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Winter term 2024/2025
Sheet 02, 31 Points

Numerical Optimization
exercise sheet

review on 30.10.2024 during the exercise class

1. *(Levenberg-Marquardt Method)*

   Consider again for a parameter $\lambda \in \mathbb{R}$ the parametrized function

$$F_\lambda : \mathbb{R} \to \mathbb{R}^2, \qquad F_\lambda(x) = \begin{pmatrix} x+1 \\ \lambda x^2 + x - 1 \end{pmatrix}$$

   and the nonlinear least squares problem

$$g(x) := \frac{1}{2}\|F_\lambda(x)\|_2^2 \to \min, \tag{1}$$

   which we have already seen on the last sheet.

   a) Prove that for every $\lambda < 1$ there exists a penalty-parameter $\mu > 0$, such that the Levenberg-Marquardt-Method converges locally to $x^* = 0$.

   b) In the material you will find the Matlab-function

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x = LevenbergMarquardt(F, dF, x0, mu, beta0, beta1, maxIt, tol),
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   which computes the solution $\mathtt{x} \in \mathbb{R}^n$ of the nonlinear least squares problem by using the Levenberg-Marquardt-Method. The damping parameter μ is controlled via the expression

$$\varepsilon_\mu := \frac{\|F(x_k)\|_2^2 - \|F(x_k + s_k)\|_2^2}{\|F(x_k)\|_2^2 - \|F(x_k) + F'(x_k)s_k\|_2^2}$$

   and the parameter $0 < \beta_0 < \beta_1 < 1$. Adjust the function such that the error in the Hessian approximation, i.e.

$$\frac{\left\|(\nabla^2 g)(x^{(k)}) - \left[\left(F'(x^{(k)})\right)^T F'(x^{(k)}) + \mu_k^2 I\right]\right\|_2}{\|(\nabla^2 g)(x^{(k)})\|_2}$$

   gets returned for all iterations as well as all damping parameter μ_k.

   c) Apply the method to the problem from above by adjusting the script from sheet 1 task 3 d) (it will be also in the material or you can use your own). You should monitor the convergence, the error of the hessian and the damping parameters μ_k for the Levenberg-Marquardt method. What do you observe for $\lambda < -1$?

   d) Copy your script from c) and adjust it, so that the circle regression from the lecture notes is solved and analysed. You can use ChatGPT to calculate the Hessian of F_i. It can provide you also a Matlab implementation, but be careful, it can make mistakes when calculating the Hessian.

(4 + 2 + 4 + 5 = 15 Points)

2. *(Convex functions)*

   a) Let $X \subset \mathbb{R}^n$ be convex. Prove:
   If $f_i : X \to \mathbb{R}$ is convex and $\alpha_i \geq 0$ for $i = 1, \ldots, m$, then

$$f(x) := \sum_{i=1}^{m} \alpha_i f_i(x)$$

   is convex on X. If at least one f_i is strictly convex and the corresponding $\alpha_i > 0$, then f is strictly convex on X.

   b) Let $X \subset \mathbb{R}^n$ be convex. Prove:
   If $g : X \to \mathbb{R}^m$ is affine and $f : \mathrm{Im}(g) \to \mathbb{R}$ is convex, then $(f \circ g)$ is convex on X, whereas $\mathrm{Im}(g)$ denotes the image of g. (Realise why $\mathrm{Im}(g)$ is convex!)

(4 + 4 = 8 Points)

3. *(Convexity and Minimizers)*

   a) Let $f : X \to \mathbb{R}$ be convex on some convex set $X \subset \mathbb{R}^n$. Then the set $S := \{x \in X : f(x) \leq f(y)\ \forall y \in X\}$ is convex, i.e. the set of global minimizers is convex.

   b) A point $x^* \in \mathcal{U} \subset \mathbb{R}^n$ is called an isolated local minimizer of $f : \mathcal{U} \to \mathbb{R}$, if there is a neighborhood $U(x^*)$ such that x^* is the only local minimizer in $U(x^*)$. Prove that isolated local minimizers are strict local minimizers.
   Hint: Prove the contraposition.
   Remark: Not every strict local minimizer is isolated, consider e.g. $x^* = 0$ of the function

$$f(x) = \begin{cases} x^4 \cos(1/x) + 2x^4, & x \neq 0 \\ 0, & \text{else} \end{cases} \in C^2(\mathbb{R}).$$

(4 + 4 = 8 Points)