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Winter term 2024/2025
Sheet 13, 26 Points

Numerical Optimization
exercise sheet

review on 29.01.2024 during the exercise class

1. *(Interior Point Method for Linear Problems I)*
Let $B \in \mathbb{R}^{m\times n}$, $m \le n$, be a matrix with full rank and $(x,\lambda,s) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ a vector with $x > 0$ and $s > 0$. Prove that the (Jacobian) matrix

$$\nabla F_0(x,\lambda,s) = \begin{pmatrix} 0 & B^T & I \\ B & 0 & 0 \\ S & 0 & X \end{pmatrix},$$

with $X = \operatorname{diag}(x_1,\dots,x_n)$ and $S = \operatorname{diag}(s_1,\dots,s_n)$, is regular.

(6 Points)

2. *(Interior Point Method for Linear Problems II,* Matlab*)*
In this task, we will work with the Interior point method for linear problems. The linear equation system in step (6) of Algorithm 4.5.1. in the lecture notes can be solved efficiently due to the sparse structure of the matrix ∇F_0. For this, let

$$r := XSe - \sigma\tau e, \qquad D^2 := XS^{-1},$$

and consider the linear equation system

$$\nabla F_0 \begin{pmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{pmatrix} = \begin{pmatrix} 0 & B^\top & I \\ B & 0 & 0 \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ XSe - \sigma\tau e \end{pmatrix}.$$

Solving the first equation for Δs, the third equation for Δx and insert this in the second equation yields:

$$\begin{aligned} \Delta\lambda &= -\left(BD^2B^\top\right)^{-1} BS^{-1}r = -\left(BD^2B^\top\right)^{-1} B(Xe - \sigma\tau S^{-1}e), \\ \Delta s &= -B^\top \Delta\lambda, \\ \Delta x &= S^{-1}r - S^{-1}X\Delta s = (Xe - \sigma\tau S^{-1}e) - S^{-1}X\Delta s. \end{aligned}$$

a) Implement a MATLAB routine `InteriorPointMethod.m`, which performs the interior point method for linear problems.
b) Test your routine by using the file `test_InteriorPointMethod.m`.

(6 + 2 = 8 Points)

3. *(Interior Point Method for Linear Problems III,* MATLAB*)*
A toy store owner sells four types of Lego boxes in his store, we name them A, B, C and D. The store owner pays for each box A \$11, for a box B \$14, \$20 for C and \$25 for D. One unit of toys A yields a profit of \$3, a unit of B yields a profit of \$4, one unit of Lego C yields a profit of \$5, while one unit of D yields a profit of \$6. The store owner estimates that no more than 1000 lego boxes will be sold every month and he can't order more than 400 boxes of one kind in a month. Further, since he also sells different toys in his store, he does not plan to invest more than \$30 000 in inventory of Lego boxes. How many units of each type should be stocked in order to maximize his monthly total profit?

   a) Formulate the task above as a linear maximization problem and determine the dual problem (a minimization problem).

   b) Apply your interior point method from above or an internal routine of MATLAB to the dual problem. How many units of each type should be stocked in order to maximize his monthly total profit?

(6 + 6 = 12 Points)