Computational Finance Exam

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1 Introduction

We are approaching the task of simulating the dynamics of our underlying S_t under three different models, Heston, Constant Elasticity of Variance (CEV) and Displaced Diffusion (DD). Moreover, we want to price European Call and Put options under those models for a grid of times and maturities using Monte Carlo techniques, and finally calibrate the volatility surfaces implied by the models, providing also a graphical representation.
We will stress mostly on the testing part, from the Martingality test on the generated underlying, to the comparison of the Monte Carlo prices with the analytical (or Quasi-analytical) solutions of the models.

2 Constant Elasticity of Variance (CEV) model

The CEV is an extension of the Black and Scholes model which incorporates the elasticity of variance in the parameter β. We will consider the case of $0.5 < \beta < 1$, this approach reflects a common empirical fact on the markets, such as, when stock prices decrease the volatility increases. We expect also that the model will be able to generate an implied volatility skew, that is the difference in implied volatility in Out-of-The-Money (OTM) and In-The-Money (ITM) options. Under the CEV model we have:

$$dS(t) = \sigma_X S^{\beta}(t) dW_t, \quad \beta > 1/2, \quad S(t_0) = S_0 \tag{1}$$

We explored two integration scheme, respectively, *met1* and *met2* in table 6:

1. $$S_{n+1} = S_n + \sigma S_n^{\beta} \sqrt{dt} \eta_n, \quad \eta_n \sim N(0,1) \tag{2}$$

2. $$S_{n+1} = S_n \exp\left(-\frac{1}{2} \sigma^2 S_n^{2(\beta-1)} dt + \sigma S_n^{\beta-1} \sqrt{dt} \eta_n \right) \tag{3}$$

We know that for $\beta < 1$ the process will reach zero for sure, this implies that in some paths from a certain t_n, all the values will be zero. Therefore, we need to impose a cut-off, such that, when $S_t < \epsilon$, we employ the following substitution:

$$\sigma \to \sigma_X \epsilon^{\beta-1}$$

We explored the following values for σ and β:

- $\sigma = 0.2, 0.35, 0.45, 0.6$
- $\beta = 0.55, 0.7, 0.8, 0.9$

The problem encountered in the definition of the cut-off value is the following; when S_{t-1} falls below a certain barrier we observed that due to the oscillations of the stochastic part of the SDE, for some paths, the value of S_t becames negative.
Therefore, the decision for ϵ has been made by lowering it till we do not observed negative values of S_t. We selected a unique ϵ for all the combinations of σ and β.
It is important to remark that for $\beta = 1$ we are in the common framework of the Black and Scholes model.
In this report we will investigate more about the combination $\sigma = 0.45$ and $\beta = 0.7$.

3 Displaced Diffusion (DD) model

The DD is a light modification of the Black and Scholes model also called Shifted Black and Scholes, the purpose of the model is to allow for negative values of S_t in order to address the presence of negative interest rates. Under the DD model we have:

$$dX_t = \sigma_X X_t dW_t \tag{4}$$

and we define:

$$S_t = X_t - \Delta \tag{5}$$

with the initial condition $X_0 = S_0 + \Delta$ and Δ constant.
The discretization scheme is the following:

$$X_{n+1} = X_n \exp(-\frac{\sigma^2}{2}dt + \sigma_X \sqrt{dt}\eta_n), \quad \eta_n \sim N(0,1) \tag{6}$$

We explored the following values for σ and Δ:

- $\sigma = 0.2, 0.35, 0.45, 0.6$
- $\Delta = 0.2, 0.4, 0.6, 0.8$

In this report we will investigate more about the combination $\sigma = 0.35$ and $\Delta = 0.4$.

4 Heston model

Under Heston we have a stochastic volatility model where the volatility follows a CIR process, it aims to addresses some limitiations of the Black and Scholes model by capturing the Volatility Smile, an empirical phenomenon where implied volatility varies with strike price and maturity in options markets, and allowing the volatility to fluctuate over time. Heston model is defined by:

$$S_t = S_0 e^{X_t} \tag{7}$$

$$dX_t = -\frac{\nu_t}{2}dt + \sqrt{\nu_t}dW_t, \quad X_0 = 0 \tag{8}$$

$$d\nu_t = \lambda(\bar{\nu} - \nu_t)dt + \eta\sqrt{\nu_t}dY_t, \quad \nu_0 = \sigma^2 \tag{9}$$

The parameters are:

$$\lambda = 7.7648, \quad \nu = 0.0601, \quad \eta = 2.0170, \quad \rho = -0.6952, \quad \nu_0 = 0.0475$$

We employed the following discretization scheme:

$$S_{n+1} = S_n \text{exp}\left(-\frac{\Delta I_n}{2} + \frac{\rho}{\eta}(\nu_{n+1} - \nu_n - \lambda\bar{\nu}\text{d}t + \lambda\Delta I_n) + \sqrt{(1-\rho^2)\Delta I_n}\xi\right), \qquad \xi \sim N(0,1) \tag{10}$$

where ν_n and $\Delta I_n = I_{n+1} - I_n \simeq \int_0^{t_{n+1}} \nu(s)\text{d}s - \int_0^{t_n} \nu(s)\text{d}s$ are discretized as:

$$\hat{\nu}_{n+1} = \nu_n + \lambda(\bar{\nu} - \nu_n)\text{d}t + \eta\sqrt{\nu_n \text{d}t}\zeta_{n+1}, \qquad \zeta \sim N(0,1) \quad , \quad \nu_{n+1} = \hat{\nu}^+_{n+1}$$

$$I_{n+1} = I_n + \frac{\nu_{n+1} + \nu_n}{2}\text{d}t, \qquad I_0 \ = \ 0$$

5 Monte Carlo approach

We followed the usual procedure and we approximated the price of the Call options as:

$$\Pi(C,T) := \frac{1}{N}\sum_{n=1}^{N} \max\left(S_n(T) - \kappa, 0\right) \tag{11}$$

where N is the number of generated paths. As far as concerned the Put options prices we employed the Put-Call parity. Whereas, some differences emerge in the computation of the MC error. For CEV and DD, after computing the Standard Deviation, the MC errors has been computed as follow:

$$\text{Err} \approx 1.65\frac{Std}{\sqrt{N}} \tag{12}$$

Nevertheless, for Heston we needed a slightly different approach. The problem to be addressed is, if we do generate N_V volatility trajectories and N_s trajectories for S_t for each volatility, we would have $N_V N_s$ values for the expectation. However, computing the error on this quantity would be incorrect since each blocks of N_s cannot be considered fully independent. Therefore, for each N_V trajectories we computed the random variable:

$$m^j := \frac{1}{N_s}\sum_{k=1}^{N_s} \max(S_k^j - K)^+, \qquad \forall j = 1, ..., NV \tag{13}$$

