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Winter term 2024/2025
Sheet 07, 30 Points

Numerical Optimization
exercise sheet

review on 04.12.2024 during the exercise class

1. *(Slater Condition)*
   Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ and let f and g be continuously differentiable. Consider a *convex* optimization problem of the form

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} & f(x) \\ \text{s.t.} & h(x) = 0, \\ & g(x) \le 0, \end{cases} \tag{1}$$

   where $x^* \in \mathbb{R}^n$ denotes a solution. We say that Problem (1) satisfies the *regularity condition of Slater*, if

$$\mathring{\mathcal{F}} := \{x \in \mathbb{R}^n : g(x) < 0,\ h(x) = 0\} \neq \emptyset.$$

   Prove: If the Problem (1) satisfies the regularity condition of Slater, then there exist Lagrange multipliers $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ such that $(x^*, \lambda^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is a KKT point of (1).

   Hint:

   - Show that the Slater Condition implies the Abadie Constraint Qualification for arbitrary $\hat{x} \in \mathcal{F}$. Then by Theorem 3.1.17 the KKT conditions hold. In order to do so, show
     i) $\mathcal{T}_{\text{lin}}(\mathcal{F}, \hat{x}) \subseteq \overline{\mathring{\mathcal{T}}_{\text{lin}}(\mathcal{F}, \hat{x})}$, where

$$\mathring{\mathcal{T}}_{\text{lin}}(\mathcal{F}, \hat{x}) := \{d \in \mathbb{R}^n : \nabla g_j(\hat{x})^\top d < 0, j \in \mathcal{A}(\hat{x}); \nabla h(\hat{x})^\top d = 0\}$$

     and
     ii) $\mathring{\mathcal{T}}_{\text{lin}}(\mathcal{F}, \hat{x}) \subseteq \mathcal{T}(\mathcal{F}, \hat{x})$.

     From that it follows

$$\mathcal{T}_{\text{lin}}(\mathcal{F}, \hat{x}) \subseteq \overline{\mathring{\mathcal{T}}_{\text{lin}}(\mathcal{F}, \hat{x})} \subseteq \overline{\mathcal{T}(\mathcal{F}, \hat{x})} = \mathcal{T}(\mathcal{F}, \hat{x})$$

     and by Lemma 3.1.10 we have the other inclusion as well. Therefore we have ACQ.

   (14 Points)

2. *(Linear independence constraint qualification (LICQ Condition))*
   Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable functions. An optimization problem of the form

$$\begin{cases} \min\limits_{x\in\mathbb{R}} & f(x) \\ \text{s.t.} & h(x) = 0 \\ & g(x) \le 0 \end{cases} \tag{2}$$

   satisfies the *LICQ condition* in a feasible point $\hat{x} \in \mathcal{F} \subset \mathbb{R}^n$, if $\nabla g_i(\hat{x}) \in \mathbb{R}^n$ and $\nabla h_j(\hat{x}) \in \mathbb{R}^n$ are linear independent for all $i \in \mathcal{A}(\hat{x})$ and for all $j = 1, \ldots, m$.

   Prove: If a local solution $x^* \in \mathcal{F}$ satisfies the LICQ condition, than the Lagrange multipliers $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ at a KKT point $(x^*, \lambda^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ of (2) are unique.

(6 Points)

3. *(Another condition and the relation of the CQs, Mangasarian-Fromovitz constraint qualification (MFCQ))*
   Let $\hat{x} \in \mathcal{F}$. We say that the MFCQ holds at $\hat{x}$ if the gradients

   $$\nabla h_j(\hat{x}), \quad j = 1, \ldots, m$$

   are linear independent and there exists a vector $d \in \mathbb{R}^n$ such that

   $$\nabla g_i(\hat{x})^T d < 0, \quad i \in \mathcal{A}(x), \quad \nabla h(\hat{x})^T d = 0.$$

   One can show: If $x \in \mathcal{F}$ fulfills MFCQ, then ACQ holds. Moreover we have:

   Theorem 1. *Let $x \in \mathcal{F}$ be given. Then the following implications hold*

   $$\begin{array}{ccccccc} \mathit{LICQ}(x) & \Rightarrow & \mathit{MFCQ}(x) & \Rightarrow & \mathit{ACQ}(x) & \Rightarrow & \mathit{GCQ}(x) \\ & & & & \Uparrow \textit{Convex problems} & & \\ & & & & \textit{Slater} & & \end{array}$$

   Remark: Note that the Slater condition implies ACQ(x) for all $x \in \mathcal{F}$.

   Prove: LICQ(x) $\quad\Rightarrow\quad$ MFCQ(x).

(4 Points)

4. *(LICQ, MFCQ and Slater)* We consider the following optimization problem with $p = 4$ constraints

   $$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & x_1^2 + x_2^2 \\ & x_1^2 + 4x_2^2 && \leq 4, \\ \text{s.t.} \quad & (x_1 - 2)^2 + x_2^2 && \leq 5, \\ & x_1,\ x_2 && \geq 0. \end{aligned} \tag{3}$$

   (a) Check, if $\hat{x} := (0, 1)^T \in \mathbb{R}^2$ fulfills LICQ and MFCQ.
   (b) Prove that the minimum $\bar{x}$ of (3) fulfills the KKT conditions.

(3 + 3 = 6 Points)