Maths for Computing

Lecture 3: The inverse of a matrix

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Today's Objective

- Introduce functions
- Define various components of a function
- Discuss properties of functions

Inverse Matrix

Given a matrix $\boldsymbol{A}$, we say that $\boldsymbol{A}^{-1}$ is its inverse if

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

Then $\boldsymbol{A}$ is said to be invertible.

Only square matrices can have inverses.

Inverse

Not all square matrices have an inverse. The inverse exists if and only if $det(\boldsymbol{A}) \neq 0$. If an inverse exists, this is unique.

Exercise

Show that $\boldsymbol{A}$ and $\boldsymbol{X}$ are inverse of each other.

$$\boldsymbol{A} = \begin{pmatrix} 5 & 6 \\ 5 & 10 \end{pmatrix} \quad \boldsymbol{X} = \begin{pmatrix} 1/2 & -3/10 \\ -1/4 & 1/4 \end{pmatrix}$$

Properties of the Inverse

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be invertible $n \times n$ matrices. Then

- $\boldsymbol{A}^{-1}$ is invertible and $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
- $\boldsymbol{AB}$ is invertible and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
- The transpose $\boldsymbol{A}^t$ is invertible and $(\boldsymbol{A}^t)^{-1} = (\boldsymbol{A}^{-1})^t$
- $(c\boldsymbol{A})^{-1} = c^{-1}\boldsymbol{A}^{-1}$ for $c \neq 0$.

Computing the Inverse of a 2 by 2 Matrix

Provided that $|\boldsymbol{A}| \neq 0$,

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \boldsymbol{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Computing the Inverse

Any $n \times n$ square matrix $\boldsymbol{A}$ has an inverse $\boldsymbol{A}^{-1}$ given by

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|} \text{adjoint}(\boldsymbol{A}),$$

with

$$\text{adjoint}(\boldsymbol{A}) = \begin{pmatrix} C_{1,1} & C_{2,1} & \cdots & C_{n,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{n,2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1,n} & C_{2,n} & \cdots & C_{n,n} \end{pmatrix}$$

where $C_{i,j}$ is the (i, j) cofactor.

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Minor of a Matrix

A submatrix of $\boldsymbol{A}$ is a matrix obtained by deleting some rows and/or some columns.

$$\begin{pmatrix} 12 & 1 & -7 & 0.5 \\ \frac{3}{4} & -4 & 8 & 1 \\ 0 & -2 & \frac{-2}{7} & 9 \end{pmatrix}$$

Minor

A minor of a matrix $\boldsymbol{A}$ is the determinant of a square submatrix of $\boldsymbol{A}$.

The minor is of *order* k if it is the determinant of a submatrix $k \times k$.

Example

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 4 & 2 & 1 \\ 5 & 0 & 4 \end{pmatrix}$$

Some minors:

- $\det(1) = 1$ is a minor of order 1, as well as $3, 0, 4, 2, \ldots$
- $\det \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = -10$ is a minor of order 2
- $\det \begin{pmatrix} 4 & 1 \\ 5 & 4 \end{pmatrix} = \ldots$ is a minor of order 2.
- $\det(A)$ is the only minor of order 3.

Rank

Rank

The rank of a matrix is the order of largest non-zero minor. The rank of $\boldsymbol{A}$ is denoted as $rank(\boldsymbol{A})$.

Properties

- $1 \leq rank(A) \leq \max\{m, n\}$ for all $m \times n$ matrices different from $\mathbf{0}$
- Let $\boldsymbol{A}$ be a $n \times n$ matrix. If $|\boldsymbol{A}| \neq 0$ then $rank(\boldsymbol{A}) = n$