Maths for Computing

Lecture 2: Functions

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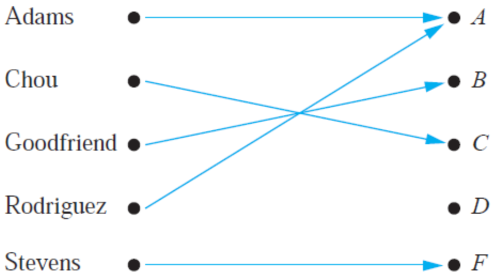
Today's Objective

- Introduce functions
- Define various components of a function
- Discuss properties of functions

Functions

In many instances we assing to each of a set a particular element of a second set. For example, suppose that each student in a mathematics class is assigned a letter grade from the set $\{A, B, C, D, F\}$. Suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez and F for Stevens. This assignment is an example of a *function*.



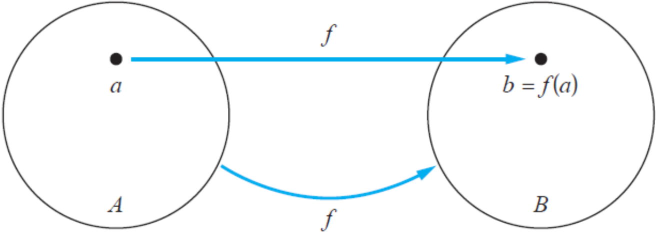
Functions

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B we write $f : A \to B$.

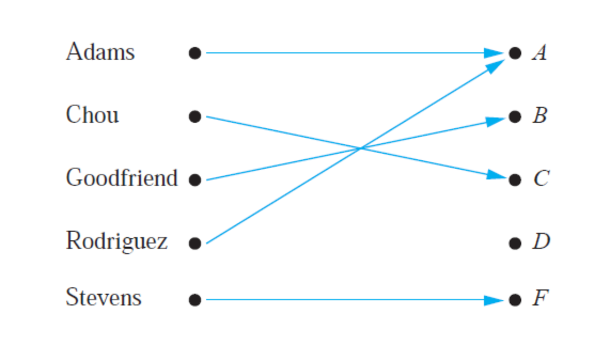
Functions are specified in many different ways. Sometimes we explicitly state the assignments, as in the maths class example. Often we give a formula, such $f(x) = x + 1$. Other times we use a computer program to specify a function.

Functions

If f is a function from A to B we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$ we say that b is the *image* of a and a is the *preimage* of b. The *image* of f is the set of all images of elements of A. Also, if f is a function from A to B we say that f *maps* A to B.

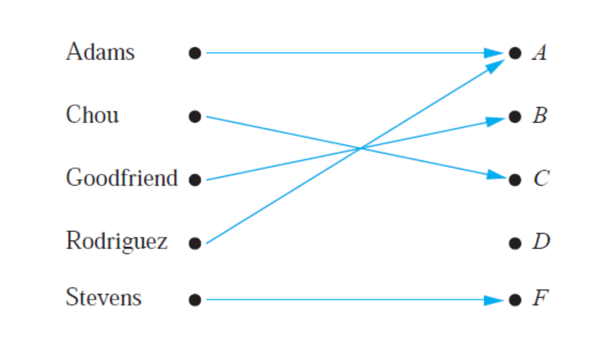


Examples



- Domain:
- Codomain:
- Image:

Examples



- Domain: { Adams, Chou, Goodfriend, Rodriguez, Stevens}
- Codomain: { A, B, C, D, F }
- Image: { A, B, C, F }

Examples

Let $f : \mathbb{Z} \to \mathbb{Z}$ assign the square of an integer to this integer.

- Domain:
- Codomain:
- Image:
- Function definition:

Examples

Let $f : \mathbb{Z} \to \mathbb{Z}$ assign the square of an integer to this integer.

- Domain: the set of all integers
- Codomain: the set of all integers
- Image: the set of all integers that are perfect squares
- Function definition: $f(x) = x^2$

Real-valued functions

A function is called *real-valued* if its codomain is the set of real numbers, and it is called *integer-valued* if its codomain is the set of integers.

Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

Real-Valued Functions

Let f_1 and f_2 be functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f_1(x) = x^2$ and $f_x(2) = x - x^2$.

- $(f_1 + f_2)(x) =$
- $(f_1 f_2)(x) =$

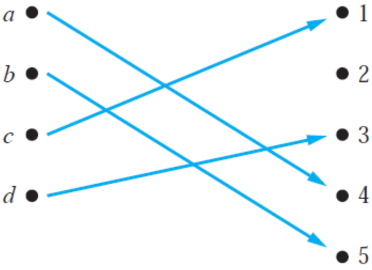
Real-Valued Functions

Let f_1 and f_2 be functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f_1(x) = x^2$ and $f_x(2) = x - x^2$.

