Statistical Modeling of Joint Extremes II
Computing & Applications

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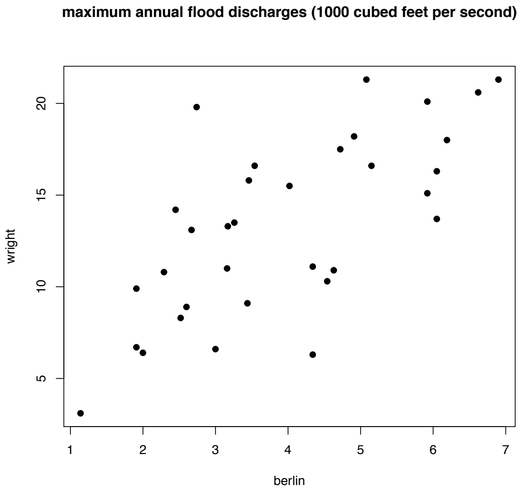
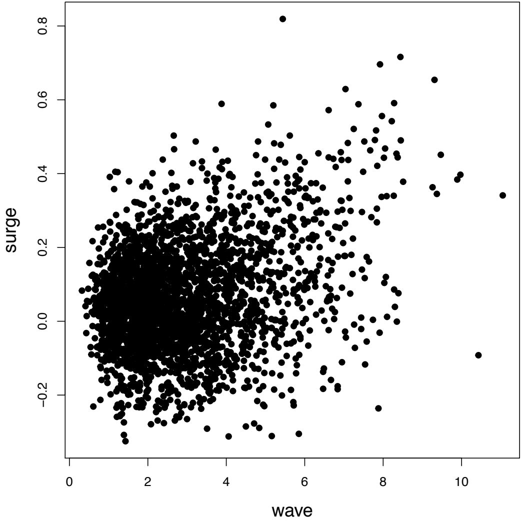
Plan for this session

1. Variable transformations for extreme analyses
2. Analysis of bivariate block maxima
   - Fitting multivariate GEVs
   - Fitting extreme value copulae
3. Analysis of bivariate threshold exceedances
   - Using the POT approach
   - Using flexible models for the angular measure
4. Dealing with asymptotic independence

Fox river data



Wave and Surge Heights in South-West England



Transforming to a common scale

A usual choice is to transform marginals to a standard Frechet distribution, but other choices can be made (uniform, Gumbel,...).

The inverse integral transform can transform data on a uniform scale to any distribution.

One difficulty is that the cdf of the marginals is usually unknown and needs to be estimated.

- For block-maxima data we can assume the margins to follow a univariate GEV;
- for peaks-over-threshold either the empirical cdf or the piece-wise method of Coles and Tawn (1991) is used.

Transforming data using GPD

For a random variable X and a chosen threshold u, the estimated cdf $\hat{F}(x)$ is

$$\hat{F}(x) = \begin{cases} 1 - \phi G(x), & \text{if } x > u \\ \tilde{F}(x), & \text{if } x \leq u \end{cases}$$

where G is the generalized Pareto cdf, $\tilde{F}$ is the empirical cdf and $\phi = (1 - \tilde{F}(u))$.

For a sample $\boldsymbol{x} = (x_i)_{i=1,\dots,n}$:

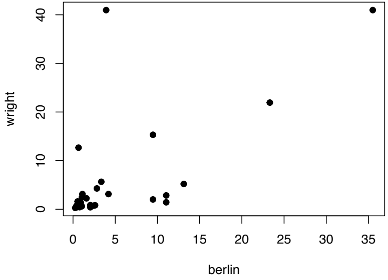
- $y = \hat{F}(\boldsymbol{x})$ is on the uniform scale
- $y = -\log(-\log(\hat{F}(x)))$ is on the Gumbel scale
- $y = -1/\log(\hat{F}(x))$

