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Winter term 2024/2025
Sheet 09, 22 Points

Numerical Optimization
exercise sheet

review on 18.12.2024 during the exercise class

1. *(Null Space Method)*
   Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and positive semidefinite matrix, $B \in \mathbb{R}^{m\times n}$ and $g \in \mathbb{R}^m$ with $m \le n$. Consider the following equality constrained optimization problem

$$\text{minimize } f(x) = \frac{1}{2}x^T Ax + a^T x \quad \text{subject to} \quad Bx = b. \tag{1}$$

   a) Let $Z \in \mathbb{R}^{n\times(n-m)}$ be a null space matrix of B, that means that the columns of Z are a basis for the null space of the matrix B. Prove: If $\mathrm{rg}(B) = m$ and $Z^T AZ$ is positive definite, then the matrix

$$K = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

   is invertible. Especially there exists a unique pair $(x^*, \lambda^*) \in \mathbb{R}^n \times \mathbb{R}^m$, which is the solution of the saddlepoint problem

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}. \tag{2}$$

   b) Let the assumption of subtask a) be fullfilled. Prove that the unique solution of (2) is the unique solution of (1).

(4 + 4 = 8 Points)

2. *(Null Space Method, (*MATLAB*))*
   Consider the linear quadratic equality constrained optimization problem (1).

   a) Apply the Matlab function `nullspace_method.m` (given in the material) to the equality constrained optimization problem (1) using

$$A := \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad a := \begin{pmatrix} 0 \\ -2 \\ -2 \\ -1 \\ -1 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}, \quad b := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

   b) Adjust the method `nullspace_method.m` to `nonlinear_nullspace_method.m` such that general nonlinear functions $f : \mathbb{R}^n \to \mathbb{R}$ can be optimized.

   c) Apply the method `nonlinear_nullspace_method.m` to the minimization problem given in a) by using the initial value $z^{(0)} = (1, 1)^T$.

(6 Points)

3. *(Hanging chain,* MATLAB*)*
In this exercise you and your group compete against the other groups. The task is as follows: We model a chain which is fixed on both ends. To get a mathematical model we consider the chain as point masses glued together by springs. This is depicted in the following:

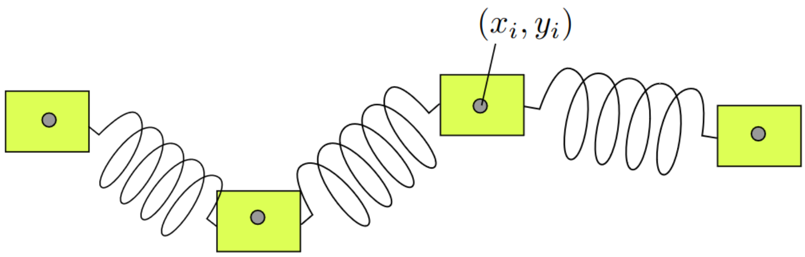

Figure 1: Chain modeled as point masses and springs.

The total energy of the system is given by $E_{\text{tot}} = E_{\text{pot}} + E_{\text{spring}}$, where $E_{\text{pot}} = mgh$ is the potential energy of the mass m with gravity $g = 9.81m/s^2$ at a height of h and $E_{\text{spring}} = \frac{1}{2}D\left((x_i - x_{i+1})^2 + (y_i - y_{i+1})^2\right)$ is the energy of a spring with spring constant D tensioned between the points (x_i, y_i) and (x_{i+1}, y_{i+1}). Now, nature tries to find the steady state of the system by minimizing the total energy. This means numerically, we face the problem

$$\begin{aligned} &\min_{x,y} E_{\text{tot}} \\ &\text{s.t. } h(x,y) = 0, \end{aligned}$$

where h models where the chain is fixed. Implement and solve this problem with MATLAB. Play around with h, D and m. Plot the resulted chain.

(8 Points)