Winter Semester 2025/26

Institute of Mathematical Finance

Prof. Dr. Alexander Lindner
Sebastian Aichmann

Time Series Analysis

Exercise Sheet 1

Exercise 1.1: Let $(W_t)_{t\in\mathbb{Z}}$ and $(Z_t)_{t\in\mathbb{Z}}$ be independent i.i.d. sequences with

$$P(W_t = 0) = P(W_t = 1) = \frac{1}{2} \qquad \text{and} \qquad P(Z_t = -1) = P(Z_t = 1) = \frac{1}{2}, \quad t \in \mathbb{Z}\,.$$

Define the time series $(X_t)_{t\in\mathbb{Z}}$ by

$$X_t = W_t(1 - W_{t-1})Z_t\ , \quad t \in \mathbb{Z}.$$

Show that $(X_t)_{t\in\mathbb{Z}}$ is white noise but not independent.

Exercise 1.2: Let $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ be two weakly stationary time series which are uncorrelated, i.e. X_s and Y_t are uncorrelated for each s and each t.

a) Show that $(X_t + Y_t)_{t\in\mathbb{Z}}$ is weakly stationary and determine its autocovariance function.

b) Now let $(X_t)_{t\in\mathbb{Z}}$ and $(Y_t)_{t\in\mathbb{Z}}$ be in addition independent and their 4th moments exist. Is $(X_tY_t)_{t\in\mathbb{Z}}$ weakly stationary?

Exercise 1.3: Let $(Z_t)_{t\in\mathbb{Z}}$ be an independent sequence of normally distributed random variables with mean 0 and variance $\sigma^2 \in (0,\infty)$. Let a, b, c be real and non-vanishing constants. Which of the following stochastic processes are weakly stationary? If there is stationarity, compute the mean and the autocovariance function.

a) $X_t = a + bZ_t + cZ_{t-1}$

b) $X_t = Z_tZ_{t-1}$

c) Find an example of a stochastic process $X = (X_t)_{t\geq 0}$, which is strictly stationary but not i.i.d. and an example of a stochastic process $Y = (Y_t)_{t\geq 0}$, which is weakly stationary, but not strictly stationary.

Exercise 1.4: Let L be a finite subset of $\mathbb{Z}$ and $a_j \in \mathbb{R}$ for $j \in L$. By a *linear filter* $\{a_j\}_{j \in L}$ we then mean the mapping that associates to a time series (x_t) the series (y_t) defined by

$$y_t = \sum_{j \in L} a_j x_{t-j}.$$

(a) Show that a linear filter $\{a_j\}_{j \in K}$ passes an arbitrary polynomial of degree k without distortion, i.e., that

$$m_t = \sum_{j \in L} a_j m_{t-j}$$

for all k'th-degree polynomials $m_t = c_0 + c_1 t + \cdots + c_k t^k$, if

$$\begin{cases} \sum_{j \in L} a_j = 1 & \text{and} \\ \sum_{j \in L} j^r a_j = 0 & \text{for } r = 1, \ldots, k. \end{cases}$$

(b) Deduce that the Spencer 15-points moving-average filter $\{a_j\}$ defined by

$$\begin{cases} a_j = a_{-j}, & |j| \le 7 \\ a_j = 0, & |j| > 7 \end{cases}$$

with

$$(a_0, a_1, \ldots, a_7) = \frac{1}{320}(74, 67, 46, 21, 3, -5, -6, -3)$$

passes arbitrary third-degree polynomials without distortion.