Then we employed the usual procedure for the statistical error on m^j by computing 1.65 $\frac{Std}{\sqrt{NV}}$ where the constant 1.65 states that we are considering the quantile equivalent to the 90% confidence level.

5.1 Pricing results

The results of the pricing procedure together with the associated MC errors and the analytical solutions are shown in Appendix A.1. More details about the analytical solutions will be presented in the model testing section. For the CEV, using $\sigma = 0.45$, $\beta = 0.7$, daily steps and 100,000 trajectories we obtained the results shown in table 6. As far as concerned the DD, using $\sigma = 0.35$, $\Delta = 0.4$, daily steps and 100,000 trajectories we obtained the results shown in table 7. Whereas, for Heston, Using the given parameters, daily steps, 1000 volatility trajectories and 1000 S_t trajectories, we obtained the results shown in table 8.
In the Jupyter Notebook can be found the prices computed for all the 16 couples of β and σ and the 16 couples of Δ and σ, we decided to not report all of them for the sake of space.

6 Implied Volatility Surface Calibration and Volatility Smiles

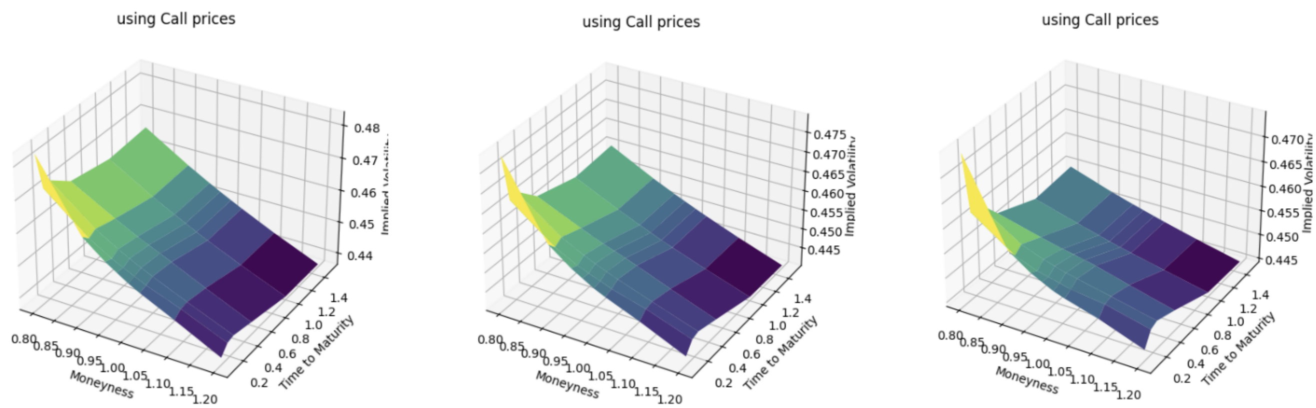
In this section we will present the results of the performed calibration of the volatility surfaces, we employed the same methodology for the three models.
First of all, we make the assumption that the obtained prices are actually the ones observed on the market. Then we applied a Bisection Numerical algorithm in order to extrapolate the volatility implied by the prices of our models. We calibrate using the Black-Scholes analytical solution, meaning that we equate the estimated prices with the BS ones and via bisection we obtain the implied volatility parameters. The Bisection algorithm starts with the definition of a range of values for sigma. For all the arbitrage free models we know for sure that the option price is between the payoff at inception and the option price computed with sigma going to infinity. At the beginning we set a high value for sigma, then we will increase it until the option, priced with that particular sigma, is above the observed market value, The algorithm proceeds by cutting in half the interval and computing the price of the option with the resulting volatility. When the resulted price is higher than the one observed, we exchange the upper bound with the new sigma, if the opposite is true we exchange the lower one. We set a specific level of tolerance i.e. $1e^{-8}$, that once is reached, stops the bisection algorithm. All the volatility surface plots are for Call options since by Put-Call parity the surface computed with Put options would be the same. Finally, after plotting the implied volatility we observed the volatility smiles for each maturity. Volatility smiles are simply the representation of the implied volatility at a fixed instant of time, they are shown in Appendix A.2 for each model, where it is possible to observe also the volatility skews, more pronounced for shorter maturity.
In the Jupyter Notebook can be found all the graphical representation of implied volatility for each combination, for each model, together with the associated volatility smiles and all the calibrated implied volatility, again we did not report all of the them for the sake of space.

6.1 CEV's implied volatilities and volatility surface

We observed that an increase in the value of β is associated with a flattening of the surface as it is shown in figure 1, the same kind relation has been observed also in *"Mathematical Modeling and Computation in Finance"* by *Cornelis W Oosterlee and Lech A Grzelak*. The calibrated implied volatility are shown in table 1.

