A BGG TYPE RESOLUTION FOR POINTED HOPF ALGEBRAS WITH ABELIAN CORADICAL.

Abstract. We define a projective resolution for a certain type of Hopf algebras and use it to study some homological properties.

1. Introduction

The Bernstein-Gelfand-Gelfand resolution of the universal enveloping of a (complex) Lie algebra $\mathfrak{g}$ was introduced in TODO:REFERENCE in order to TODO:REFERENCE. It principal ingredients are a triangular decomposition of $\mathfrak{g}=: \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and the action of the Weyl group on the weight space of $\mathfrak{g}$. Later it was realised that for quantised universal enveloping algebras at a non-root of unity an almost identical resolution can be given.

1.1. **Motivation and main results.** In this paper we intend to generalise this procedure further to the setting of pointed Hopf algebras with abelian coradical.

1.2. **Outline.**

2. Pointed Hopf algebras with abelian coradical

We work over an algebraically closed field k of characteristic zero; 'dim' and '$\otimes$' ought to be understood as dimension and tensor product over k. Standard notation for Hopf algebras, as in e.g. [Mon93, Rad12], is freely used. Given a Hopf algebra H we write $\mathrm{Gr}(H)$ for its group of group-likes, $\mathrm{Pr}(H)$ for its space of primitive elements and H° for its (finite) dual Hopf algebra. The antipode of H is denoted by $S\colon H \to H$, its counit by $\epsilon\colon H \to k$ and its coproduct by $\Delta\colon H \to H \otimes H$. For calculations involving the coproduct of H or the coaction of some H comodule M we rely on reduced Sweedler notation. For example we write $h_{(1)} \otimes h_{(2)}\colon =\Delta(h)$ for $h \in H$. An element $x \in H$ whose coproduct is $\Delta(x) = 1 \otimes x + x \otimes g$, with $g \in \mathrm{Gr}(H)$ is called a twisted primitive.

The *Cartier-Kostant-Milnor-Moore theorem*, see [Swe69, Mon93], states, that any cocommutative Hopf algebra H over an algebraically closed field of characteristic zero is isomorphic to the smash product $U(P)\#kG$ of the universal enveloping algebra of its Lie algebra of primitive elements with the group algebra of its group-like elements. This has some remarkable consequences like the existence of a PBW-basis. The Hochschild cohomology ring of such smash products was study in [BW07].

In this paper we want to consider a generalisation of the previous class of Hopf algebras and study their Hochschild (co-)homology.

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Definition 2.1 ([Rad12]). A Hopf algebra H is called *pointed* if every simple comodule is one-dimensional.

The connection between (finite-dimensional) pointed Hopf algebras and cocommutative Hopf algebras was establised by ANDRUSKIEWITSCH and SCHNEIDER in conjectural form.

Conjecture. *Every finite-dimensional pointed Hopf algebra is generated by group-likes and twisted primitives.*

As a consequence of [AGI19] and the classification of Nichols algebras of diagonal type [Hec09, AA17] the conjecture holds in case the group of group-like elements is abelian.

An important tool for studying pointed Hopf algebras is the coradical filtration. It is defined in analogy with the Jacobson radical of a ring.

Definition 2.2 ([Rad12]). Let C be a coalgebra. The *coradical* of C is

$$C_0 \colon = \sum D, \text{ with } D \subset C \text{ simple comodule.} \tag{1}$$

The *coradical filtration* of C is the ascending filtration $C_0 \subseteq C_1 \subseteq C_2 \subseteq \dots$ whose n+1-th term is inductively defined by

$$C_{n+1} \colon = \{c \in C \mid \Delta(c) \in C_n \otimes C + C \otimes C_0\}$$

We say C is *coradically graded* if as vector spaces

$$C = \sum_{n \geq 0} C(n) \text{ with } \sum_{n=0}^{m} C(n) = C_m$$

We note two important observations about the coradial filtration without a proof.

Lemma 2.3 ([AS02, Definition 1.13]). *The coradical filtration of a coalgebra C is an exhaustive filtration. That is, $C = \cup_{n\geq 0} C_n$.*

Definition 2.4. The coradically graded coalgebra associated to a coalgebra C is

$$\operatorname{gr} C = \oplus_{n\geq 0} C_n \setminus C_{n-1} \text{ with } C_{-1} \colon = \{0\}.$$

In case H is a pointed Hopf algebra its coradical H_0 is a sub-Hopf algebra and the coradical filtration is a coalgebra as well as an algebra filtration. This equips $\operatorname{gr} H$ with the structure of a graded Hopf algebra containing H_0 as a sub-Hopf algebra in degree zero.

Similarly to the aforementioned Cartier-Kostant-Milnor-Moore theorem we want to write the coradically graded Hopf algebra $\operatorname{gr} H$ associated to a pointed Hopf algebra H as a (version of) a smash product.