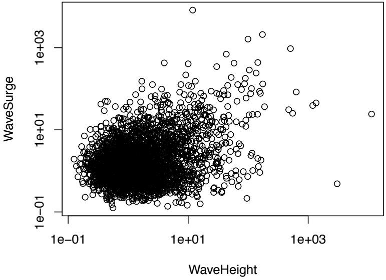
S.G. Coles, J.A. Tawn (1991) Modelling extreme multivariate events. *Journal of the Royal Statistical Society Series B*, 53:377-392.

```
> Uni_transform_GPD <- function(x,thresh=quantile(x,0.95)){
+   par <- fevd(x,threshold = thresh, type="GP")$results$par
+   exc <- length(x[x>thresh])/length(x)
+   u <- ifelse(x>rep(thresh,length(x)),
+          1-exc*(1+(par[2]/par[1])*(x-thresh))^(-1/par[2]),
+          rank(x)/(length(x)+1))
+   return(u)
+ }
```

```
> Frechet_transform <- function(x,thresh=quantile(x,0.95)){
+   -1/log(Uni_transform_GPD(x,thresh))
+ }
```

Normalized plots





Fitting bivariate block-maxima

The function `fbvevd` from the `evd` package allows us to easily fit bivariate extreme value models.

Let's look at the help.

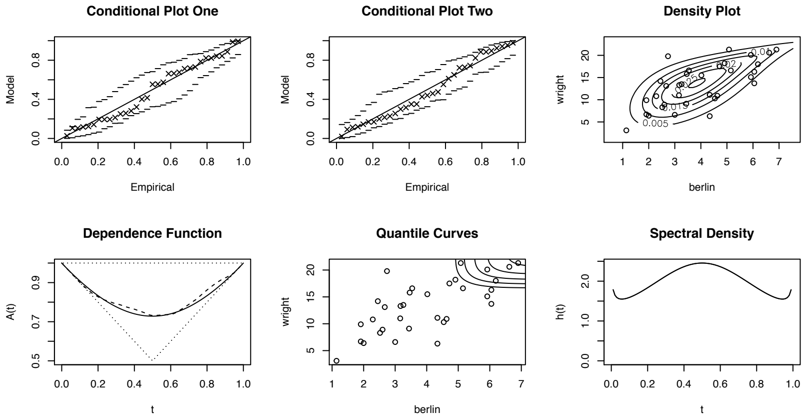
The definition of the models available for fitting is given here.

S.G. Coles, J.A. Tawn. (1994) Statistical methods for multivariate extremes: an application to structural design. *Applied Statistics* 43:1-48.

```
> fbvevd(fox,model="log")
Call: fbvevd(x = fox, model = "log")
Deviance: 295.3364
AIC: 309.3364
Dependence: 0.543351

Estimates
   loc1   scale1   shape1     loc2   scale2   shape2      dep
 3.3533   1.4058  -0.1894  11.8250   4.8532  -0.3649   0.5427

Standard Errors
   loc1   scale1   shape1     loc2   scale2   shape2      dep
0.29987  0.21783  0.19002  0.96897  0.69109  0.15580  0.09348

Optimization Information
  Convergence: successful
  Function Evaluations: 36
  Gradient Evaluations: 13
```

Diagnostics

```
plot(fbvevd(fox,model="log"))
```



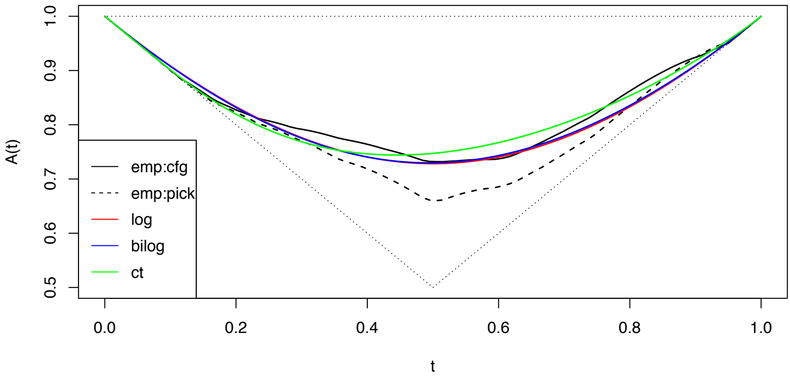
Model choice

Model	AIC	α	β
Logistic	309.33	0.54 (0.09)	
Bilogistic	311.3	0.53 (0.19)	0.56 (0.23)
Coles-Tawn	310.9	0.49 (0.54)	13.84 (140)

- A deviance test can also be performed between nested models: for instance Logistic vs Bilogistic.
- In this case the test statistics is very small (0.263) compared to the critical value of $\chi^2(0.95, 1) = 3.84$.