Table 1: Implied volatilities with $\beta = 0.70$ and $\sigma = 0.45$

	K=0.800	K=0.900	K=0.950	K=0.975	K=1.000	K=1.025	K=1.100	K=1.200
1M	0.483154	0.462307	0.456866	0.454622	0.452416	0.450558	0.445567	0.438604
2M	0.471342	0.46022	0.456094	0.454388	0.452774	0.451126	0.447184	0.441262
3M	0.472128	0.461362	0.457292	0.455434	0.453682	0.452076	0.447521	0.441751
6M	0.467122	0.459556	0.456061	0.454465	0.4529	0.451301	0.446437	0.440266
1Y	0.464653	0.456629	0.452975	0.451224	0.449471	0.447819	0.443171	0.437398
18M	0.465603	0.457174	0.453363	0.451544	0.449795	0.448083	0.443206	0.437261



(a) $\beta = 0.7$ and $\sigma = 0.45$ (b) $\beta = 0.8$ and $\sigma = 0.45$ (c) $\beta = 0.9$ and $\sigma = 0.45$

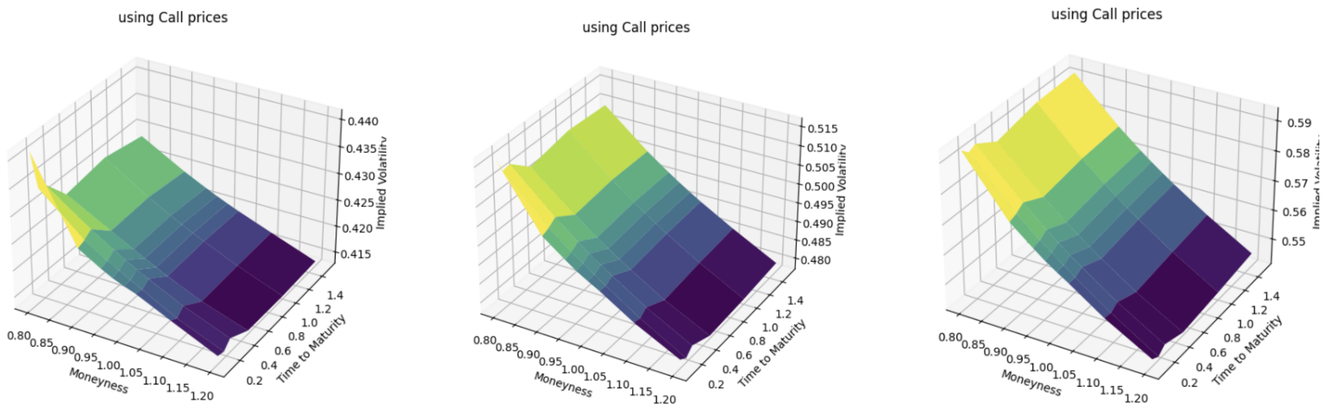
Figure 1: Implied Volatility surface evolution with CEV model

6.2 DD's implied volatilities and volatility surface

We observed that an increase in Δ is associated with an increase in the surface slope as it is shown in figure 2. The calibrated implied volatility are shown in table 2.

Table 2: Implied volatilities with $\Delta = 0.40$ and $\sigma = 0.35$

	K=0.800	K=0.900	K=0.950	K=0.975	K=1.000	K=1.025	K=1.100	K=1.200
1M	0.516343	0.500324	0.496037	0.493987	0.492279	0.490685	0.485247	0.479415
2M	0.512185	0.501065	0.496527	0.494545	0.492504	0.490506	0.48479	0.478216
3M	0.512646	0.5024	0.497817	0.495645	0.493604	0.491666	0.486151	0.479942
6M	0.508021	0.498908	0.494674	0.492648	0.49068	0.488809	0.48355	0.477823
1Y	0.509916	0.500014	0.49568	0.493663	0.491713	0.489839	0.484602	0.478429
18M	0.509232	0.499864	0.495757	0.493835	0.492014	0.490258	0.485403	0.479425



(a) $\Delta = 0.2$ and $\sigma = 0.35$ (b) $\Delta = 0.4$ and $\sigma = 0.35$ (c) $\Delta = 0.6$ and $\sigma = 0.35$

Figure 2: Implied Volatility surface evolution with DD model

6.3 Heston's implied volatilities and volatility surface

The implied volatility surface is shown in figure 3 and the calibrated implied volatility are shown in table 3.

Table 3: Implied volatilities

	K=0.800	K=0.900	K=0.950	K=0.975	K=1.000	K=1.025	K=1.100	K=1.200
1M	0.284211	0.294172	0.246697	0.218598	0.188976	0.163377	0.162262	0.194917
2M	0.340093	0.270764	0.229914	0.208284	0.186539	0.16689	0.151694	0.176373
3M	0.310802	0.254672	0.222413	0.205581	0.188816	0.173218	0.151058	0.164268
6M	0.293451	0.244121	0.220433	0.208979	0.197959	0.187618	0.163692	0.154906
1Y	0.262087	0.232691	0.219055	0.21251	0.206182	0.200105	0.183812	0.16829
18M	0.260148	0.235981	0.225081	0.219922	0.214974	0.210227	0.197212	0.182982

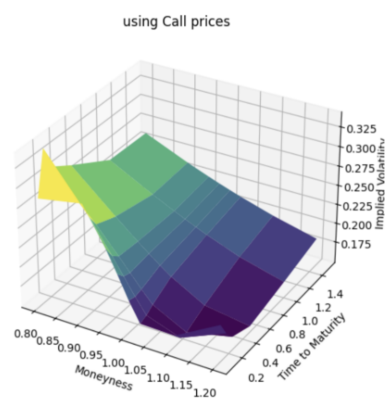


Figure 3: Implied Volatility surface with Heston model

7 Models Testing

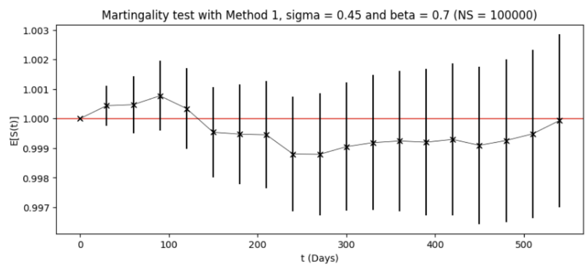
Finally, we will focus on the most important section of this report. The developed testing procedure is based on martingality test, comparison with Black-Scholes prices and analytical solutions of the models.

