Maths for Computing

Lecture 3: Introduction to Matrices

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Today's Objective

- Introduce functions
- Define various components of a function
- Discuss properties of functions

Matrices

A matrix is a table of real numbers. We say that a matrix has dimension $n \times m$ if it has n rows and m columns.

$$\boldsymbol{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{pmatrix}$$

$\boldsymbol{A}$ is often expressed as $\boldsymbol{A} = (a_{i,j})_{n \times m}$ or simply $\boldsymbol{A} = (a_{i,j})$ where $a_{i,j}$ is the element of $\boldsymbol{A}$ in the i-th row and j-th column.

A vector is a matrix with only one row (**row vector**) or only one column (**column vector**). Vectors are usually denoted by small bold letters like $\boldsymbol{x}$ or $\boldsymbol{y}$.

Exercise

Construct the 4×3 matrix $\boldsymbol{A} = (a_{i,j})_{4\times 3}$ with $a_{i,j} = 2i - j$.

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$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{pmatrix}$$

Some Special Matrices

- The *zero matrix* $\mathbf{0}$ denotes the $n \times m$ matrix consisting of only zeros
- A *square* matrix has $n = m$, i.e. same number of rows and columns
- In a square matrix $\boldsymbol{A} = (a_{i,j})_{n \times n}$, the elements $a_{1,1}, a_{2,2}, \ldots, a_{n,n}$ constitute the *main diagonal*
- A square matrix $\boldsymbol{A} = (a_{i,j})_{n \times n}$ is *symmetric* if $a_{i,j} = a_{j,i}$ for all $i \neq j$, i.e. it is symmetric about the main diagonal
- The *identity matrix* of order n denoted by $\boldsymbol{I}_n$ or simply $\boldsymbol{I}$ is the $n \times n$ matrix having ones along the main diagonal and zeros elsewhere.
- A square matrix is called *lower triangular* (or upper) if the elements above (or below) the main diagonal are zero.

Examples

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$$

$$U = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

Matrix addition and multiplication by a scalar

Consider two matrices $\boldsymbol{A} = (a_{i,j})_{n\times m}$ and $\boldsymbol{B} = (b_{i,j})_{n\times m}$.

- $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be of the same order (same dimension)
- $\boldsymbol{A} = \boldsymbol{B}$ if $a_{i,j} = b_{i,j}$ for all i and j. Otherwise $\boldsymbol{A} \neq \boldsymbol{B}$.
- The sum $\boldsymbol{A} + \boldsymbol{B}$ is defined as

$$\boldsymbol{A} + \boldsymbol{B} = (a_{i,j} + b_{i,j})_{n\times m}$$

- If $\alpha \in \mathbb{R}$

$$\alpha\boldsymbol{A} = (\alpha a_{i,j})_{n\times m}$$

Properties of summation and multiplication by scalar

Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be $n \times m$ matrices and let $\alpha, \beta \in \mathbb{R}$.

- $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$
- $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
- $\boldsymbol{A} + \mathbf{0} = \boldsymbol{A}$
- $\boldsymbol{A} + (-1)\boldsymbol{A} = \mathbf{0}$
- $(\alpha + \beta)\boldsymbol{A} = \alpha\boldsymbol{A} + \beta\boldsymbol{A}$
- $\alpha(\boldsymbol{A} + \boldsymbol{B}) = \alpha\boldsymbol{A} + \alpha\boldsymbol{B}$

Matrix Product

Compatible Matrices

The product of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted as $\boldsymbol{AB}$ can be defined if the dimensions of the two matrices are $m \times n$ ($\boldsymbol{A}$) and $n \times p$ ($\boldsymbol{B}$), i.e. the number of columns of $\boldsymbol{A}$ is the same as the number of rows of $\boldsymbol{B}$.

Call $\boldsymbol{C} = \boldsymbol{AB}$, then $\boldsymbol{C}$ is $m \times p$ matrix $\boldsymbol{C} = (c_{i,j})_{m \times p}$ with entries

$$c_{i,j} = \sum_{r=1}^{n} a_{i,r} b_{r,j} = a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \cdots + a_{i,n} b_{n,j}$$

Notice that if $\boldsymbol{AB}$ exists, it does not follow that $\boldsymbol{BA}$ does as well.