Pickands dependence function

Can be computed using `abvnonpar` (empirical) and `abvevd` (model-based).



Fitting extreme value copulae

- The package `copula` provides a large number of features to perform inference over multivariate data modeled via copulae.
- In the context of extremes we are particularly interested in extreme-value copulae.
- These are formally defined as the limit copulae of component-wise maxima.

G. Gudendorf, J. Segers (2010). Extreme-value copulas. In *Copula theory and its applications* (pp. 127-145). Springer, Berlin, Heidelberg.

J. Yan (2007) Enjoy the joy of copulas: with a package `copula`. *Journal of Statistical Software* 21:1-21.

Fitting extreme value copulae

The `fitCopula(copula,x)` commands fits the specific `copula` class to the dataset `x`.

Here we fit the Gumbel, extreme-T and Husler-Reiss copulae using:

```
tev <- fitCopula(tevCopula(),pobs(fox))
gumb <- fitCopula(gumbelCopula(),pobs(fox))
hr <- fitCopula(huslerReissCopula(),pobs(fox))
```

The command `pobs` transforms the data to a uniform scale.

Model	α	df
Gumbel	2.148	
Husler-Reiss	1.99	
t-EV	0.97	18.28

Test of goodness of fit

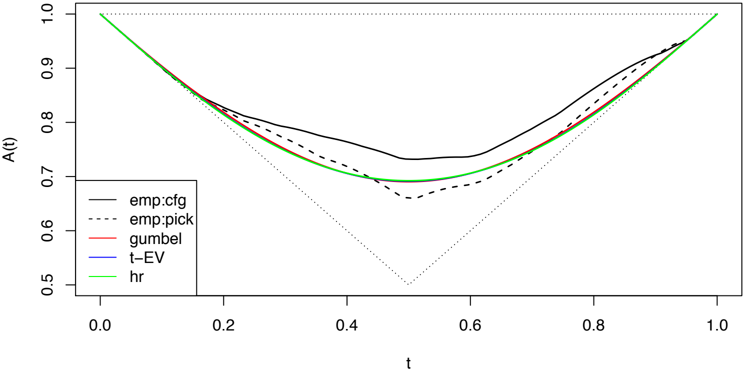
The function `gofEVCopula` performs a goodness of fit test comparing the non-parametric dependence function estimation with the model-based one.

```
gofEVCopula(gumbelCopula(),fox)
gofEVCopula(tevCopula(),fox,N=100)
gofEVCopula(huslerReissCopula(),fox)
```

In all cases the fit is good (p-values 0.453, 0.302 and 0.235 respectively)

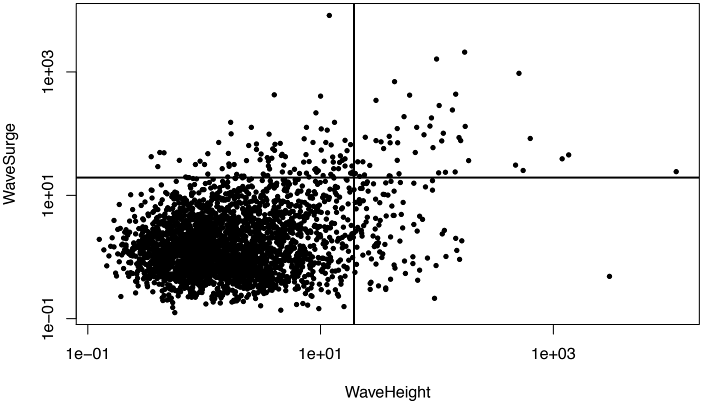
Pickands dependence function

The function `A` computes the Pickands dependence function of EV-copulae.