7.1 Martingality test

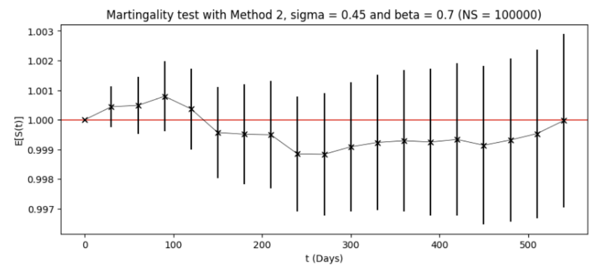
By performing the martingality test we want to access if our generated underlying respects the following condition:

$$E[S(t)] = S_0, \quad \forall t \tag{14}$$

If is not true then our estimated prices will not be arbitrage free. Therefore the validity of this condition was fundamental in the development of the models.

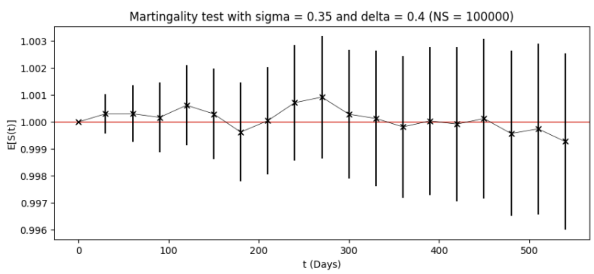


(a) Martingality test for the first CEV's integration scheme

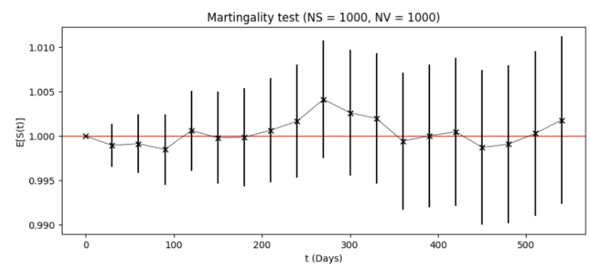


(b) Martingality test for the second CEV's integration scheme

Figure 4: Martingality tests for different CEV's integration schemes



(a) Martingality test for DD model



(b) Martingality test for Heston model

Figure 5: Martingality tests for DD and Heston models

7.2 Black-Scholes prices comparison

As far as concern the comparison with BS prices, this was feasible for the CEV and the DD since, respectively, by fixing $\beta = 1$ in equation (1) and $\Delta = 0$ in equation (5) we get exactly the BS expression.
In table table 4 and table 5 are presented the results for a unique strike for the sake of space, all the other strikes and combinations for both models can be found in the Jupyter Notebook.

Table 4: CEV prices computed using MC vs prices computed using BS with $\beta = 1.00$ and $\sigma = 0.45$

S_0	Strike	t	Put_met1	Call_met1	Err	Put_BS	Call_BS
1.000	1.200	0.083	0.2052	0.0052	+/- 1.4e-04	0.2052	0.0052
1.000	1.200	0.167	0.2173	0.0173	+/- 3.1e-04	0.2170	0.0170
1.000	1.200	0.250	0.2296	0.0296	+/- 4.7e-04	0.2290	0.0290
1.000	1.200	0.500	0.2617	0.0617	+/- 8.6e-04	0.2610	0.0610
1.000	1.200	1.000	0.3110	0.1110	+/- 1.5e-03	0.3114	0.1114
1.000	1.200	1.500	0.3508	0.1508	+/- 2.1e-03	0.3514	0.1514

Table 5: DD prices computed using MC Vs prices computed using BS with $\Delta = 0.00$ and $\sigma = 0.35$

S_0	Strike	t	Put_MC	Call_MC	Err	Put_BS	Call_BS
1.000	1.200	0.083	0.2015	0.0015	+/- 6.0e-05	0.2016	0.0016
1.000	1.200	0.167	0.2075	0.0075	+/- 1.7e-04	0.2075	0.0075
1.000	1.200	0.250	0.2148	0.0148	+/- 2.8e-04	0.2147	0.0147
1.000	1.200	0.500	0.2361	0.0361	+/- 5.5e-04	0.2362	0.0362
1.000	1.200	1.000	0.2723	0.0723	+/- 9.9e-04	0.2728	0.0728
1.000	1.200	1.500	0.3024	0.1024	+/- 1.4e-03	0.3030	0.1030

7.3 Analytical solutions

7.3.1 CEV's analytical solution

The results can be observed in table 6. We employed the following solution, exploiting $F_{\chi^2}(x;\delta,\lambda)$ that is the noncentral chi-squared cumulative distribution function with noncentrality parameter λ and degree of freedom δ. Equation (15) has been retrieved from *"Mathematical Modeling and Computation in Finance"* by *Cornelis W Oosterlee and Lech A Grzelak*, the proof of this result can be found in *"Computing the constant elasticity of variance option pricing formula"* by *M. Schröder (1989)*.

$$\begin{aligned}\Pi(C,T) &= S_0\left(1 - F_{\chi^2}(a, b+2, c)\right) - K \cdot F_{\chi^2}(c, b, a)\\ \Pi(P,T) &= -S_0 \cdot F_{\chi^2}(a, b+2, c) + K \cdot \left(1 - F_{\chi^2}(c, b, a)\right)\end{aligned} \tag{15}$$

$$a = \frac{K^{2(1-\beta)}}{(1-\beta)^2\sigma^2 T}, b \quad = \frac{1}{1-\beta}, c = \frac{S_0^{2(1-\beta)}}{(1-\beta)^2\sigma^2 T}.$$

7.3.2 DD's analytical solution

The results can be observed in table 7, we employed the following expression which is a simple Shifted Black-Scholes closed form solution:

$$\begin{aligned}\Pi(C,T) &= (S_0 + \Delta)\Phi(d_1) - (K + \Delta)\Phi(d_2)\\ \Pi(P,T) &= -(S_0 + \Delta)\Phi(-d_1) + (K + \Delta)\Phi(-d_2)\end{aligned} \tag{16}$$

$$\begin{aligned}d_1 &= \frac{\log\left(\frac{S_0+\Delta}{K+\Delta}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\\ d_2 &= d_1 - \sigma\sqrt{T}\end{aligned}$$

7.3.3 Heston: A Fourier approach

The employed methodology exploits Fast Fourier Transform in order to discretize the Fourier Transform leading to many computational advantages. The general idea is to treat option price analogous to a probability density function, taking advantage from the representation of the characteristic function. The code is based on the SINC algorithm and the results are shown in table 8.

