Chair of Mathematical Finance
Department of Mathematics
Technical University of Munich

Financial Mathematics 2
Prof. Dr. Rudi Zagst, Matthias König, Tobias Lausser

Exercise sheet 2

The exercise sheet will be discussed in one group of in-person exercise sessions on 07.05.2025. For details, see the course's Moodle announcements and TUMonline. You should try to solve the exercises at home before the exercise.

Exercise 2.1

Let $X \sim \mathcal{N}_d(\mu, \Sigma)$ follow a multivariate normal distribution.

a) Let $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^m$. Show that $Y := AX + b \sim \mathcal{N}_m(A\mu + b, A\Sigma A^T)$.
Hint: You may use that $X \in \mathbb{R}^d$ *has multivariate normal distribution if and only if* $a^T X$ *is (univariate) normally distributed for any* $a \in R^d$.

b) Let $W = \{W_t\}_{t\geq 0}$ be a Brownian motion. For some given $0 \leq t_1 < .... < t_d$ define $X_i = W_{t_i}$. Show that $X \sim \mathcal{N}_d(0, \Sigma)$ with $\Sigma_{ij} = \min(t_i, t_j)$ for all $1 \leq i, j \leq d$.

c) Assume Σ is invertible. Show that $X_1, ..., X_d$ are independent if and only if Σ is a diagonal matrix.

d) Find an example for a random vector $(X, Y) \in \mathbb{R}^2$ such that the marginals X and Y are normally distributed and $Cov(X, Y) = 0$, but $X + Y$ is not normally distributed.

Exercise 2.2

Let $Z \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 > 0$. For given $\gamma \in \mathbb{R}$ define $L = \exp\big(-\frac{1}{2}\gamma^2\sigma^2 - \gamma Z\big)$ and $\tilde{\mathbb{Q}} : \mathcal{F} \to \mathbb{R}$ as

$$\tilde{\mathbb{Q}}(A) = \mathbb{E}\big[\mathbb{1}_A \cdot L\big].$$

a) Show that $\tilde{\mathbb{Q}}$ is a probability measure.

b) Show that $\tilde{\mathbb{Q}}$ is equivalent to $\mathbb{Q}$.

c) Calculate the moment generating function

$$M_Z^{\tilde{\mathbb{Q}}}(t) = \mathbb{E}_{\tilde{\mathbb{Q}}}[\exp(t \cdot Z)] \qquad \forall t \in \mathbb{R},$$

of Z with respect to $\tilde{\mathbb{Q}}$.

d) Determine the distribution of Z with respect $\tilde{\mathbb{Q}}$.

Exercise 2.3

Let $W = \{W_t\}_{t\geq 0}$ be a Brownian motion. Show that for each of the following definitions of X_t, the stochastic process $X = \{X_t\}_{t\geq 0}$ is a Brownian motion.

a) $X_t := -W_t$

b) $X_t := tW_{\frac{1}{t}}$ for $t > 0$ and $X_0 := 0$.
Hint: For verifying the continuity of X*, you may use that* $\lim_{t\to\infty} \frac{1}{t} W_t = 0$ *holds* $\mathbb{Q}$*-a.s.*

c) $X_t := W_{t+\tau} - W_\tau$, where τ is a positive constant.

Exercise 2.4

Let $W = \{W_t\}_{t\geq 0}$ be a Brownian motion.

a) Write a Python-function `def BM(T, n)` that simulates W on the interval $[0, T]$, using the grid $\triangle = T/n$. The function should return the path of W as an $n+1$-dimensional vector. To this end recall that $W_{t_i} - W_{t_{i-1}} \sim \sqrt{t_i - t_{i-1}}\, \mathcal{N}(0, 1)$.
Hint: Use `np.random.default_rng, rng.normal, np.cumsum`.

b) Set $T = 10$, $n = 1000$ and plot several simulated paths of the Brownian motion in one figure.
Hint: Use `matplotlib.pyplot`.

c) Simulate n different paths of a Brownian motion. Take a value t, with $0 < t \leq T$. Use the realizations $W_t^{(1)}, W_t^{(2)}, \ldots, W_t^{(n)}$ to determine the empirical distribution of W_t. Verify that $W_t \sim \mathcal{N}(0, t)$, by comparing the empirical density with the real density of a normal distribution.
Hint: Use `scipy.stats.gaussian_kde`.

d) For $0 \leq s \leq t < \infty$ write a similar Python-function `def CovBM(s,t,m)` that simulates m times the values W_s and W_t and calculates the empirical covariance of the samples:

$$Q := \frac{1}{m-1} \sum_{i=1}^{m} (W_s^i - \bar{W}_s)(W_t^i - \bar{W}_t),$$

where the index i denotes the i-th realization and $\bar{W}_s$ and $\bar{W}_t$ are the respective empirical means. Set $s = 1$, $t = 2$ and $m = 10000$ to illustrate numerically that $\mathrm{Cov}(W_s, W_t) = \min\{s, t\}$.
Hint: Use `np.cov`.