Example

It is often called row-column product

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -7 \end{pmatrix}$$

Exercise

Consider the following 2 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Compute $\boldsymbol{C} = \boldsymbol{AB}$. Is $\boldsymbol{BA}$ defined?

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & -1 & 6 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Exercise

Consider the following 2 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Compute $\boldsymbol{C} = \boldsymbol{AB}$. Is $\boldsymbol{BA}$ defined?

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & -1 & 6 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} -1 & 2 \\ 8 & 5 \\ 5 & 14 \end{pmatrix}$$

$\boldsymbol{BA}$ is not defined

Powers of Matrices

If $\boldsymbol{A}$ is a square matrix, because of associativity we can write $\boldsymbol{AA} = \boldsymbol{A}^2$ and $\boldsymbol{AAA} = \boldsymbol{A}^3$, and so on. In general

$$\boldsymbol{A}^n = \underbrace{\boldsymbol{AA}\cdots\boldsymbol{A}}_{n\ times}$$

Transpose

The transpose matrix is the matrix where row and columns are interchanged.

If $\boldsymbol{A} = (a_{i,j})_{n\times m}$ then its transpose $\boldsymbol{A}^t = (a^t_{i,j})_{m\times n}$.

The (i,j) element of $\boldsymbol{A}$ is equal to the (j,i) element of $\boldsymbol{A}^t$,
i.e. $a_{i,j} = a^t_{j,i}$.

Example

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 7 & -3 & 6 \end{pmatrix}$$

$$A^t = \begin{pmatrix} -1 & 7 \\ 3 & -3 \\ 2 & 6 \end{pmatrix}$$

The Determinant of Order 2

The determinant is a number associated to any square matrix.

2×2

$$det(\boldsymbol{A}) = |\boldsymbol{A}| = det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

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The Determinant of Order 3

3×3

$$det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} =$$

$$a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{2,1}a_{3,2}a_{1,3}$$

$$-(a_{1,3}a_{2,2}a_{3,1} + a_{1,2}a_{2,1}a_{3,3} + a_{1,1}a_{3,2}a_{2,3})$$

This already gets messy!!

Expansion by Cofactors

$$
\begin{aligned}
|\boldsymbol{A}| &= \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \\
&= a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}
\end{aligned}
$$

- Cofactor 1 is determined by deleting the first row and the first column
- Cofactor 2 is determined by deleting the first row and the second column
- Cofactor 3 is determined by deleting the first row and the third column

General Rule for Determinants

Let $\boldsymbol{A}$ be a $n \times n$ matrix.

The expansion of $|\boldsymbol{A}|$ in terms of the elements of the ith row is given by

$$|\boldsymbol{A}| = a_{i,1}C_{i,1} + a_{i,2}C_{i,2} + \cdots + a_{i,n}C_{i,n},$$

where $C_{i,j}$ is a *cofactor*.

A cofactor $C_{i,j}$ can be found as follows:

- Delete the i-th row and the j-th column from $\boldsymbol{A}$ and compute its determinant
- Multiply the determinant by the factor $(-1)^{i+j}$

Exercise

Consider the matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 & 0 & 2 \\ 6 & 1 & c & 2 \\ -1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 3 \end{pmatrix}$$

- What is $C_{2,3}$?
- What is $|\boldsymbol{A}|$?

Properties of Determinants

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices and $\alpha \in \mathbb{R}$.

- If all elements in a row (or column) of $\boldsymbol{A}$ are 0, then $|\boldsymbol{A}| = 0$
- $|\boldsymbol{A}| = |\boldsymbol{A}^t|$
- $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$
- $|\alpha \boldsymbol{A}| = \alpha^n |\boldsymbol{A}|$
- $|\boldsymbol{I}| = 1$
- If $\boldsymbol{A}$ is triangular (or diagonal) $|\boldsymbol{A}| = \prod_{i=1}^{n} a_{i,i}$

Exercise

Compute the determinant of the following matrix

$$\begin{pmatrix} a_1 - x & a_2 & a_3 & a_4 \\ 0 & -x & 0 & 0 \\ 0 & 1 & -x & 0 \\ 0 & 3 & 1 & -x \end{pmatrix}$$