Appendix A

A.1: Pricing tables

Table 6: Prices computed with $\beta = 0.70$ and $\sigma = 0.45$ and analytic solutions

S_0	Strike	t	Put_met1	Call_met1	MC Err	Put_met2	Call_met2	MC Err	Put_analytic	Call_analytic
1.000	0.800	0.083	0.0029	0.2029	+/- 6.6e-04	0.0029	0.2029	+/- 6.6e-04	0.0024	0.2024
1.000	0.800	0.167	0.0105	0.2105	+/- 8.9e-04	0.0104	0.2104	+/- 8.9e-04	0.0100	0.2100
1.000	0.800	0.250	0.0194	0.2194	+/- 1.1e-03	0.0194	0.2194	+/- 1.1e-03	0.0187	0.2187
1.000	0.800	0.500	0.0437	0.2437	+/- 1.4e-03	0.0436	0.2436	+/- 1.4e-03	0.0433	0.2433
1.000	0.800	1.000	0.0836	0.2836	+/- 1.9e-03	0.0836	0.2836	+/- 2.0e-03	0.0839	0.2839
1.000	0.800	1.500	0.1168	0.3168	+/- 2.4e-03	0.1168	0.3168	+/- 2.4e-03	0.1168	0.3168
1.000	0.900	0.083	0.0155	0.1155	+/- 5.7e-04	0.0154	0.1154	+/- 5.7e-04	0.0151	0.1151
1.000	0.900	0.167	0.0320	0.1320	+/- 7.7e-04	0.0319	0.1319	+/- 7.7e-04	0.0316	0.1316
1.000	0.900	0.250	0.0461	0.1461	+/- 9.2e-04	0.0461	0.1461	+/- 9.2e-04	0.0454	0.1454
1.000	0.900	0.500	0.0789	0.1789	+/- 1.3e-03	0.0789	0.1789	+/- 1.3e-03	0.0784	0.1784
1.000	0.900	1.000	0.1260	0.2260	+/- 1.8e-03	0.1260	0.2260	+/- 1.8e-03	0.1263	0.2263
1.000	0.900	1.500	0.1631	0.2631	+/- 2.2e-03	0.1631	0.2631	+/- 2.2e-03	0.1633	0.2633
1.000	0.950	0.083	0.0301	0.0801	+/- 5.0e-04	0.0300	0.0800	+/- 5.0e-04	0.0297	0.0797
1.000	0.950	0.167	0.0500	0.1000	+/- 6.9e-04	0.0500	0.1000	+/- 6.9e-04	0.0496	0.0996
1.000	0.950	0.250	0.0660	0.1160	+/- 8.4e-04	0.0659	0.1159	+/- 8.4e-04	0.0653	0.1153
1.000	0.950	0.500	0.1015	0.1515	+/- 1.2e-03	0.1014	0.1514	+/- 1.2e-03	0.1008	0.1508
1.000	0.950	1.000	0.1508	0.2008	+/- 1.7e-03	0.1508	0.2008	+/- 1.7e-03	0.1511	0.2011
1.000	0.950	1.500	0.1891	0.2391	+/- 2.2e-03	0.1891	0.2391	+/- 2.2e-03	0.1894	0.2394
1.000	0.975	0.083	0.0401	0.0651	+/- 4.6e-04	0.0401	0.0651	+/- 4.6e-04	0.0398	0.0648
1.000	0.975	0.167	0.0612	0.0862	+/- 6.5e-04	0.0611	0.0861	+/- 6.5e-04	0.0607	0.0857
1.000	0.975	0.250	0.0776	0.1026	+/- 8.0e-04	0.0776	0.1026	+/- 8.0e-04	0.0769	0.1019
1.000	0.975	0.500	0.1139	0.1389	+/- 1.1e-03	0.1139	0.1389	+/- 1.2e-03	0.1132	0.1382
1.000	0.975	1.000	0.1640	0.1890	+/- 1.7e-03	0.1640	0.1890	+/- 1.7e-03	0.1644	0.1894
1.000	0.975	1.500	0.2028	0.2278	+/- 2.1e-03	0.2028	0.2278	+/- 2.1e-03	0.2031	0.2281
1.000	1.000	0.083	0.0521	0.0521	+/- 4.2e-04	0.0521	0.0521	+/- 4.2e-04	0.0518	0.0518
1.000	1.000	0.167	0.0736	0.0736	+/- 6.1e-04	0.0736	0.0736	+/- 6.1e-04	0.0732	0.0732
1.000	1.000	0.250	0.0903	0.0903	+/- 7.6e-04	0.0903	0.0903	+/- 7.6e-04	0.0896	0.0896
1.000	1.000	0.500	0.1272	0.1272	+/- 1.1e-03	0.1272	0.1272	+/- 1.1e-03	0.1265	0.1265
1.000	1.000	1.000	0.1778	0.1778	+/- 1.6e-03	0.1778	0.1778	+/- 1.6e-03	0.1781	0.1781
1.000	1.000	1.500	0.2170	0.2170	+/- 2.1e-03	0.2170	0.2170	+/- 2.1e-03	0.2173	0.2173
1.000	1.025	0.083	0.0659	0.0409	+/- 3.7e-04	0.0660	0.0410	+/- 3.8e-04	0.0657	0.0407
1.000	1.025	0.167	0.0875	0.0625	+/- 5.6e-04	0.0875	0.0625	+/- 5.6e-04	0.0870	0.0620
1.000	1.025	0.250	0.1042	0.0792	+/- 7.1e-04	0.1042	0.0792	+/- 7.1e-04	0.1034	0.0784
1.000	1.025	0.500	0.1412	0.1162	+/- 1.1e-03	0.1412	0.1162	+/- 1.1e-03	0.1404	0.1154
1.000	1.025	1.000	0.1922	0.1672	+/- 1.6e-03	0.1922	0.1672	+/- 1.6e-03	0.1925	0.1675
1.000	1.025	1.500	0.2316	0.2066	+/- 2.0e-03	0.2316	0.2066	+/- 2.0e-03	0.2320	0.2070
1.000	1.100	0.083	0.1180	0.0180	+/- 2.5e-04	0.1180	0.0180	+/- 2.5e-04	0.1178	0.0178
1.000	1.100	0.167	0.1365	0.0365	+/- 4.3e-04	0.1366	0.0366	+/- 4.4e-04	0.1360	0.0360
1.000	1.100	0.250	0.1519	0.0519	+/- 5.8e-04	0.1519	0.0519	+/- 5.9e-04	0.1511	0.0511
1.000	1.100	0.500	0.1876	0.0876	+/- 9.4e-04	0.1876	0.0876	+/- 9.4e-04	0.1868	0.0868
1.000	1.100	1.000	0.2383	0.1383	+/- 1.5e-03	0.2383	0.1383	+/- 1.5e-03	0.2386	0.1386
1.000	1.100	1.500	0.2780	0.1780	+/- 1.9e-03	0.2780	0.1780	+/- 1.9e-03	0.2784	0.1784
1.000	1.200	0.083	0.2046	0.0046	+/- 1.2e-04	0.2047	0.0047	+/- 1.2e-04	0.2046	0.0046
1.000	1.200	0.167	0.2160	0.0160	+/- 2.9e-04	0.2161	0.0161	+/- 2.9e-04	0.2157	0.0157
1.000	1.200	0.250	0.2277	0.0277	+/- 4.3e-04	0.2278	0.0278	+/- 4.3e-04	0.2271	0.0271
1.000	1.200	0.500	0.2585	0.0585	+/- 7.7e-04	0.2585	0.0585	+/- 7.8e-04	0.2579	0.0579
1.000	1.200	1.000	0.3064	0.1064	+/- 1.3e-03	0.3065	0.1065	+/- 1.3e-03	0.3067	0.1067
1.000	1.200	1.500	0.3452	0.1452	+/- 1.7e-03	0.3452	0.1452	+/- 1.7e-03	0.3457	0.1457

