Mathematics Fundamentals

Lecture 2: Differential Calculus

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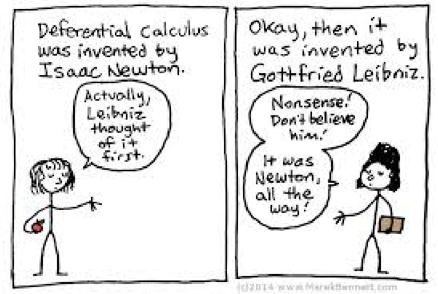
What is Calculus?

Calculus is a branch of mathematics that deals with change. It is divided into two main areas:

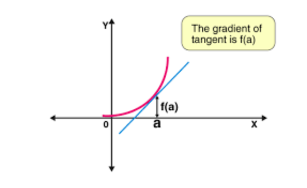
- **Differential Calculus:** Focuses on the concept of a derivative, which represents the rate of change of a function.
- **Integral Calculus:** Focuses on the concept of an integral, which represents the accumulation of quantities, such as areas under a curve.



Differential vs Integral Calculus

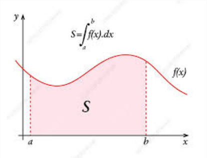
Differential Calculus

- Concerned with the slope of curves.
- Determines instantaneous rates of change.
- Applications include velocity, acceleration, and optimization problems.



Integral Calculus

- Concerned with the area under curves.
- Determines total accumulated quantities.
- Applications include calculating areas, volumes, and total accumulated change.



Applications of Differential Calculus

Differential calculus is used in a wide range of fields:

- **Physics:** To determine velocity and acceleration, and to solve problems involving motion.
- **Economics:** To find marginal cost and revenue, and to optimize profit.
- **Biology:** To model population growth and decay, and to study the rates of reaction in biochemistry.
- **Computer Science:** In machine learning to optimize loss functions through gradient descent.
- **Engineering:** To analyze and design systems that change dynamically, such as electrical circuits or control systems.

Example: Physics - Velocity and Acceleration

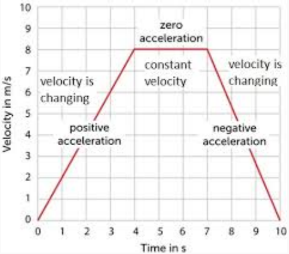
- **Velocity:** The rate of change of position with respect to time.

$$v(t) = \frac{ds(t)}{dt}$$

- **Acceleration:** The rate of change of velocity with respect to time.

$$a(t) = \frac{dv(t)}{dt}$$

- These concepts are essential in understanding motion, force, and energy.



Limits

Limits are fundamental to calculus and describe the behavior of a function as the input approaches a particular value.

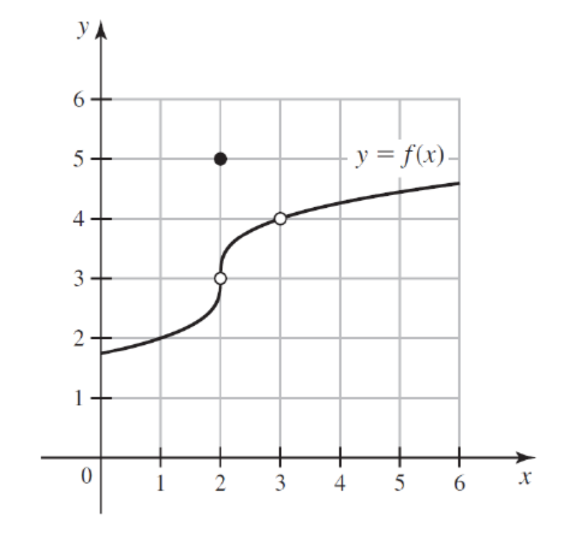
Definition

The limit of $f(x)$ as x approaches a is the value L that $f(x)$ gets closer to as x gets closer to a:

