# Mathematics Fundamentals

Lecture 1: Introduction and Functions

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# About Me and the Organization of the Course

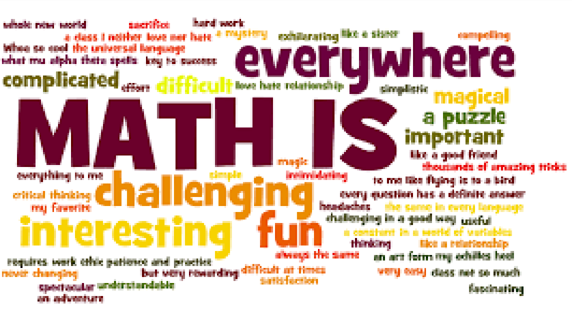
- I am Manuele Leonelli (Statistician, but most of my research is closer to Machine Learning and AI).
- I am always glad to provide help and get feedback from you. Reach out to me with any question or doubt you might have.
- Email: mleonelli@faculty.ie.edu
- Study material, review questions, communications will be via Blackboard.

## Material and Assessment


- For each class I provide the slides, with some additional technical details we will quickly look at (or skip).
- A notebook with code, showcasing a practical application of the mathematical concepts we learn.
- Multiple choice questions with solutions so that you can assess your level.
- I also provide overall quizzes based on the full content of the course.
- No assessment for this course!!!

## What is Mathematics?

- Mathematics is the study of numbers, quantities, shapes, and abstract concepts.
- It provides tools to model and solve real-world problems.
- More than just calculations, it's the language of patterns and structures.
- Essential in understanding algorithms and data structures.



## Mathematical Reasoning and Skills

- **Logical Thinking:** Crucial for programming and algorithm design.
- **Problem-Solving:** Key in algorithms, data analysis, and software development.
- **Abstraction:** Helps in designing scalable and efficient systems.
- **Precision:** Avoids errors in programming and data interpretation.

# Mathematics in Computer Science: Key Examples

- **AlphaGo and AI:**
  - AI relies on probability, game theory, and deep learning.
  - Link to Nature article on AlphaGo
- **PageRank Algorithm:**
  - Based on linear algebra and probability theory.
  - Link to Wikipedia article on PageRank
- **Large Language Models:**
  - Built on statistics, linear algebra, and optimization.
  - Link to New York Times article on GPT-3

# Why Mathematics is Crucial for Your Career

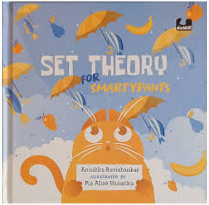
- **Foundation for Algorithms:** Algorithms are grounded in mathematical principles.
- **Data Science & Machine Learning:** Rely on statistics, probability, and linear algebra.
- **Cryptography & Security:** Encryption algorithms are based on number theory and algebra.
- **Career Flexibility:** A strong math background opens doors in various tech roles.

**Definition**

A set is a collection of objects called elements of the set. We write $a \in A$ to denote that $a$ is an element of the set $A$. The notation $a \notin A$ denotes that $a$ is not an element of the set $A$.

**Examples**

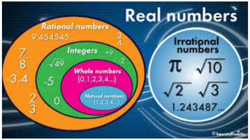
- The set $V$ of all vowels is $V = \{a, e, i, o, u\}$.
- The set $O$ of odd positive integers less than 100 is $O = \{1, 3, 5, 7, 9\}$.



# Sets of Numbers

We are often interested in sets of numbers:

- $\mathbb{N} = \{0, 1, 2, ...\}$ is the set of natural numbers. The ones that we use to count, including 0.
- $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3...\}$ the set of integer numbers.
- $\mathbb{Q}$ is the set of rational numbers. The set of numbers that can be written as $\frac{a}{b}$ where $a$ and $b$ are integers. Example: $\frac{3}{4}, \frac{6}{1009}$.
- $\mathbb{R} = \{\text{all numbers}\}$. Think of all decimal expressions, like 1.3432234... or 5.667754333....



# Functions

In many instances, we assign to each element of a set a particular element of a second set. For example, consider a recommendation system that suggests movies to users. This assignment is an example of a function.

**Example: Movie Recommendations**

Suppose we have the following users and movies:

- Users: Alice, Bob, Carol, Dave
- Movies: Inception, Avatar, The Matrix, Interstellar

If the function assigns movies as follows:

- Alice $\rightarrow$ Inception
- Bob $\rightarrow$ The Matrix
- Carol $\rightarrow$ Interstellar
- Dave $\rightarrow$ Avatar

# Functions

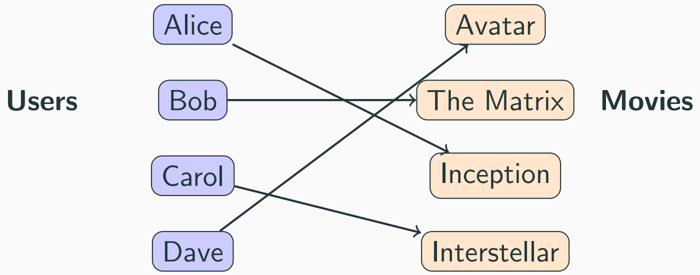
**Definition**

Let $A$ and $B$ be sets. A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$. We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element $a$ of $A$. If $f$ is a function from $A$ to $B$ we write $f : A \to B$.

Functions are specified in many different ways:

- Explicitly stating the assignments, as in the maths class example;
- Through a formula, such as $f(x) = x + 1$;
- Through a computer program, or an algorithm.