S. Demarta, A.J. McNeil (2005). The t copula and related copulas. *International statistical review* 73:111-129.

Bivariate exceedances - the censored approach



Fitting exceedances

- The function `fbvpot` fits bivariate extreme models using the censored approach. It has a similar syntax to the `fbvevd` function we used before.
- No widely available threshold choice diagnostics exist for bivariate exceedances. Goodness of fit assessed after model-fitting.
- `fbvpot` fits simultaneously marginal and joint exceedance models. The same threshold is used marginally and jointly.
- If this assumption does not appear to be tenable, it is possible to first fit marginal models, transform the data to the Frechet scale and then fit the joint model with common marginals.
- Here we do not deal with threshold choice issues and assume the choice made is *sensible*.

```
> thresh <-c(quantile(wave,0.95),quantile(surge,0.95))
> fbvpot(wavesurge,threshold = thresh)
Call: fbvpot(x = wavesurge, threshold = thresh)
Likelihood: censored
Deviance: 2036.076
AIC: 2046.076
Dependence: 0.307285

Threshold: 6.08 0.322
Marginal Number Above: 144 144
Marginal Proportion Above: 0.0498 0.0498
Number Above: 49
Proportion Above: 0.0169

Estimates
   scale1     shape1     scale2     shape2        dep
 1.261341  -0.134651   0.091877   0.008904   0.759339
```

Model choice

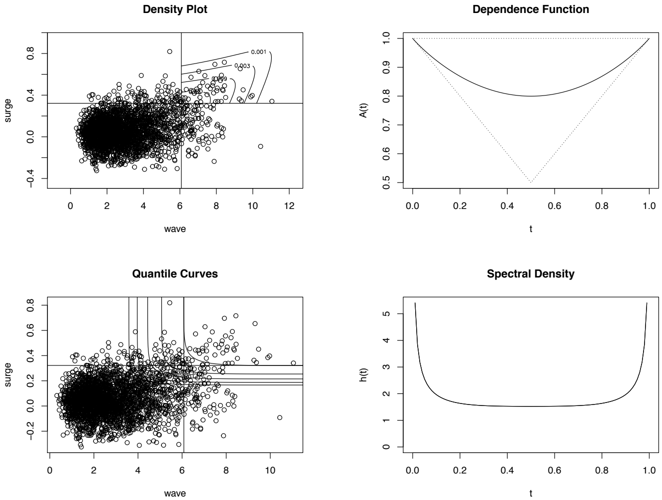
Censored approach

Model	AIC	α	β
Logistic	2046	0.76 (0.03)	
Bilogistic	2047	0.79(0.05)	0.73(0.07)
Coles-Tawn	2047	0.43 (0.18)	0.33 (0.13)

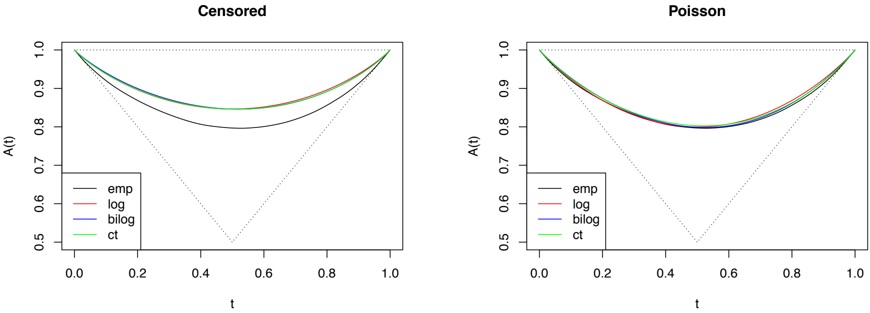
Poisson approach

Model	AIC	α	β
Logistic	2663	0.67 (0.01)	
Bilogistic	2661	0.72 (0.02)	0.62 (0.03)
Coles-Tawn	2644	0.79 (0.13)	0.46 (0.06)