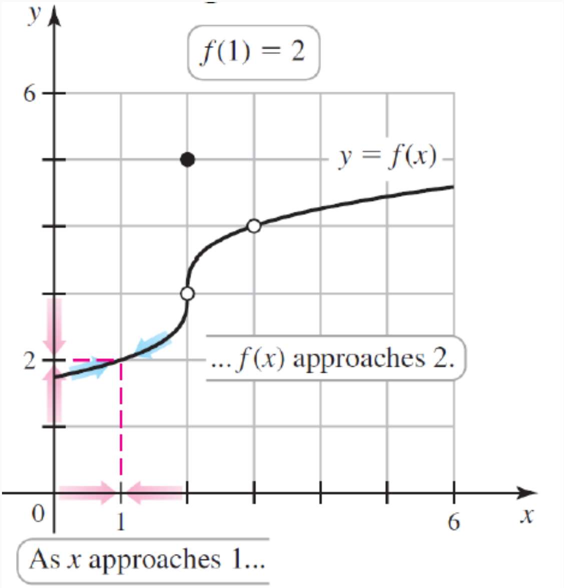
$$\lim_{x \to a} f(x) = L$$

- **Example:** For $f(x) = 2x$, $\lim_{x \to 3} f(x) = 6$.

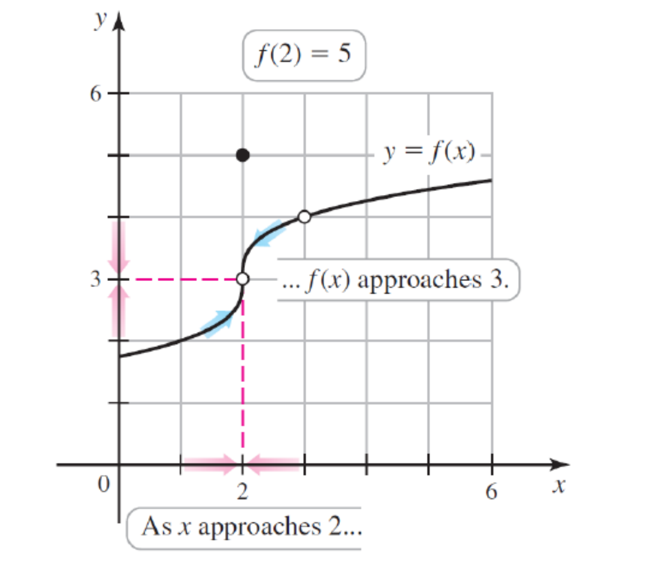
Limits - Example



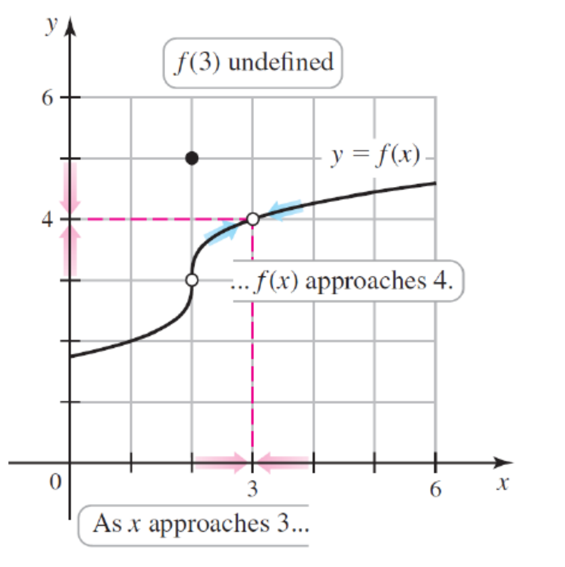
Limits - Example



Limits - Example



Limits - Example



Continuity

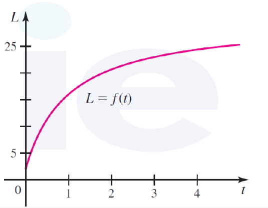
A function is **continuous** if it has no abrupt changes.

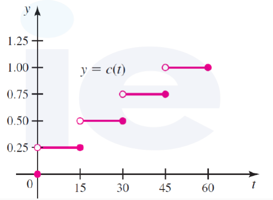
Definition

A function $f(x)$ is continuous at a point $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

If it is continuous at all points in its domain it is said to be continuous.