Lemma 2.5. *Let H be a pointed Hopf algebra. The inclusion $\iota \colon H_0 \hookrightarrow \operatorname{gr} H$ has a retraction $\pi \colon \operatorname{gr} H \twoheadrightarrow H_0$.*

Lemma 2.6. *Let H be a pointed Hopf algebra. The vector space $R \colon = \{h \in \operatorname{gr} H \mid (\mathrm{id} \otimes \pi)\Delta(h) = h \otimes 1\}$ can be equipped with the structure of a braided Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}$.*

2.1. **Nichols algebras.**

3. Weyl groupoids and their geometry

4. The Hochschild homology of pointed Hopf algebras with abelian coradical

——-TODO: READ articles——-

(1) Hochschild and cyclic homology: [AAPW], [BW07], [EGST19], [HK07], [KK14], [Lod98], [MW09, MPSW10], [Wit19]
(2) (Pointed) Hopf algebras: [AS02, ARS10, AA17], [Ang13], [AGI19] [Lau16] [Mon93] [PV16] [Rad12] [Vay19]
(3) Quantum groups: [Jos95], [Kas98]
(4) Kac-Moody algebras: [Kac90], [Kum02],
(5) Weyl group(oids): [CH15, CMW19] [AY18], [Hec09, HS10, HW11]
(6) Tensor categories: [EGNO15]

5. The weight space and root system of a finite-dimensional Nichols algebra of Cartan type

We put ourselves into a nice world and work over an algebraically closed field of characteristic zero. Within this setting - as a consequence of the theorems of Maschke and Schur- every module over (the group algebra of) an abelian group is semisimple and the simple modules are one-dimensional.

5.1. **Generalised Cartan matrices.** In the following we want to introduce a setup which allows us to describe the representation theory of a (special) finite abelian group by geometric means. The geometric information is governed by a root system obtained from a generalised Cartan matrix.

Definition 5.1. [Kum02, 1.1.1 and 1.1.5] A *generalised Cartan matrix* (short GCM) of size N is a square matrix $A := (a_{ij})_{1\leq i,j\leq N}$ with integral coefficients such that for each pair (i,j) the following conditions hold:

(1) $a_{ii} = 2$
(2) $a_{ij} \leq 0$ if $i \neq j$
(3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

The matrix A is called *symmetriseable* if an invertible diagonal matrix D with entries in $\mathbb{Q}$ exists such that $D^{-1}A$ is a symmetric matrix.

Lemma 5.2. *[Kum02, 1.1.5] Let A be a symmetrisable generalised Cartan matrix of size N. There exists a unique invertible diagonal matrix $D = diag(e_1, \ldots, e_N)$ such that*

(1) *$D^{-1}A$ is symmetric*
(2) *every entry of D is a non-negative integer,*
(3) *Ifx $D' = (e'_1, \ldots, e'_N)$ is another matrix satisfying the first two conditions then $e_i \leq e'_i$ for all $1 \leq i \leq N$.*

We call such a matrix the minimal diagonal *matrix associated to the symmetrisable Cartan matrix A.*

Definition 5.3. [Kum02, p. 4] Let $A := (a_{ij})_{1\leq i,j\leq N}$ be a generalised Cartan matrix of size N. It is called *indecomposable* if no partition of $\{1, \ldots N\}$ into two disjoint subsets I_1 and I_2 exists such that $a_{ij} = 0 = a_{ji}$ whenever $i \in I_1$ and $j \in I_2$. Otherwise A is called *decomposable*.

For the following definition we need a good source(!). Maybe [Kac90, Chapter 4].

Definition 5.4. We say that A is of finite type if $D^{-1}A$ is positive definite. In this case we call A an *Cartan matrix.*

Lemma 5.5. *If the generalised Cartan matrix A of size N is of finite type the symmetric matrix $D^{-1}A$ induces a inner product on the space $\mathbb{R}^N$.*

Let us recall the definition of an abstract root system Δ.

Definition 5.6. Let E be a finite-dimensional $\mathbb{R}$ vector space with an inner product denoted $(\cdot|\cdot)$. A subset $\Delta \subset E$ is called a root system of E if

1. Δ spans E,
2. $\alpha \in \Delta$ implies that $\mathbb{R}\alpha \cap \Delta = \{\pm\alpha\}$,
3. For $\alpha, \beta \in \Delta$ the reflection of β along α is contained in Δ. In formulas this means $\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Delta$ and
4. for $\alpha, \beta \in \Delta$ we have $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

We refer to E as the abstract *weight space* associated to Δ. Its elements are called *weights.*

Remark. Given a weight space E and weights $\alpha, \beta \in E$ we write

$$\langle \alpha, \beta \rangle := 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} \tag{2}$$

Definition 5.7. A system of *simple roots* of a root system $\Delta \subset E$ is a subset $\Phi \subset \Delta$ such that

1. Φ is a basis of the weight space E and
2. every root $\alpha \in \Delta$ can be written solely non-negative or non-positive integral linear combination of simple roots.