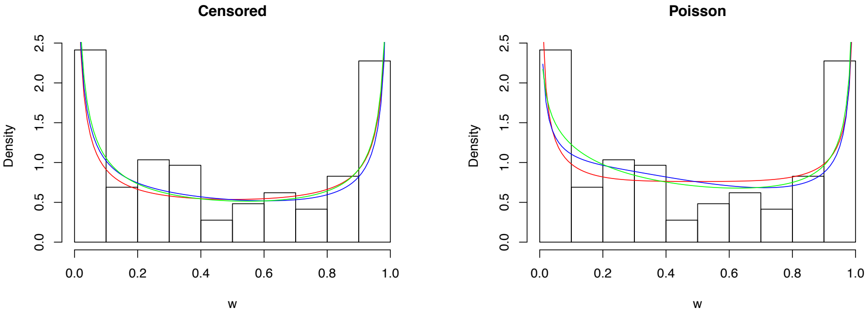
Diagnostics



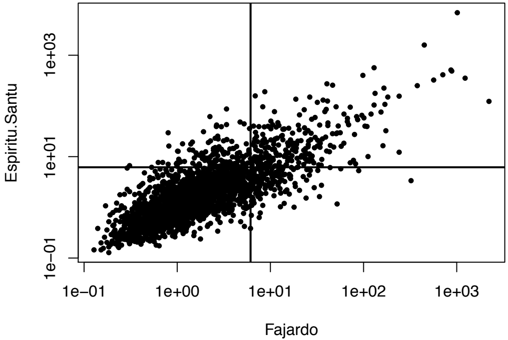
Pickands dependence



Spectral density



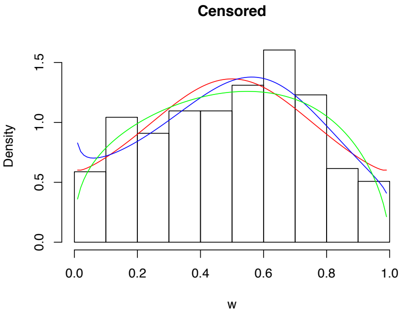
River flows

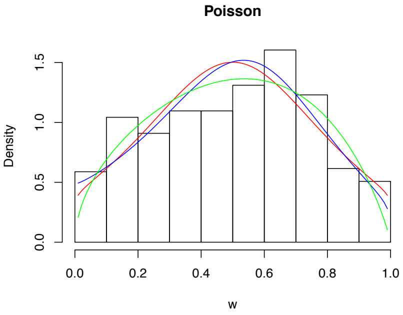


Espiritu.Santu
1e−01
1e+01
1e+03
1e−01
1e+00
1e+01
1e+02
1e+03
Fajardo

Spectral density

Stronger dependence: for the logistic model $\alpha = 0.511$ (censored) and $\alpha = 0.481$ (Poisson).

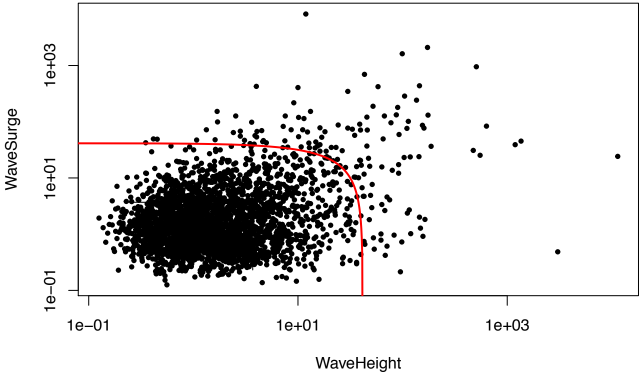




A different threshold strategy

The censored approach is widely used, but it is not the only possible modelling strategy.

One can consider extreme only observations that exceed a specified radius.