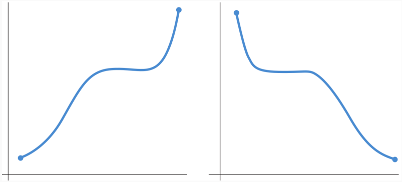


Increasing/Decreasing Functions

A function's behavior can be described as increasing or decreasing over an interval.

Key Idea

- $f(x)$ is **increasing** on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
- $f(x)$ is **decreasing** on an interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

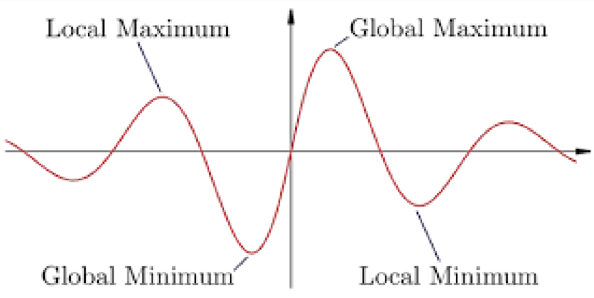


Global/Local Maxima/Minima, Unbounded Functions

Key Concepts

- A **local maximum** is the highest point in a small neighborhood of the function.
- A **global maximum** is the highest point over the entire domain of the function.
- A **local minimum** is the lowest point in a small neighborhood of the function.
- A **global minimum** is the lowest point over the entire domain of the function.
- A function is **unbounded** if it does not have a maximum or minimum value.

Global/Local Maxima/Minima, Unbounded Functions



Weierstrass Theorem

If a function $f : [a, b] \to \mathbb{R}$ is continuous on a closed interval $[a, b]$, then f attains both a global maximum and a global minimum on $[a, b]$. If it is increasing, then the maximum is in b and the minimum is in a.

Slope of a Line as Rate of Change

Slope represents the rate of change of a line and measures how much the y-value changes for a unit change in the x-value.

Definition

The slope m of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by:

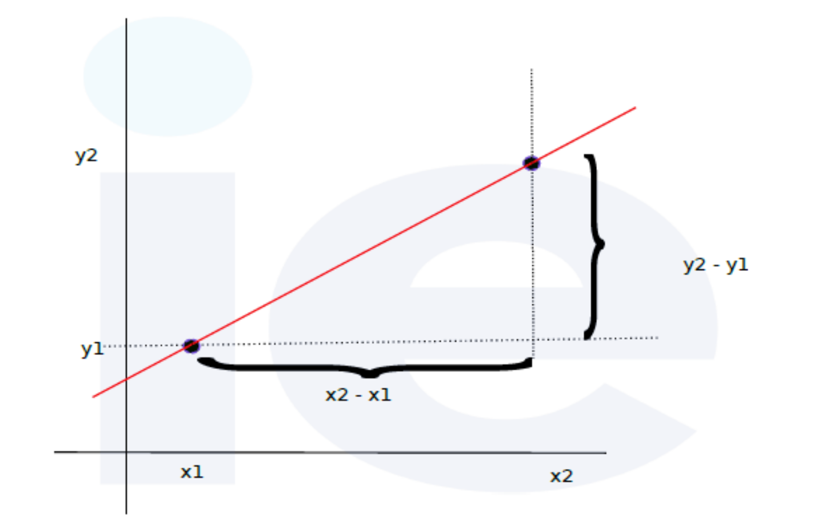
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- **Example:** For points $(1, 2)$ and $(3, 6)$, the slope m is:

  $$m = \frac{6 - 2}{3 - 1} = 2$$

  This means for every 1 unit increase in x, y increases by 2 units.

Slope of a Line as Rate of Change



Generalizing Slope to Functions

For a function $f(x)$, the rate of change between two points x_1 and x_2 on the curve is analogous to the slope of a line.

