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Sheet 11, 28 Points

Numerical Optimization
exercise sheet

review on 15.01.2024 during the exercise class

1. *(Active set method)*
The active set method is an optimization mehtod for quadratic problems (QP) with (affine) linear equality and inequality constraints of the form

$$\begin{cases} \min\limits_{x} f(x) := \frac{1}{2}x^T Ax + a^T x \\ \text{s.t. } Bx = b,\ Cx \le c, \end{cases} \tag{1}$$

where A is symmetric positive definite on $\ker B$ and B has full rank. The idea is to reduce the problem to a sequence of QPs with only equality constraints. At an iterate $x^{(k)} \in \mathcal{F}$ not being the solution x^*, we seek a *feasible descent direction* $d^{(k)}$ by solving (see (3.6.19))

$$\begin{cases} \min\limits_{d^{(k)}} f(x^{(k)} + d^{(k)}) \\ \text{s.t. } Bd^{(k)} = 0,\ C_j d^{(k)} = 0,\ j \in \mathcal{A}(x^{(k)}) \end{cases}$$

which is equivalent to

$$\begin{cases} \min\limits_{d^{(k)}} \frac{1}{2}(d^{(k)})^T A d^{(k)} + (d^{(k)})^T (Ax^{(k)} + a) \\ \text{s.t. } D^{(k)} d^{(k)} = 0. \end{cases} \tag{2}$$

We know from the lecture, that there can now arise three possibilities.

Case 1: $d^{(k)} = 0$, $\mu^{(k)} \ge 0$

Case 2: $d^{(k)} = 0$, $\mu_j^{(k)} < 0$ for at least on $j \in \mathcal{A}(x^{(k)})$

Case 3: $d^{(k)} \neq 0$ is a feasible direction

a) Derive the KKT system for (1) and (2).

b) Let (1) be convex and let $x^{(k)} \in \mathcal{F}$ be the current iterate. Show that, if for the solution $d^{(k)}$ and the Lagrange multiplier $\mu^{(k)}$ of (2) holds Case 1, then $x^{(k)}$ is a solution of (1).

c) Assume that we are in Case 2 and the Inactivation step has been performed, i.e. we have $d^{(k)} = 0$ and $\mu_j^{(k)} < 0$ for at least one $j \in \mathcal{A}^{(k)}$. Further, we have $\tilde{d}^{(k)}$, $\tilde{\lambda}^{(k)}$ and $\tilde{\mu}^{(k)}$ as the solution of problem (2) w.r.t. the set $\tilde{\mathcal{A}}^{(k)} := \mathcal{A}^{(k)} \setminus \{j\}$ as described in the lecture notes. Show that $\tilde{d}^{(k)}$ is a feasible direction at $x^{(k)}$ by showing $\tilde{d}^{(k)} \in \mathcal{L}(\mathcal{F}, x^{(k)})$.
Hint: Exploit the optimality conditions for $d^{(k)}$ and $\tilde{d}^{(k)}$.

d) Assume everything from c) holds and further assume that $D^{(k)}$ has full rank for all $k \in \mathbb{N}$. Show that $\tilde{d}^{(k)} \neq 0$.

(2 + 4 + 6 + 4 = 16 Points)

2. *(Active set method,* MATLAB*)*
In this exercise we want to visualize the iterations of the active set method.

a) Download the MATLAB function `activeset.m` from Moodle and adjust the method such that the `nullspace_method.m` is used to solve the subproblems.

b) Apply the active set method to the problem

$$\begin{aligned} \min_{x\in\mathbb{R}^2} f(x) &= (x_1-1)^2 + (x_2-2.5)^2 \\ \text{s.t. } x_1 - 2x_2 + 2 &\geq 0 \\ -x_1 - 2x_2 + 6 &\geq 0 \\ -x_1 + 2x_2 + 2 &\geq 0 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

with the initial value $x_0 = (2,0)^T$ and the associated active set $\mathcal{A}(x_0)$.

c) Plot the contours of f with the constraints and the path of the iterations of the active set method.

(2 + 4 + 6 = 12 Points)