Mathematics Fundamentals
Lecture 4: Matrices

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Introduction to Matrices

What is a Matrix?

- A **matrix** is a rectangular array of numbers or symbols arranged in rows and columns.
- The size of a matrix is defined by the number of rows and columns, denoted as $m \times n$, where m is the number of rows and n is the number of columns.
- Each element in a matrix is called an entry and is denoted by a_{ij}, where i is the row number and j is the column number.
- Example: A 2×3 matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Here, $a_{12} = 2$ is the entry in the first row and second column.

Real-World Example 1: Data Matrix in Machine Learning

- In **machine learning**, a dataset is often represented as a matrix, where each row represents an instance (e.g., a customer, patient, or product), and each column represents a feature (e.g., age, income, or height).
- For example, a matrix X with 4 instances and 3 features could look like this:

$$X = \begin{pmatrix} 25 & 50000 & 5.6 \\ 30 & 60000 & 5.9 \\ 22 & 45000 & 5.4 \\ 27 & 52000 & 5.7 \end{pmatrix}$$

- This matrix can be used in various algorithms to predict outcomes, classify data, or discover patterns.

Real-World Example 2: Social Network Adjacency Matrix

- In a **social network**, the relationships between individuals can be represented using an **adjacency matrix**.
- If we consider a small network of 4 people, the adjacency matrix A might look like this:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

- Here, $A_{ij} = 1$ indicates that person i is connected to person j, while $A_{ij} = 0$ indicates no direct connection.
- Such matrices are crucial in analyzing social structures, identifying influencers, and understanding network dynamics.

Types of Matrices

Types of Matrices (Part 1)

- **Identity Matrix:** A square matrix with ones on the diagonal and zeros elsewhere.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- **Zero Matrix:** A matrix where all elements are zero.

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- **Square Matrix:** A matrix with the same number of rows and columns.

Types of Matrices (Part 2)

- **Diagonal Matrix:** A square matrix where all off-diagonal elements are zero.

$$D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$$

- **Triangular Matrices:**
    - **Upper Triangular:** All elements below the diagonal are zero.

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

    - **Lower Triangular:** All elements above the diagonal are zero.

$$L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

Symmetric Matrix

- **Symmetric Matrix:** A matrix that is equal to its reflection across the diagonal.

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{pmatrix}$$

- Here, $s_{ij} = s_{ji}$ for all i and j.
- Symmetric matrices are common in applications like physics, statistics (covariance matrices), and more.

Matrix Operations

What is a Vector?

- A **vector** is a special type of matrix with only one row or one column.
- Example of a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

- Vectors are essential for operations like the dot product, which is used in matrix multiplication.

Matrix Operations (Part 1)

- **Matrix Addition:** Adding two matrices of the same size by adding their corresponding elements.

$$A+B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

- **Scalar Multiplication:** Multiplying every element of a matrix by a scalar.

$$cA = c \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} c \times a_{11} & c \times a_{12} \\ c \times a_{21} & c \times a_{22} \end{pmatrix}$$

Matrix Operations (Part 2)

- **Introduction to Dot Product:** The dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- **Matrix Multiplication:** The product of two matrices A and B is another matrix C, where each element c_{ij} is the dot product of the i-th row of A and the j-th column of B.

$$C = A \cdot B = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 \end{pmatrix}$$

where $\mathbf{a}_i$ is the i-th row of A and $\mathbf{b}_j$ is the j-th column of B.

When Does Matrix Multiplication Exist?

- Matrix multiplication is only defined when the number of columns in the first matrix matches the number of rows in the second matrix.

$$A_{m \times n} \cdot B_{n \times p} = C_{m \times p}$$

- Example:

$$A_{3\times3} \cdot B_{3\times2} \quad \text{is valid, but} \quad B_{3\times2} \cdot A_{3\times3} \quad \text{is not.}$$

- Even when both products are defined, AB and BA may yield different results or may not both exist.

Matrix Operations (Part 3)

- **Example of Matrix Multiplication:**

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix}$$

$$C = A \cdot B = \begin{pmatrix} (1 \cdot 7 + 2 \cdot 9 + 3 \cdot 11) & (1 \cdot 8 + 2 \cdot 10 + 3 \cdot 12) \\ (4 \cdot 7 + 5 \cdot 9 + 6 \cdot 11) & (4 \cdot 8 + 5 \cdot 10 + 6 \cdot 12) \end{pmatrix}$$

$$C = \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}$$

Matrix Operations (Part 4)

- **Transpose of a Matrix:** The transpose of a matrix is formed by swapping the rows and columns of the matrix.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \quad \Rightarrow \quad A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

Properties of Matrix Operations

- **Commutativity (Addition):**

$$A + B = B + A \quad \text{(True for addition)}$$

$$AB \neq BA \quad \text{(Generally false for multiplication)}$$

- **Associativity:**

$$(A + B) + C = A + (B + C) \quad \text{(Addition)}$$

$$(AB)C = A(BC) \quad \text{(Multiplication)}$$

- **Distributive Properties:**

$$A(B + C) = AB + AC \quad \text{(Left Distributive)}$$

$$(A + B)C = AC + BC \quad \text{(Right Distributive)}$$

- **Null and Identity Elements:**
  - $A + 0 = A$, where 0 is the zero matrix.
  - $AI = A$ and $IA = A$, where I is the identity matrix.