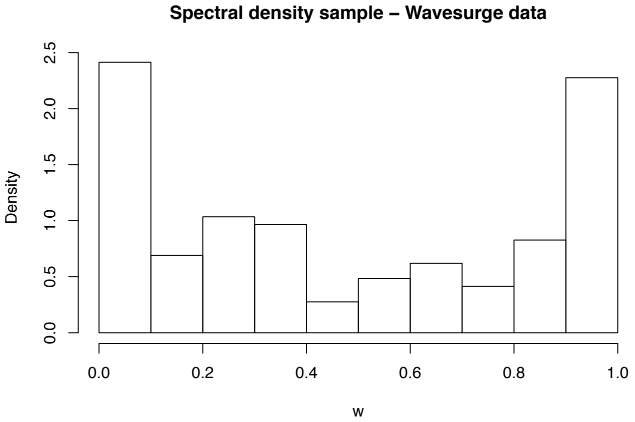


Empirical angular measure

- Suppose $(x_i, y_i)_{i=1,\dots,n}$ is a sample with Frechet margins
- Construct $(w_i, r_i) = (\frac{x_i}{x_i+y_i}, x_i + y_i)$
- Choose a high threshold such that $\{w_i : i \in I_n\}$ where $I_n = \{i = 1, \dots, N : r_i > u\}$ can be regarded as a sample from the spectral measure.

Empirical angular measure - Wavesurge data

```
radius <- apply(wavesurge_trans,1,sum)
w<- (wavesurge_trans[,1]/radius)
hist(w[radius >quantile(radius,0.95)])
```



Non-parametric angular estimation

The function `angdensity` of the package `extremis` gives a smooth non-parametric estimator of the spectral density respecting the mean constraint.

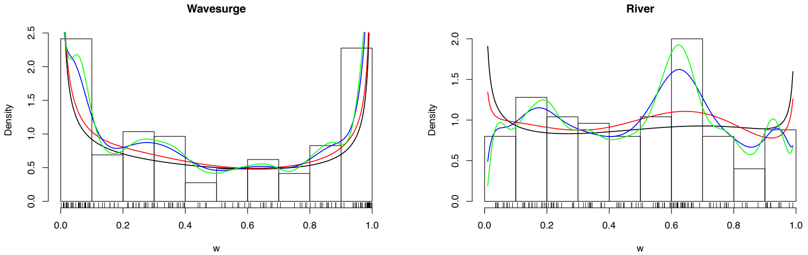


Figure: Angular density non-parametric estimation: nu=5 (black), nu=10 (red), nu=50 (blue), nu=100 (green).

Non-parametric angular cdf

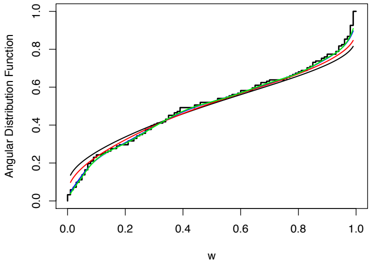


Figure: Angular cdf non-parametric estimation: nu=5 (black), nu=10 (red), nu=50 (blue), nu=100 (green). Left: wavesurge data; Right: river data.

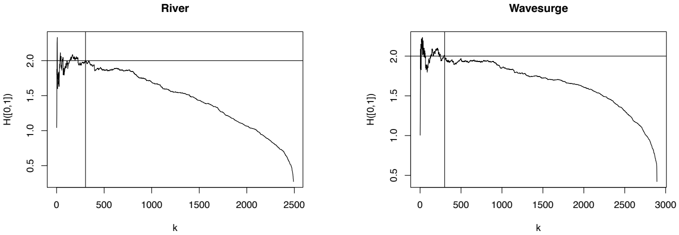
Shiny: angular density

There is a freely-available shiny app that illustrates the angular density estimator at this link.

