the nonlinear least squares-problem (1) using the initial value $x_0 = 10$ and various values for $\lambda$ with $\lambda < 7/16$. Plot the error $|0 - x^{(k)}|$ over the iterations $k$ in a loglog-plot and also plot the relative error of the Hessian over the iterations. What do you observe?

(2 + 2 + 3 + 5 = 12 Points)