Chair of Mathematical Finance
Department of Mathematics
Technical University of Munich

Financial Mathematics 2
Prof. Dr. Rudi Zagst, Matthias König, Tobias Lausser

Exercise sheet 7

The exercise sheet will be discussed in two groups of in-person exercise sessions on 25.06.2025. For details, see the course's Moodle announcements and TUMonline. You should try to solve the exercises at home before the exercise.

Exercise 7.1

Consider the Black-Scholes model (Definition 4.1), defined on $(\Omega, \mathcal{F}, \mathbb{F}_W, \mathbb{Q})$, with market dynamics given by

$$\begin{aligned} dB_t &= rB_t\,dt, \quad B_0 = 1, \\ dP_t &= P_t(b\,dt + \sigma\,dW_t), \quad P_0 = p_0 > 0, \end{aligned}$$

with constant coefficients $b \in \mathbb{R}$, r, $\sigma \in \mathbb{R}_+$. A cash-or-nothing call option is a contingent claim with terminal payoff

$$D = C \cdot \mathbb{1}_{\{P_T > K\}}, \quad C \in \mathbb{R}_{>0}.$$

a) Draw a payoff chart at maturity T.

b) Determine an arbitrage-free price of the cash-or-nothing call option, i.e. of D, at time $t = 0$.

Exercise 7.2

In this exercise we are going to explicitly derive the formulas for the Greeks of European call-options in the Black-Scholes model.

a) Prove Lemma 4.18, i.e. show that

$$\Delta_{\text{Call}} := \frac{\partial \text{Call}}{\partial P_0} = N(d_1).$$

Hint: As an intermediate step, show that $n(d_1 - \sigma\sqrt{T}) = n(d_1)\frac{P_0}{K}e^{rT}$ and use this for a simplifying substition.

b) Prove Lemma 4.21, i.e. show that

$$\Gamma := \frac{\partial^2 \text{Call}}{\partial P_0^2} = \frac{n(d_1)}{P_0\sigma\sqrt{T}}.$$

c) Prove Lemma 4.23, i.e. show that

$$\Theta_{\text{Call}} := \frac{\partial \text{Call}}{\partial T} = \frac{P_0\sigma n(d_1)}{2\sqrt{T}} + Kre^{-rT}N(d_2)$$

d) Prove Lemma 4.24, i.e. show that

$$\mathcal{V} := \frac{\partial \text{Call}}{\partial \sigma} = P_0\sqrt{T}n(d_1).$$

Exercise 7.3

In this exercise we prove the equivalence of 1) and 3) in Theorem 4.10:

Assume a generalized Black-Scholes model as in Definition 4.4 and let the market filtration be given as $\mathbb{F} = \mathbb{F}^W$. Provided that Novikov's condition (cf. Remark 2.107) holds for each entry σ_{ij} of the volatility matrix σ, show that the following statements are equivalent:

1) There exists a unique EMM $\tilde{\mathbb{Q}} \sim \mathbb{Q}$ for the discounted price processes $\tilde{P}^{(i)}$, $i = 1, \ldots, d$.
3) Equality $n = d$ holds (the number of assets d equals the number of sources of risk n) and the volatility matrix $\sigma(t, \omega)$ is $(\lambda[0, T] \otimes \mathbb{Q})$-a.e. non-singular.

Exercise 7.4

Consider a stock price in a Black-Scholes market with initial value `P0`, drift μ, risk-neutral drift r, volatility σ and a European-type call option with time to maturity T and strike K.

a) Write an Python-function `def Call(P0, r, sigma, T, K)` that returns the analytical value of a call option with initial value `P0`, risk-neutral drift `r`, volatility `sigma`, time to maturity `T`, and strike `K`. Call your function for the parameters `P0` $= 1$, `r` $= 0.05$, `sigma` $= 0.2$, `T` $= 2$ and `K` $= 0.8$.

b) Now we are interested in inverting the Black-Scholes-Call formula for the Black-Scholes-volatility `sigma`. For this purpose write an R-function `def ImpliedVola(P0, r, T, K, C)` that returns the implied volatility of a call option with initial value `P0`, drift `r`, time to maturity `T`, strike `K` and value `C`. Double-check your function using the call price from (a).

   Hint: Use `scipy.optimize.root` to find the values of σ in $[0, 5]$, where the theoretical call price coincides with `C`.

c) Load the file *OptionValue.csv* into your Python. The first line of the table contains 6 possible maturities: 1, 2, 3, 6, 9, 12 months. The annualized risk-less interest rates r corresponding to these maturities are given in the second line. The remaining lines contain the prices of European call options written on the EuroStoxx 50 index with the following strikes 2700, 2800, 2900, 3000, 3100, 3200, 3300. E.g. the third line contains the option prices for strike 2700 and for each of the maturities above. Assume that the EuroStoxx 50 is traded at $P_0 = 2890.62$. Compute the implied volatilities for all entries of the given EuroStoxx 50 call option matrix.
   Hint: Read the option data using the function `pd.read_csv`. It might be useful to define separate arrays with the maturities and interest rates.

d) Plot the implied volatility surface in a 3-dimensional plot. What do you observe?
   Hint: Use the axis configuration `ax = fig.add_subplot(111, projection='3d')` to plot the volatility surface.

The exercise sheet will be discussed in two groups of in-person exercise sessions on 25.06.2025. For details, see the course's Moodle announcements and TUMonline. You should try to solve the exercises at home before the exercise.