- $(f_1 + f_2)(x) = x$
- $(f_1 f_2)(x) = x^3 - x^4$

Injective Functions

A function f is said to be *injective* or *one-to-one* if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.



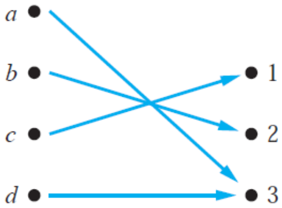
Increasing Functions

A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$ whenever $x < y$ and x and y are in the domain of f.
Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$ whenever $x < y$ and x and y are in the domain of f.

A function that is either strictly increasing or strictly decreasing must be one-to-one.
A function that is increasing, but not strictly increasing, or decreasing, but not strictly decreasing, is not one-to-one.

Surjective Functions

A function f from A to B is called *surjective* or *onto* if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.



Bijective Functions

The function f is *bijective* if it is both one-to-one and onto.

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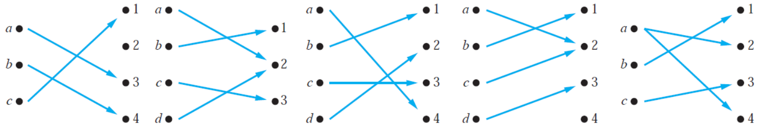

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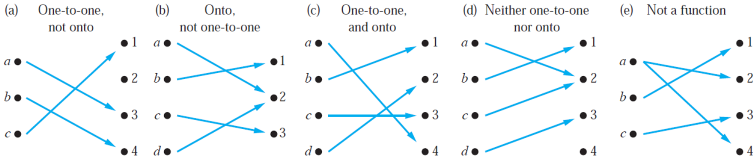

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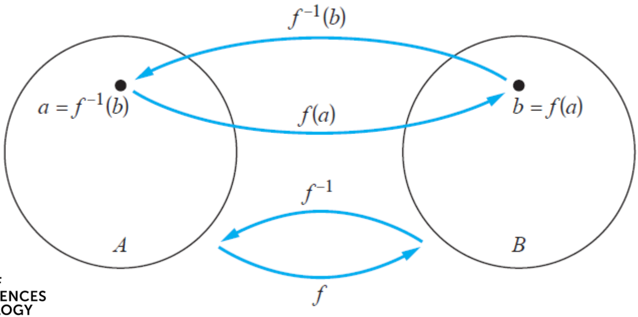
- Consider the function $f(x) = x^2$ from the set of integers to the set of integers. This is not injective since for instance $f(1) = f(-1) = 1$ but $1 \neq -1$. It is not surjective since for instance there is no integer x with $x^2 = -1$. Therefore is not bijective.
- Consider the function $f(x) = x + 1$ from the set of integers to the set of integers. It is injective since $x + 1 \neq y + 1$ when $x \neq y$. It is surjective since for every integer y there is an integer x such that $f(x) = y$. Therefore it is bijective.
- Consider the function $f : A \to A$, such that $f(x) = x$. This is called *identity* function. It is bijective.

Let f be a bijective function from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence $f^{-1}(b) = a$ when $f(a) = b$. A bijective function is called *invertible* since we can define its inverse.



Inverse Functions

- Let $f : \mathbb{Z} \to \mathbb{Z}$ be such that $f(x) = x + 1$. It is invertible since it is bijective. To find the inverse, suppose that y is the image of x so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbb{Z}$ that is sent to y by f. Thus $f^{-1}(y) = y - 1$.
- Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. It is not invertible since it is not bijective.
- Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(x) = x^2$. One can show that it is bijective and therefore it is invertible. Its inverse can be derived as $f^{-1}(y) = \sqrt{y}$.

Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a, b) | a \in A \text{ and } f(a) = b\}$.

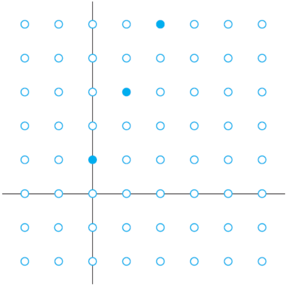


FIGURE 8 The Graph of $f(n) = 2n + 1$ from **Z** to **Z**.

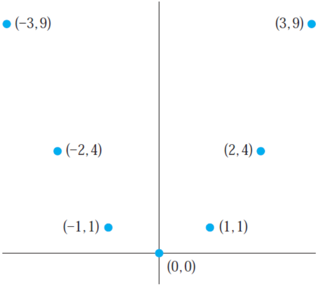


FIGURE 9 The Graph of $f(x) = x^2$ from **Z** to **Z**.

The Graph of a Function