Prof. Dr. Karsten Urban
M. Sc. Lewin Ernst

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Sheet 08, 30 Points

Numerical Optimization
exercise sheet

review on 11.12.2024 during the exercise class

1. *(Linear programming and KKT)*
We are going to show necessary and sufficient conditions for linear problems. From the Optimization lecture we know that every linear program can be written as

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} c^T x \\ \text{s.t. } &Ax = b \\ &x \geq 0 \end{aligned} \tag{1}$$

which has the associated dual problem

$$\begin{aligned} &\max_{(\lambda,\mu) \in \mathbb{R}^m \times \mathbb{R}^n} b^T \lambda \\ \text{s.t. } &A^T \lambda + \mu = c \\ &\mu \geq 0. \end{aligned} \tag{2}$$

Note that $\mu = c - A^T \lambda$ can be seen as a slack variable. For those linear problems, one can show

Theorem 1 (Necessary and sufficient optimality conditions)**.** *The following statements are equivalent:*

i) The primal problem (1) *has a solution* $\bar{x}$*.*

ii) The dual problem (2) *has a solution* $(\bar{\lambda}, \bar{\mu})$*.*

iii) The optimality system

$$A^T \lambda + \mu = c, \quad Ax = b, \quad x^T \mu = 0, \quad x \geq 0, \quad \mu \geq 0 \tag{3}$$

has a solution $(\bar{x}, \bar{\lambda}, \bar{\mu})$*.*

The tasks are:

a) Derive the dual problem (2) from the primal problem (1) with the definition of the dual problem (D) and the dual feasible region $\mathcal{F}_D$ of Section 3.2.1 in the lecture notes.

b) Proof Theorem 1.
Hint: Some hints are in order:
- The saddle point property (Theorem 3.2.3) is important.
- Show ACQ for such linear programs (you only have to prove it for the primal problem).
- Linear programs are convex, so you may use Theorem 3.1.20 (KKT for convex problems) in some way or another.
- How do the KKT conditions for problem (1) and problem (2) look like?

(14 Points)

2. *(Penalty Method)*
   Consider for $f \in C(\mathbb{R}^n)$ and $h \in C(\mathbb{R}^n; \mathbb{R}^m)$ the constrained optimization problem

$$\begin{cases} \min & f(x) \\ \text{s.t.} & h(x) = 0 \end{cases} \tag{4}$$

   with

$$\mathcal{F} := \{x \in \mathbb{R}^n : h(x) = 0\} \neq \emptyset.$$

   Let $(\tau_k)_{k\in\mathbb{N}} \subset \mathbb{R}_{>0}$ be a strictly monotonically increasing sequence with $\lim_{k\to\infty} \tau_k = \infty$. Prove that for a sequence $(x^{(k)})_{k\in\mathbb{N}}$ generated by the penalty method, it holds:

   a) The sequence $(\mathcal{L}_{\tau_k}(x^{(k)}))_{k\in\mathbb{N}}$ is monotonically increasing.
   b) The sequence $(\|h(x^{(k)})\|)_{k\in\mathbb{N}}$ is monotonically decreasing.
   c) The sequence $(f(x^{(k)}))_{k\in\mathbb{N}}$ is monotonically increasing.
   d) It holds $\lim_{k\to\infty} h(x^{(k)}) = 0$.
   e) Every accumulation point of $(x^{(k)})_{k\in\mathbb{N}}$ is a (global) solution of (4).

(8 Points)

3. *(Penalty Method and Lagrange Method)*

   a) Write a MATLAB routine

```
x = penalty_method(f, h, x0, tau0, rho, tol, maxIt)
```

   which performs the penalty method with a suitable termination criteria. The input parameter are given by:

   - `f`: The objective function $f : \mathbb{R}^n \to \mathbb{R}$ (function handle).
   - `h`: The constraint function $h : \mathbb{R}^n \to \mathbb{R}^m$ with $m \leq n$ (function handle).
   - `x0`: Initial value $x^{(0)} \in \mathbb{R}^n$.
   - `tau0`: Initial penalty parameter τ_0.
   - `rho`: Parameter for increasing the penalty parameter τ.
   - `tol`: Tolerance for termination criterion.
   - `maxIt`: Maximal number of iterations for termination criterion.

   The output parameter is given by:

   - `x`: Vector containing the evolution of all solutions computed by the penalty method.

   For minimizing the penalty Lagrange function $\mathcal{L}_\tau : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathcal{L}_\tau(x) = f(x) + \frac{1}{2}\tau\|h(x)\|^2$$

   you can use `fminsearch` or `fminuc`.

   b) Write a MATLAB routine

```
x = lagrange_method(f, h, x0, lambda0, tau0, rho, tol, maxIt)
```

   which performs the method of lagrange multipliers (Section 3.5 in the lecture notes) with a suitable termination criteria. In addition to the notations from subtask a) the input parameter are given by

- `lambda0`: Initial lagrange parameter $\lambda^{(0)}$.

The output parameter is given by:

- `x`: Vector containing the evolution of all solutions computed by the penalty method.

For minimizing the function $\mathcal{G}_\tau : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\mathcal{G}_\tau(x, \lambda) = f(x) + \lambda^\top h(x) + \frac{1}{2}\tau \|h(x)\|^2$$

you can use `fminsearch` or `fminuc`.

c) Apply the MATLAB routine `penalty_method.m` and `lagrange_method.m` to the function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x_1, x_2) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2$$

subject to the constraint $h(x_1, x_2) = 0$, where

$$h(x_1, x_2) = (x_1 + 0.5)^2 + (x_2 + 0.5)^2 - 0.25,$$

by using the initial value $x^{(0)} = (1, 1)^\top$ and the parameter $\tau^{(0)} = 0.1$, $\rho = 6$ and $\lambda^{(0)} = 10$, respectively. Create a plot displaying the contour lines of f together with the solution of each iteration of the penalty method and the method of lagrange multipliers. Discuss your results.

(8 Points)