Mathematics Fundamentals

Lecture 5: More on Matrices

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Overview of Lecture 5

Determinants

Matrix Inverses

Rank of a Matrix

Matrix Decompositions

Determinants

Introduction to Determinants

Determinants are a scalar value that can be computed from the elements of a square matrix. They provide key information about the matrix, including whether it is invertible.

- Determinants have important applications in areas such as linear algebra, geometry, and differential equations.
- Geometric interpretation: The determinant of a 2x2 matrix can be interpreted as the area scaling factor for linear transformations.
- If the determinant of a matrix is zero, the matrix is **singular**, meaning it does not have an inverse.

Calculating Determinants

2x2 Matrix:

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$$

3x3 Matrix: $\det(A) = aei + bfg + cdh - ceg - bdi - afh$

4x4 Matrix:

$$\begin{aligned}\det(A) = a(f(kp - lo) - g(jp - ln) + h(jo - kn)) \\ - b(e(kp - lo) - g(ip - mn) + h(io - km)) \\ + c(e(jp - ln) - f(ip - mn) + h(io - jm)) \\ - d(e(jo - kn) - f(io - km) + g(io - jm))\end{aligned}$$

- Determinants become increasingly complex to compute as the size of the matrix increases, often requiring advanced techniques like cofactor expansion or matrix decomposition.

Properties of Determinants

Determinants have several important properties:

- **Product Rule:** $\det(AB) = \det(A) \times \det(B)$
- **Transpose:** $\det(A^T) = \det(A)$
- **Identity Matrix:** $\det(I) = 1$ for any identity matrix I
- **Triangular Matrix:** The determinant of a triangular matrix (upper or lower) is the product of its diagonal elements: $\det(A) = a_{11} \times a_{22} \times \cdots \times a_{nn}$
- **Zero Row/Column:** If a matrix has a row or a column consisting entirely of zeros, then $\det(A) = 0$

Matrix Inverses

Introduction to Matrix Inverses

A square matrix A has an **inverse** A^{-1} if and only if:

$$AA^{-1} = A^{-1}A = I$$

Where I is the identity matrix. Inverses have several key properties:

- A matrix is invertible only if its determinant is non-zero.
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(A^{-1})^{-1} = A$

Connection Between Inverses and Determinants

For a 2x2 matrix A, the inverse is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- The determinant plays a crucial role in determining whether an inverse exists.
- For larger matrices, the calculation of the inverse relies on methods like Gaussian elimination or LU decomposition.

Rank of a Matrix

Linear Independence

- A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is linearly independent if the only solution to the equation:

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

  is $c_1 = c_2 = \cdots = c_n = 0$.
- If any of the coefficients c_i can be non-zero while still satisfying the equation, the vectors are linearly dependent.

Example:

- Vectors $v_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}$ are linearly independent because the only solution to $c_1 v_1 + c_2 v_2 = 0$ is $c_1 = 0$ and $c_2 = 0$.
- Vectors $v_1 = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 & 4 \end{pmatrix}$ are linearly dependent because $v_2 = 2v_1$, meaning $c_1 = 2$ and $c_2 = -1$ is a non-trivial solution to $c_1 v_1 + c_2 v_2 = 0$.

Rank of a Matrix

The **rank** of a matrix is the maximum number of linearly independent rows (or columns) in the matrix.

- A matrix is full-rank if all its rows (or columns) are linearly independent.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

- The rows of A are linearly dependent because the third row can be written as a linear combination of the first two rows ($R_3 = 2R_2 - R_1$).
- The rank of A is 2, as only two of the rows (or columns) are linearly independent.

Matrix Decompositions

Matrix Decompositions

Matrix decompositions are methods to factorize a matrix into products of simpler matrices. These are useful for simplifying matrix operations.

- **LU Decomposition:** $A = LU$ where L is lower triangular and U is upper triangular.
- **Cholesky Decomposition:** For positive definite matrices, $A = LL^T$.
- **QR Decomposition:** $A = QR$, where Q is orthogonal and R is upper triangular.
- Decompositions simplify calculations of determinants, inverses, and solutions to linear systems.

Determinant Computation Using LU Decomposition

If a matrix A can be decomposed into an LU decomposition, such that $A = LU$, then the determinant of A can be computed as:

$$\det(A) = \det(L) \times \det(U)$$

Since the determinant of a triangular matrix is the product of its diagonal elements:

$$\det(L) = l_{11} \times l_{22} \times \cdots \times l_{nn}, \quad \det(U) = u_{11} \times u_{22} \times \cdots \times u_{nn}$$

Therefore:

$$\det(A) = \left(\prod_{i=1}^{n} l_{ii} \right) \times \left(\prod_{i=1}^{n} u_{ii} \right)$$

Introduction to Spectral Decomposition

Spectral decomposition, also known as eigenvalue decomposition, is a method where a matrix is expressed in terms of its eigenvalues and eigenvectors.

- Applicable primarily to square matrices, and especially useful for symmetric matrices.
- If A is a symmetric matrix, it can be decomposed as:

$$A = Q\Lambda Q^T$$

  where:
  - Q is an orthogonal matrix containing the eigenvectors of A as columns.
  - Λ is a diagonal matrix with the eigenvalues of A on the diagonal.

Eigenvalues and Eigenvectors

Eigenvalues and **eigenvectors** are fundamental in understanding the properties of a matrix.

- For a matrix A, a scalar λ is called an **eigenvalue** if there exists a non-zero vector v (the **eigenvector**) such that:
$$Av = \lambda v$$
- The equation can be rearranged to:
$$(A - \lambda I)v = 0$$
- The eigenvalues λ are found by solving the characteristic equation:
$$\det(A - \lambda I) = 0$$
- Once the eigenvalues are determined, the corresponding eigenvectors can be found by solving the linear system $(A - \lambda I)v = 0$.