Table 7: Prices computed with $\Delta = 0.40$ and $\sigma = 0.35$ and analytic solutions

S_0	Strike	t	Put_MC	Call_MC	MC Err	Put_analytic	Call_analytic
1.000	0.800	0.083	0.0039	0.2039	+/- 7.1e-04	0.0036	0.2036
1.000	0.800	0.167	0.0136	0.2136	+/- 9.5e-04	0.0132	0.2132
1.000	0.800	0.250	0.0242	0.2242	+/- 1.1e-03	0.0235	0.2235
1.000	0.800	0.500	0.0519	0.2519	+/- 1.5e-03	0.0519	0.2519
1.000	0.800	1.000	0.0977	0.2977	+/- 2.2e-03	0.0976	0.2976
1.000	0.800	1.500	0.1338	0.3338	+/- 2.7e-03	0.1345	0.3345
1.000	0.900	0.083	0.0186	0.1186	+/- 6.2e-04	0.0184	0.1184
1.000	0.900	0.167	0.0374	0.1374	+/- 8.3e-04	0.0370	0.1370
1.000	0.900	0.250	0.0531	0.1531	+/- 1.0e-03	0.0524	0.1524
1.000	0.900	0.500	0.0888	0.1888	+/- 1.4e-03	0.0888	0.1888
1.000	0.900	1.000	0.1416	0.2416	+/- 2.0e-03	0.1417	0.2417
1.000	0.900	1.500	0.1817	0.2817	+/- 2.5e-03	0.1825	0.2825
1.000	0.950	0.083	0.0342	0.0842	+/- 5.4e-04	0.0339	0.0839
1.000	0.950	0.167	0.0562	0.1062	+/- 7.5e-04	0.0558	0.1058
1.000	0.950	0.250	0.0736	0.1236	+/- 9.1e-04	0.0729	0.1229
1.000	0.950	0.500	0.1118	0.1618	+/- 1.3e-03	0.1119	0.1619
1.000	0.950	1.000	0.1668	0.2168	+/- 1.9e-03	0.1670	0.2170
1.000	0.950	1.500	0.2084	0.2584	+/- 2.4e-03	0.2092	0.2592
1.000	0.975	0.083	0.0445	0.0695	+/- 5.0e-04	0.0443	0.0693
1.000	0.975	0.167	0.0675	0.0925	+/- 7.1e-04	0.0672	0.0922
1.000	0.975	0.250	0.0854	0.1104	+/- 8.7e-04	0.0848	0.1098
1.000	0.975	0.500	0.1244	0.1494	+/- 1.3e-03	0.1245	0.1495
1.000	0.975	1.000	0.1803	0.2053	+/- 1.9e-03	0.1805	0.2055
1.000	0.975	1.500	0.2224	0.2474	+/- 2.4e-03	0.2232	0.2482
1.000	1.000	0.083	0.0566	0.0566	+/- 4.6e-04	0.0564	0.0564
1.000	1.000	0.167	0.0801	0.0801	+/- 6.6e-04	0.0797	0.0797
1.000	1.000	0.250	0.0982	0.0982	+/- 8.3e-04	0.0976	0.0976
1.000	1.000	0.500	0.1377	0.1377	+/- 1.2e-03	0.1379	0.1379
1.000	1.000	1.000	0.1942	0.1942	+/- 1.8e-03	0.1945	0.1945
1.000	1.000	1.500	0.2368	0.2368	+/- 2.4e-03	0.2376	0.2376
1.000	1.025	0.083	0.0705	0.0455	+/- 4.1e-04	0.0703	0.0453
1.000	1.025	0.167	0.0939	0.0689	+/- 6.2e-04	0.0936	0.0686
1.000	1.025	0.250	0.1120	0.0870	+/- 7.9e-04	0.1115	0.0865
1.000	1.025	0.500	0.1518	0.1268	+/- 1.2e-03	0.1520	0.1270
1.000	1.025	1.000	0.2086	0.1836	+/- 1.8e-03	0.2090	0.1840
1.000	1.025	1.500	0.2516	0.2266	+/- 2.3e-03	0.2524	0.2274
1.000	1.100	0.083	0.1217	0.0217	+/- 2.9e-04	0.1215	0.0215
1.000	1.100	0.167	0.1421	0.0421	+/- 4.9e-04	0.1420	0.0420
1.000	1.100	0.250	0.1592	0.0592	+/- 6.6e-04	0.1588	0.0588
1.000	1.100	0.500	0.1980	0.0980	+/- 1.1e-03	0.1983	0.0983
1.000	1.100	1.000	0.2548	0.1548	+/- 1.7e-03	0.2553	0.1553
1.000	1.100	1.500	0.2985	0.1985	+/- 2.2e-03	0.2992	0.1992
1.000	1.200	0.083	0.2066	0.0066	+/- 1.5e-04	0.2066	0.0066
1.000	1.200	0.167	0.2201	0.0201	+/- 3.4e-04	0.2201	0.0201
1.000	1.200	0.250	0.2338	0.0338	+/- 5.0e-04	0.2335	0.0335
1.000	1.200	0.500	0.2683	0.0683	+/- 8.9e-04	0.2686	0.0686
1.000	1.200	1.000	0.3226	0.1226	+/- 1.5e-03	0.3232	0.1232
1.000	1.200	1.500	0.3658	0.1658	+/- 2.0e-03	0.3666	0.1666

