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Winter term 2024/2025
Sheet 06, 32 Points

Numerical Optimization
exercise sheet

review on 27.11.2024 during the exercise class

1. *(Fun with cones and tangential cones)*

   a) Let $\mathcal{S}$[1] $\subset \mathbb{R}^n$ be a set and let $x \in \mathcal{S}$ be given. Show that the tangential cone $\mathcal{T}(\mathcal{S}, x)$ is actually a cone and is a closed set.

   b) The conical hull of a set $\mathcal{S} \subset \mathbb{R}^n$ at $x \in \mathcal{S}$ is given by $\operatorname{cone}(\mathcal{S} - x) := \{\alpha(y - x) : y \in \mathcal{S}, \alpha > 0\}$. Let $\mathcal{S} \subset \mathbb{R}^n$ be convex and let $x \in \mathcal{S}$ be given. Show that the tangential cone $\mathcal{T}(\mathcal{S}, x)$ is the closure of the conical hull of $\mathcal{S}$ in x, i.e. $\mathcal{T}(\mathcal{S}, x) = \overline{\operatorname{cone}(\mathcal{S} - x)}$.

(6 + 6 = 12 Points)

2. *(More cone fun: tangential cone versus linearized tangential cone)*
   Consider the following cosntrained optimization problems

$$
(I)\begin{cases} \min\limits_{x\in\mathbb{R}^2} & f(x) \\ \text{s.t.} & g_1(x) \le 0, \\ & g_2(x) \le 0, \end{cases}
\qquad\qquad
(II)\begin{cases} \min\limits_{x\in\mathbb{R}^2} & f(x) \\ & g_1(x) \le 0, \\ \text{s.t.} & g_2(x) \le 0, \\ & g_3(x) \le 0, \end{cases}
$$

with

$$
f(x) := (x_1 - 2)^2 + x_2^2, \qquad g_1(x) := (x_1 - 1)^3 + x_2, \qquad g_2(x) := -x_2, \qquad g_3(x) := x_1 - 1.
$$

Solve the following task for (I) and (II).

   (a) Determine the unique minimum $x^* \in \mathbb{R}^2$. Note: A proof is not necessary.
   (b) Draw the feasible region $\mathcal{F}$.
   (c) Determine the tangential cone $\mathcal{T}(\mathcal{F}, x^*)$ and the linearized tangential cone $\mathcal{T}_{\text{lin}}(\mathcal{F}, x^*)$ (it is enough to derive $\mathcal{T}(\mathcal{F}, x^*)$ for one of the problems and the other can be given without explanations).
   (d) Draw the tangential cone $\mathcal{T}(\mathcal{F}, x^*)$ and the linearized tangential cone $\mathcal{T}_{\text{lin}}(\mathcal{F}, x^*)$.
   (e) Check the necessary conditions for a minimum of Theorem 3.1.8, i.e.

$$
x^* \in \mathcal{F} \text{ is a local solution} \qquad \Rightarrow \qquad \nabla f(x^*)^T d \ge 0 \qquad \forall d \in \mathcal{T}(\mathcal{F}, x^*).
$$

   What do you observe if you replace $\mathcal{T}(\mathcal{F}, x^*)$ with $\mathcal{T}_{\text{lin}}(\mathcal{F}, x^*)$? Is this condition still necessary? How is this related to Abadie Constraint Qualification in Definition 3.1.11?

(2 + 2 + 6 + 2 + 2 = 14 Points)

[1] We use this notation to emphasize that it does not have to be a cone.

3. *(Constraint optimization)*
Consider the objective function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x, y) = 2x^2 + xy + 3y^2 - 4x - y.$$

How should the parameter $\alpha, \beta \in \mathbb{R}$ be selected, such that the point $x^* = (3, -2)^T \in \mathbb{R}$ is a minimum subject to the constrain $h(x, y) = 0$, where

$$h(x, y) := 3x + \alpha y - \beta?$$

(6 Points)