Aside - One threshold diagnostic

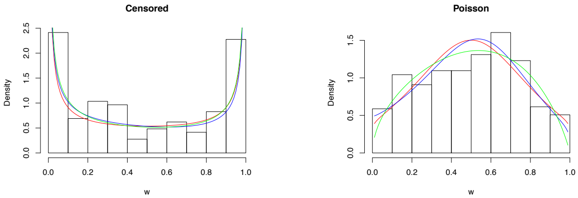
Beirlant et al. introduced one heuristic routine to give a possible choice of threshold:

- Assume data is on the same scale (e.g. Frechet)
- Consider the k largest radii r_i from a sample of size $n > k$.
- An estimator of the spectral measure $H([0,1)] = 2$ is $(k/n)r_{(n-k)}$
- A possible threshold is then the largest k for which $(k/n)r_{(n-k)}$ is close to two.



Asymptotic independence

- In practical applications the degree of dependence is often observed to decrease as the rarity of the corresponding events increases.
- This means that extreme events may be independent between the two margins. Such scenario is usually called *asymptotic independence*.
- All the model we applied so far assume the variables to be asymptotically dependent. However this assumption needs to be checked.
- Multivariate Gaussians are asymptotically independent.



Chi-plots

The function `chiplot` of the `evd` package produces the measures $\chi(u)$ and $\bar{\chi}(u)$ measuring the strength of (sub)-asyptotic dependence.

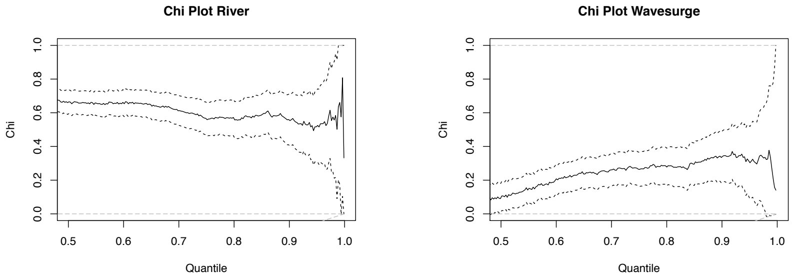


Figure: Plot of $\chi(u)$ as a function of u: Left: wavesurge data; Right: river data.

Estimates of χ

An estimate of χ is given by $2(1 - \hat{A}(0.5))$ where $\hat{A}$ is an estimator of the Pickands dependence function.

Method	Wavesurge	River
Empirical	0.201	0.657
Logistic (censored)	0.307	0.575
Logistic (Poisson)	0.401	0.604

Chibar-plots

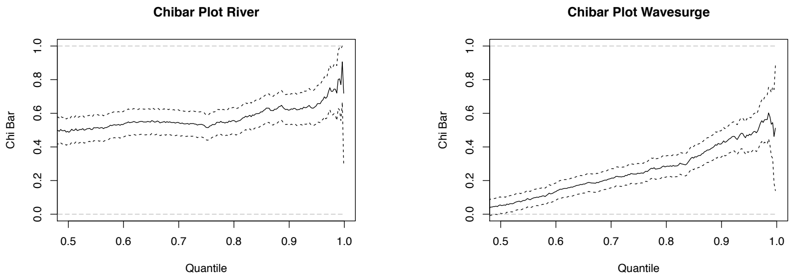


Figure: Plot of $\bar{\chi}(u)$ as a function of u: Left: wavesurge data; Right: river data.

Modelling asymptotic independence

- Standard extreme value theory does not apply in the case of asymptotic independence.
- Bortot et al (2000) introduced a model to deal with asymptotic independence only.
- Ramos and Ledford (2009), Wadsworth and Tawn (2012) and Wadsworth et al (2017) have introduced models that bridge both scenarios.
- However, either regime only occurs at the boundary of the parameter space.

Bortot, P., Coles, S., & Tawn, J. (2000). The multivariate gaussian tail model: An application to oceanographic data. *Journal of the Royal Statistical Society Series C* 49:31-49.

Ramos, A., & Ledford, A. (2009). A new class of models for bivariate joint tails. *Journal of the Royal Statistical Society Series B* 71:219-241

Wadsworth, J. L., & Tawn, J. A. (2012). Dependence modelling for spatial extremes. *Biometrika* 99:253-272.

