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Winter term 2024/2025
Sheet 12, 28 Points

Numerical Optimization
exercise sheet

review on 22.01.2024 during the exercise class

1. *(SQP method for nonlinear programs with equality constraints)*
We consider the following nonlinear program with nonlinear equality constraints

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} & f(x), \quad f:\mathbb{R}^n\to\mathbb{R} \\ \text{s.t.} & h(x)=0, \quad h:\mathbb{R}^n\to\mathbb{R}^m \end{cases} \tag{1}$$

and we derive the SQP method for this problem. But first we look at the so-called Lagrange-Newton method. If x^* is a local solution of the problem above and a CQ holds, the KKT conditions hold, i.e. there exists $\lambda^* \in \mathbb{R}^m$ with

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\ h(x^*) &= 0. \end{aligned}$$

To solve problem (1) it seems to be a good idea to solve the KKT system to determine (x^*, λ^*). We do this by applying Newton's method to the system

$$F(x,\lambda) := \begin{pmatrix} \nabla_x \mathcal{L}(x,\lambda) \\ h(x) \end{pmatrix} = 0$$

Therefore, let $x^{(k)}$ and $\lambda^{(k)}$ be iterates and let f as well as h be two times continuously differentiable. Then the Newton update is

$$\begin{pmatrix} \nabla^2_{xx}\mathcal{L}(x^{(k)},\lambda^{(k)}) & (\nabla h(x^{(k)}))^T \\ \nabla h(x^{(k)}) & 0 \end{pmatrix} \begin{pmatrix} d_x^{(k)} \\ d_\lambda^{(k)} \end{pmatrix} = \begin{pmatrix} -\nabla_x \mathcal{L}(x^{(k)},\lambda^{(k)}) \\ -h(x^{(k)}) \end{pmatrix} \text{[1]} \tag{2}$$

and we can update our iterates $x^{(k)}$ and $\lambda^{(k)}$.

a) Write down a pseudo algorithm which realizes the above procedure.

b) To apply the convergence theorem of Newton's method to your algorithm in a) we have to ensure, that the matrix of the Newton update (2) is regular. Therefore, prove that if $\nabla h(x)$ has full rank and $s^T \nabla^2_{xx}\mathcal{L}(x,\lambda)s > 0$ for all $s \in \mathbb{R}^n \setminus \{0\}$ with $\nabla h(x)s = 0$ holds, then the matrix of the Newton update (2) is regular.

c) Derive a quadratic program (QP) with affine linear equality constraints, such that the KKT system of this QP is (2).

d) Write down the SQP pseudo algorithm with your QP in c).

(2 + 4 + 6 + 2 = 14 Points)

[1] We make here the agreement, that the Jacobian of $h : \mathbb{R}^n \to \mathbb{R}^m$ is given by the matrix $(\nabla h(x))_{i,j} = \partial_{x_j} h_i(x)$ for $1 \le i \le m, 1 \le j \le n$, i.e. $\nabla h(x) \in \mathbb{R}^{m\times n}$.

2. *(Interior point methods, QPs)*
Consider the quadratic program (QP)

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} & \frac{1}{2}x^T Ax + a^T x, \\ \text{s.t.} & Bx = b, \\ & Cx \leq c, \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite and $B \in \mathbb{R}^{m\times n}$ has full rank. Derive an analogue of the Newton correction (4.1.3).

(4 Points)

3. *(Central Path)*
Consider the convex quadratic problem

$$\begin{cases} \min\limits_{x\in\mathbb{R}^2} & f(x) := \frac{1}{2}x_1^2 + x_2 \\ \text{s.t.} & g_1(x) := -x_1 \leq 0,\ g_2(x) := -x_2 \leq 0. \end{cases} \tag{3}$$

a) Write down the *barrier problem* (4.2.2). Moreover, calculate the gradient and the Hessian matrix of the objective function for this problem.
b) Calculate the solution x_μ of the barrier problem and sketch the central path.
c) Prove, that $x^* = \lim_{\mu\to 0^+} x_\mu$ exists and solves (3).
d) In how far is the central path changing, if the constraint $-x_2 \leq 0$ of problem (3) occures M-times.

$(4 + 2 + 2 + 2 = 10$ Points)