# Functions

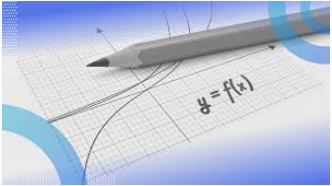

Users
Alice
Bob
Carol
Dave
Avatar
The Matrix
Inception
Interstellar
Movies

# Examples of Functions

## Functions All Around Us

We interact with functions every day, often without realizing it:

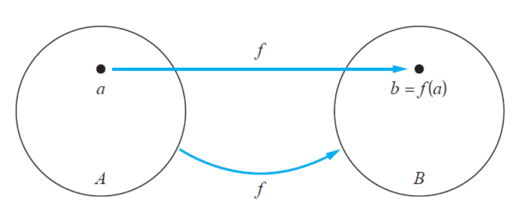
- **Spotify Recommendations:** A function of your listening history, liked songs, and trending music.
- **Instagram Filters:** A function of the original image, selected filter, and filter settings.
- **Gaming Physics Engines:** The trajectory of a projectile in a game.



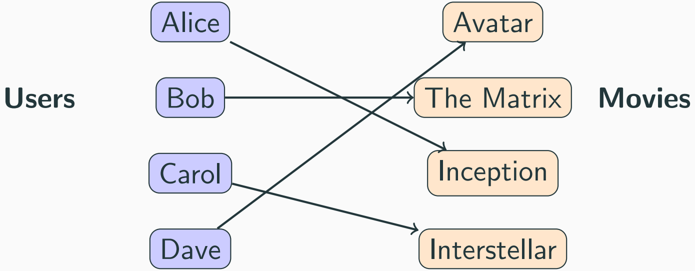
# Functions

## Definition

- If $f$ is a function from $A$ to $B$, we say that $A$ is the domain of $f$ and $B$ is the codomain of $f$.
- If $f(a) = b$, we say that $b$ is the image of $a$.
- The image of $f$ is the set of the images of all elements of $A$.



# Functions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# Properties of Functions

## Injective (One-to-One) Function

A function $f$ is said to be injective (or one-to-one) if and only if $f(a) = f(b)$ implies that $a = b$ for all $a$ and $b$ in the domain of $f$.

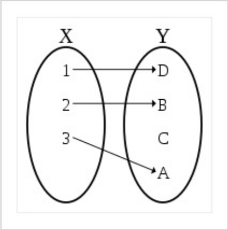
## Surjective (Onto) Function

A function $f$ from $A$ to $B$ is called surjective (or onto) if and only if for every element $b \in B$, there is an element $a \in A$ such that $f(a) = b$.

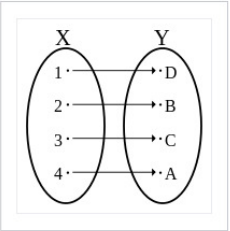
## Bijective Function

A function $f$ is bijective if it is both injective and surjective.

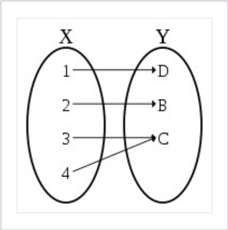
# Properties of Functions



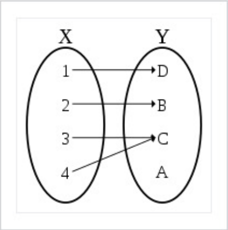

An **injective** non-surjective function (injection, not a bijection)




An **injective** surjective function (bijection)




A non-injective surjective function (surjection, not a bijection)

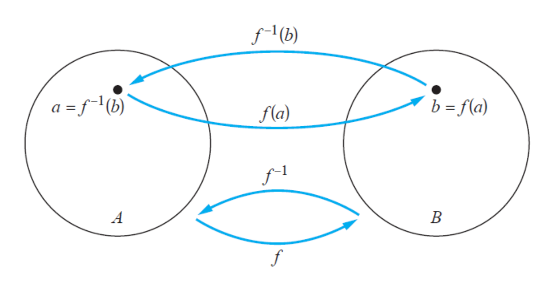



A non-injective non-surjective function (also not a bijection)

# Inverse Functions

**Definition**

Let $f$ be a bijective function from the set $A$ to the set $B$. The *inverse function* of $f$ is the function that assigns to an element $b$ belonging to $B$ the unique element $a$ in $A$ such that $f(a) = b$. The inverse function of $f$ is denoted by $f^{-1}$. Hence $f^{-1}(b) = a$ when $f(a) = b$. A bijective function is called *invertible* since we can define its inverse.



# Real-Valued Functions

**Real-valued functions** map real numbers to real numbers and are fundamental in the study of calculus.

**Definition**

A function $f : \mathbb{R} \to \mathbb{R}$ assigns to each element $x \in \mathbb{R}$ a unique element $y = f(x) \in \mathbb{R}$.

- **Example:** The function $f(x) = x^2$ assigns to each real number $x$ its square $x^2$.
- **Example:** The function $f(x) = \sin(x)$ maps each real number $x$ to its sine value.

## Graphing Functions

The graph of a function $f(x)$ is a visual representation of the relationship between $x$ and $f(x)$.

**Key Idea**

The **horizontal axis (x-axis)** represents the input values, the **vertical axis (y-axis)** represents the output values $y = f(x)$.

- **Example:** The graph of $f(x) = x^2$ is a parabola opening upwards.