Wadsworth, J. L., Tawn, J. A., Davison, A. C., & Elton, D. M. (2017). Modelling across extremal dependence classes. *Journal of the Royal Statistical Society Series B* 79:149-175.

The Huser-Wadsworth model

- Recently Huser and Wadsworth (2018) proposed the a spatial model bridging both asymptotic regimes smoothly.
- This can be used also in non-spatial multivariate cases and more specifically bivariate.
- Define the copula model
$$\boldsymbol{X} = R^{\delta}\boldsymbol{W}^{1-\delta},$$
where R is standard Pareto distribution, $\boldsymbol{W}$ is an asymptotic independent copula (e.g. Gaussian), $\delta \in (0, 1)$ and R and $\boldsymbol{W}$ are independent.
- If $\delta > 0.5$ extremes are dependent, otherwise independent.

Huser, R., & Wadsworth, J. L. (2018). Modeling spatial processes with unknown extremal dependence class. *Journal of the American Statistical Association*, 1-11.

Estimation of the δ parameter

- The package `SpatialADAI` available here fits the model of Huser and Wadsworth (2018).
- Data needs to be first transformed into uniform margins (either using empircal cdf or the Coles-Tawn procedure).
- Estimation is carried out either via a censored approach or by selecting only observations exceeding both thresholds.
- The function `fit.cop.2dim` fits the model and `chiu2` computes estimates of $\chi(u)$ from the fitted model.

Estimation of the δ parameter

Estimation of the parameter δ using a 0.95 threshold in each marginal.

Dataset	Model	Method	δ
wavesurge	IEVL	Censored	0.50
wavesurge	Gaus	Censored	0.52
river	IEVL	Censored	0.33
river	Gaus	Censored	0.46
river	Gauss	Partial(0.9)	0.65

Data on the uniform scale

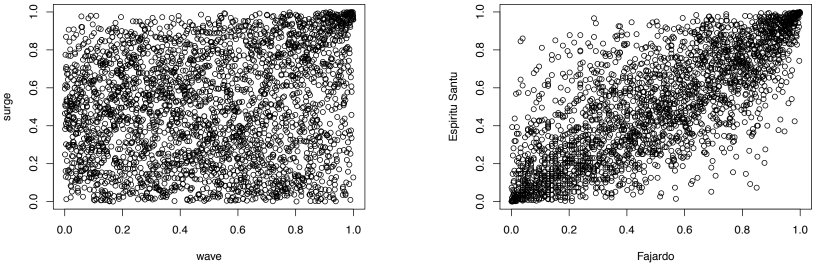
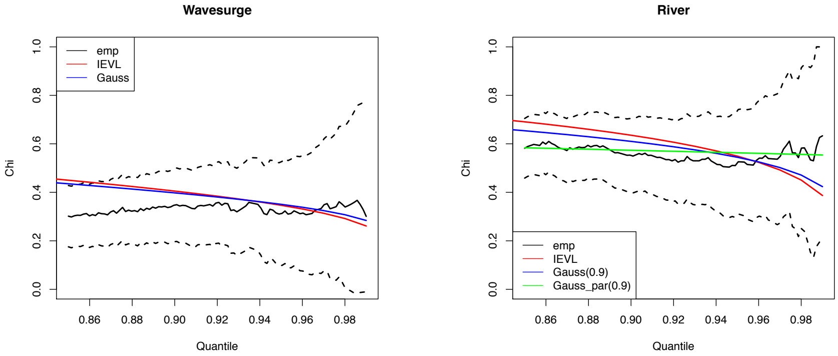


Figure: Data on the uniform scale: Left: wavesurge data; Right: river data.

Coefficient of asymptotic dependence



Conclusions

- In this session we have analysed extreme dependence using a variety of approaches:
  - block-maxima;
  - threshold exceedances;
  - angular densities;
- We have discussed the difference between asymptotic dependence/independence and measures to quantify either regime.
- Brief discussion of model for asymptotically independent extremes.