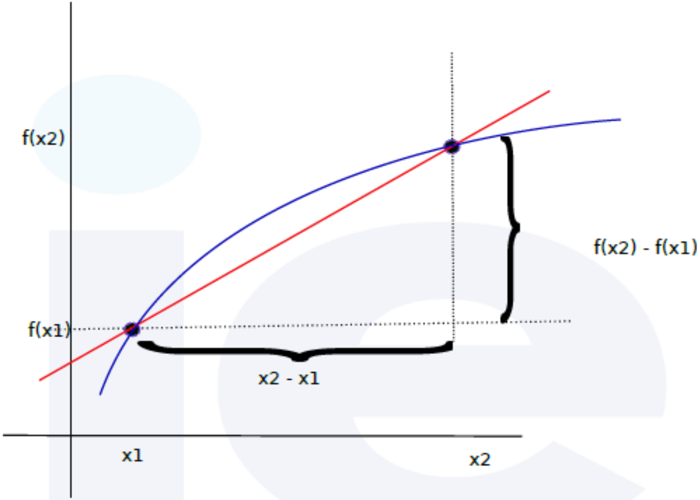
Secant Line

The slope of the secant line between points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the curve of $f(x)$ is:

$$m_{\text{secant}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- This represents the average rate of change of the function over the interval $[x_1, x_2]$.

Generalizing Slope to Functions



Instantaneous Rate of Change/Derivative

To find the **instantaneous rate of change** of a function $f(x)$ at a specific point $x = a$, we consider the limit as x_2 approaches x_1.

Derivative

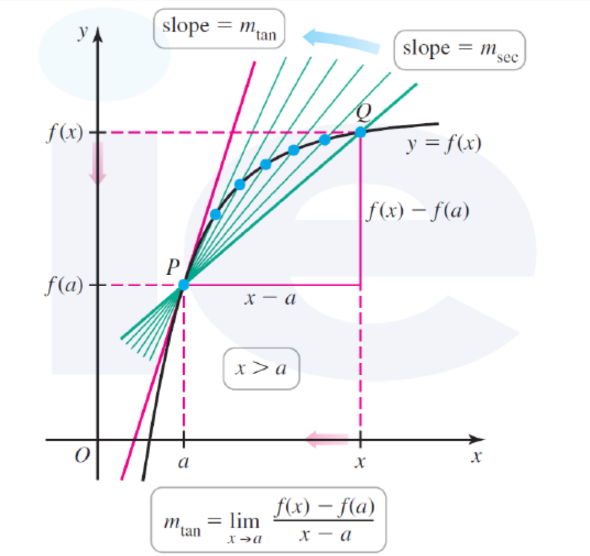
The derivative $f'(a)$ is defined as:

$$f'(a) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

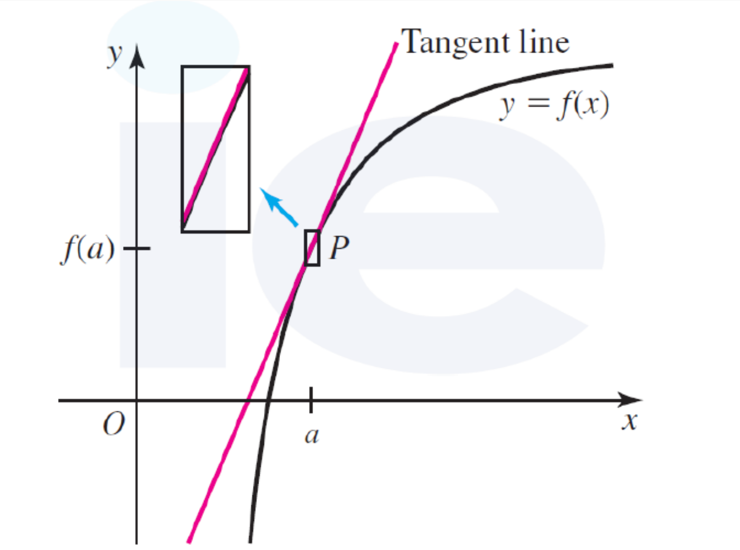
This represents the slope of the tangent line to the curve at $x = a$.

- The derivative $f'(a)$ gives us the exact rate of change of $f(x)$ at the point $x = a$.