Table 8: Prices computed with Heston model and Fourier method

S_0	Strike	t	Put	Call	MC Err	Fourier Put	Fourier Call
1.000	0.800	0.083	0.0001	0.2001	+/- 2.2e-03	0.0009	0.2009
1.000	0.800	0.167	0.0028	0.2028	+/- 2.6e-03	0.0033	0.2033
1.000	0.800	0.250	0.0047	0.2047	+/- 2.9e-03	0.0058	0.2058
1.000	0.800	0.500	0.0133	0.2133	+/- 3.7e-03	0.0131	0.2131
1.000	0.800	1.000	0.0256	0.2256	+/- 4.9e-03	0.0266	0.2266
1.000	0.800	1.500	0.0404	0.2404	+/- 5.9e-03	0.0391	0.2391
1.000	0.900	0.083	0.0042	0.1042	+/- 1.6e-03	0.0045	0.1045
1.000	0.900	0.167	0.0095	0.1095	+/- 1.8e-03	0.0099	0.1099
1.000	0.900	0.250	0.0138	0.1138	+/- 2.0e-03	0.0146	0.1146
1.000	0.900	0.500	0.0271	0.1271	+/- 2.6e-03	0.0271	0.1271
1.000	0.900	1.000	0.0468	0.1468	+/- 3.7e-03	0.0480	0.1480
1.000	0.900	1.500	0.0663	0.1663	+/- 4.6e-03	0.0656	0.1656
1.000	0.950	0.083	0.0096	0.0596	+/- 1.1e-03	0.0098	0.0598
1.000	0.950	0.167	0.0168	0.0668	+/- 1.2e-03	0.0170	0.0670
1.000	0.950	0.250	0.0227	0.0727	+/- 1.4e-03	0.0232	0.0732
1.000	0.950	0.500	0.0388	0.0888	+/- 1.9e-03	0.0388	0.0888
1.000	0.950	1.000	0.0623	0.1123	+/- 3.0e-03	0.0635	0.1135
1.000	0.950	1.500	0.0837	0.1337	+/- 3.9e-03	0.0832	0.1332
1.000	0.975	0.083	0.0143	0.0393	+/- 7.8e-04	0.0145	0.0395
1.000	0.975	0.167	0.0225	0.0475	+/- 9.2e-04	0.0226	0.0476
1.000	0.975	0.250	0.0292	0.0542	+/- 1.1e-03	0.0295	0.0545
1.000	0.975	0.500	0.0465	0.0715	+/- 1.6e-03	0.0465	0.0715
1.000	0.975	1.000	0.0717	0.0967	+/- 2.6e-03	0.0727	0.0977
1.000	0.975	1.500	0.0938	0.1188	+/- 3.5e-03	0.0934	0.1184
1.000	1.000	0.083	0.0218	0.0218	+/- 4.4e-04	0.0218	0.0218
1.000	1.000	0.167	0.0304	0.0304	+/- 5.9e-04	0.0306	0.0306
1.000	1.000	0.250	0.0376	0.0376	+/- 7.4e-04	0.0378	0.0378
1.000	1.000	1.000	0.0821	0.0821	+/- 2.2e-03	0.0831	0.0831
1.000	1.000	1.500	0.1047	0.1047	+/- 3.2e-03	0.1044	0.1044
1.000	1.025	0.083	0.0341	0.0091	+/- 2.1e-04	0.0341	0.0091
1.000	1.025	0.167	0.0418	0.0168	+/- 3.3e-04	0.0420	0.0170
1.000	1.025	0.250	0.0489	0.0239	+/- 4.5e-04	0.0490	0.0240
1.000	1.025	0.500	0.0670	0.0420	+/- 9.0e-04	0.0670	0.0420
1.000	1.025	1.000	0.0938	0.0688	+/- 1.9e-03	0.0946	0.0696
1.000	1.025	1.500	0.1167	0.0917	+/- 2.8e-03	0.1165	0.0915
1.000	1.100	0.083	0.1004	0.0004	+/- 3.3e-05	0.1004	0.0004
1.000	1.100	0.167	0.1017	0.0017	+/- 1.0e-04	0.1018	0.0018
1.000	1.100	0.250	0.1039	0.0039	+/- 1.5e-04	0.1039	0.0039
1.000	1.100	0.500	0.1140	0.0140	+/- 2.7e-04	0.1139	0.0139
1.000	1.100	1.000	0.1370	0.0370	+/- 9.9e-04	0.1374	0.0374
1.000	1.100	1.500	0.1586	0.0586	+/- 1.8e-03	0.1585	0.0585
1.000	1.200	0.083	0.2000	0.0000	+/- 2.0e-06	0.2000	0.0000
1.000	1.200	0.167	0.2001	0.0001	+/- 1.8e-05	0.2001	0.0001
1.000	1.200	0.250	0.2004	0.0004	+/- 3.6e-05	0.2004	0.0004
1.000	1.200	0.500	0.2024	0.0024	+/- 9.5e-05	0.2024	0.0024
1.000	1.200	1.000	0.2130	0.0130	+/- 3.0e-04	0.2131	0.0131
1.000	1.200	1.500	0.2287	0.0287	+/- 8.4e-04	0.2286	0.0286