Instantaneous Rate of Change/Derivative

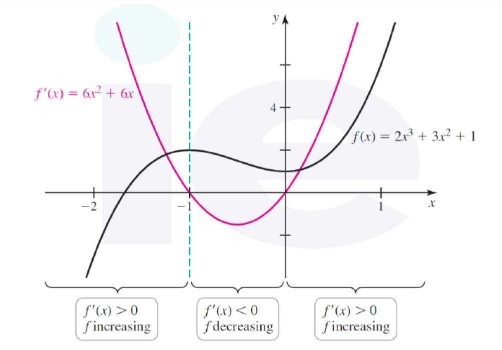


Instantaneous Rate of Change/Derivative



Derivative and Increasing/Decreasing Functions

- **If $f'(x) > 0$:** The function $f(x)$ is **increasing** on the interval.
- **If $f'(x) < 0$:** The function $f(x)$ is **decreasing** on the interval.



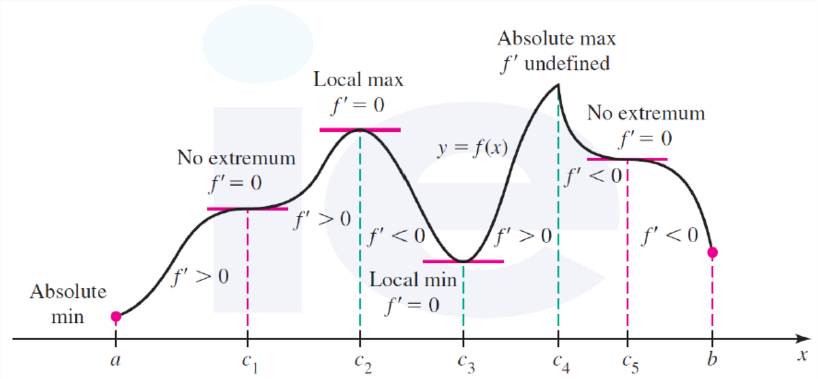
Critical Points

Definition: A point x is called a **critical point** if $f'(x) = 0$.

Key Points

- A critical point is a **necessary** condition for a local maximum or minimum, but it is not **sufficient** on its own.
- To determine whether a critical point is a local maximum or minimum, we must check the sign of the first derivative before and after the point.
- The global maximum or minimum of a function can only occur at the boundary of the domain, at non-differentiable points, or at critical points.

Critical Points



Secondr Derivatives

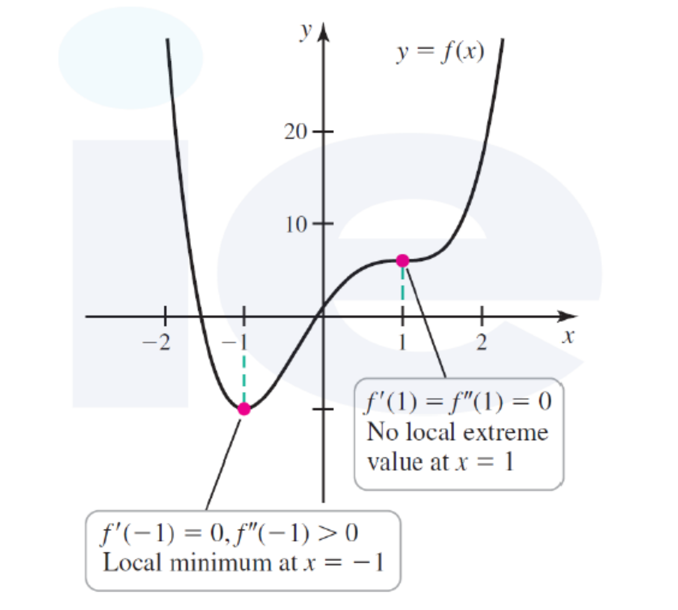
Second Derivative

The second derivative, if it exists, is defined as the derivative of the derivative and is denoted as

$$f''(x) = (f'(x))' = \frac{d^2 f}{d^2}(x)$$

- If x_0 is a stationary point ($f'(x_0) = 0$) then
  - if $f''(x_0) > 0$ it is a local minimum (convex function)
  - if $f''(x_0) < 0$ it is a local maximum (concave function)
  - if $f''(x_0) = 0$, we have to check the sign of $f'(x_0)$

An Example



y
$y = f(x)$
20
10
-2
-1
1
2
x
$f'(1) = f''(1) = 0$
No local extreme
value at $x = 1$
$f'(-1) = 0, f''(-1) > 0$
Local minimum at $x = -1$