A.2: Volatility smiles

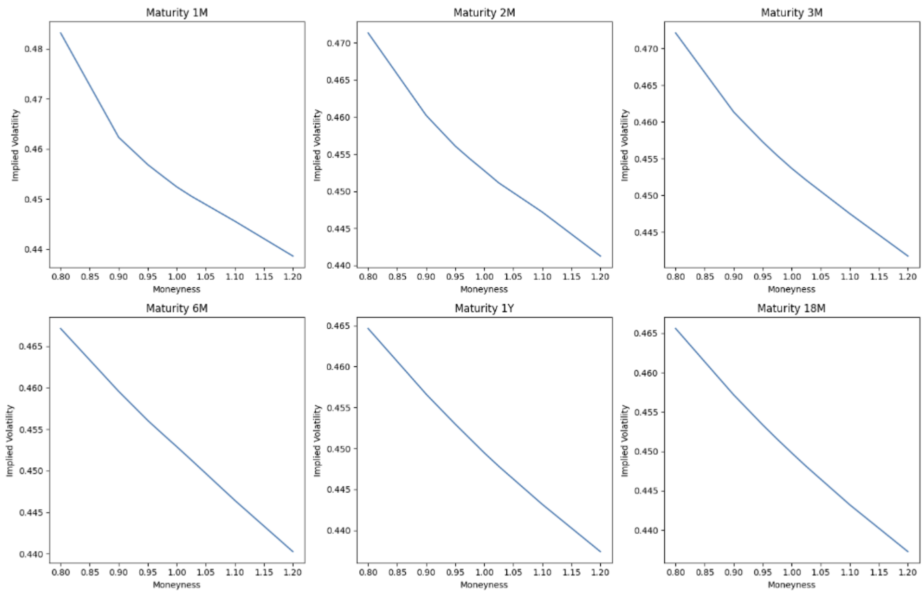


Figure 6: Volatility smiles with CEV model for $\beta = 0.7$ and $\sigma = 0.45$

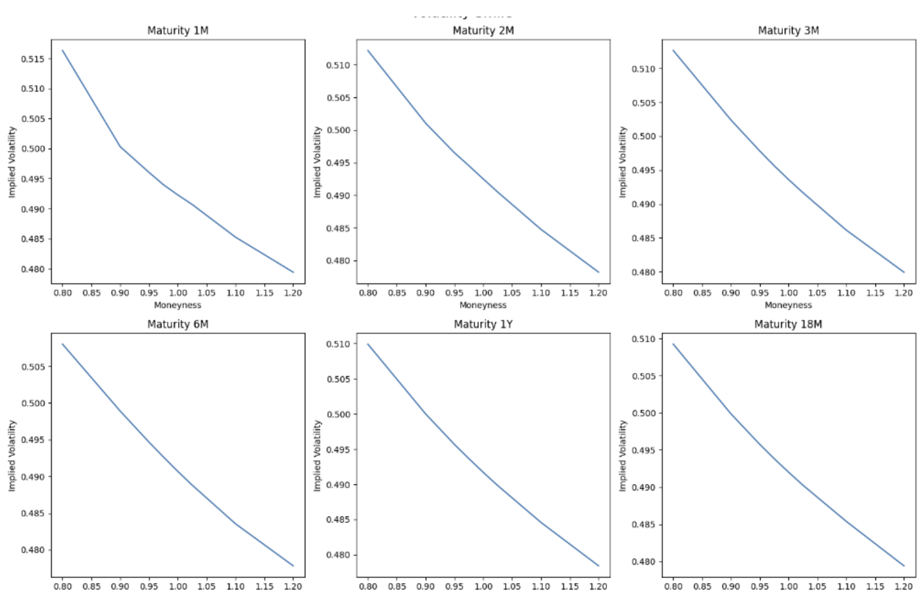


Figure 7: Volatility smiles with DD model for $\Delta = 0.4$ and $\sigma = 0.35$

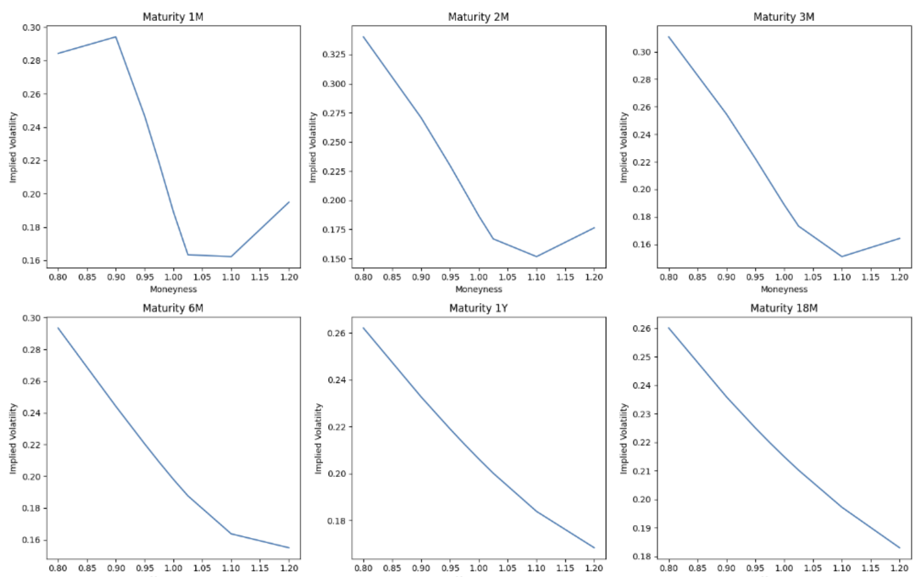


Figure 8: Volatility smiles with Heston model