The subset of roots obtainable by non-negative integral combinations of simple roots is called the set of *positive roots* $\Delta^+ \subset \Delta$.

Definition 5.8. Let $\Delta \subset \mathbb{R}^N$ be a root system and $\Phi = \{\alpha_1, \dots, \alpha_N\}$ a system of simple roots. The *Weyl group* of Δ is the group of reflections along roots $W \subset Aut(\mathbb{R}^N)$. It is generated by the reflections $s_{\alpha_1}, \dots, s_{\alpha_N}$ along simple roots.

There is a correspondence between abstract root systems and Cartan matrices of finite type.

Theorem 5.9. *Let A be a Cartan matrix. Then $D^{-1}A$ induces an inner product on $\mathbb{R}^N$ and the standard basis $\{e_1, \dots, e_N\}$ of $\mathbb{R}^N$ form a set of simple roots.*

Proof. TODO $\square$

The root system associated to a Cartan matrix A is obtained by considered the closure of the reflection of the simple roots.

Theorem 5.10. *Suppose E is a weight space of dimension N for the root system $\Delta \subset E$. Given a system of simple roots $\Phi = \{\alpha_1, \dots, \alpha_N\}$ we obtain a Cartan matrix $A := (a_{ij})_{1 \le i,j \le N}$ via*

$$a_{ij} = \langle \alpha_j, \alpha_i \rangle \qquad \text{for } 1 \le i, j \le N. \tag{3}$$

Moreover any other choice of system of simple roots yields the same Cartan matrix up to permutation.

Proof. TODO □

Theorem 5.11. *Suppose that E, F are weight spaces of the root systems $\Delta \subset E$ and $\Delta' \subset F$. There exists an isomorphism of inner product spaces $f : E \to F$ such that $f(\Delta) = f(\Delta')$ if and only if the Cartan matrices of Δ and Δ' agree up to permutation.*

Proof. TODO □

By the previous theorems the geometry of the weight space is governed by its Cartan matrix. Let us consider some weights with special properties

Definition 5.12. Let E be the weight space of a root system Δ and Φ a system of simple roots. Suppose $\beta \in E$ to be a weight. We call β

(1) *integral* if $\langle \beta, \alpha \rangle \in \mathbb{Z}$ for every root $\alpha \in \Delta$.
(2) *fundamental* if $2\langle \beta, \alpha \rangle$ is 1 for exactly one simple root $\alpha_i \in \Phi$ and 0 otherwise.

The set $\Omega = \{\omega_1, \dots \omega_N\}$ of fundamental weights forms another basis of E. In particular they generate the lattice of integral weights.

We write $\rho := \omega_1 + \dots \omega_N$ for the sum of fundamental weights.

A choice of positive roots gives us a nice partial order on the weight space.

Definition 5.13. Let $\Phi \subset \Delta$ be a sytem of simple roots of a root system Δ of the weight space E. Given two weights $\beta, \gamma \in E$ we say $\beta \leq \gamma$ if $\gamma - \beta$ can be written as a positive integral combination of simple roots

5.2. **Nichols algebras of Cartan type.** We fix a complex N-dimensional vector space V, together with an ordered basis $\{x_1, \dots, x_N\}$ of V an invertible matrix $\mathfrak{q} := (q_{ij})_{1 \leq i,j \leq N} \in \mathrm{GL}(V)$, called the *matrix of the braiding* whose entries lie on the unit circle $S_1 \subset \mathbb{C}$.

For every entry q_{ij} of $\mathfrak{q}$ exists a unique number $r_{ij} \in [0, 1)$ such that $q_{ij} = e^{2\pi i r_{ij}}$.

Definition 5.14. We call the matrix $\mathfrak{h}$ whose entries are

$$h_{ij} = e^{\pi i r_{ij}}. \tag{4}$$

the *root of the braiding matrix*.

Definition 5.15. A realisation of $\mathfrak{q}$ is a triple $(\Gamma, (K_i)_{1 \leq i \leq N}, (\chi_i)_{1 \leq i \leq N})$ comprising an abelian group Γ, a collection of elements $(K_i)_{1 \leq i \leq N} \in \Gamma$ and a collection $(\chi_i)_{1 \leq i \leq N} \in \widehat{\Gamma}$ of characters of Γ such that

$$q_{ij} = \chi_j(K_i), \qquad \text{for } 1 \leq i, j \leq N. \tag{5}$$

Remark. Any realisation $(\Gamma, (K_i)_{1 \leq i \leq N}, (\chi_i)_{1 \leq i \leq N})$ of $\mathfrak{q}$ gives V the structure of a left-left Yetter-Drinfeld module over Γ by letting

$$V = \oplus_{1 \leq i \leq N} k x_i,$$

where kx_i denotes the one-dimensional Γ-Yetter-Drinfeld module whose action is given by χ_i and whose coaction is given by K_i.

In the following we want to work with a particularly nice realisation which allows us to use Cartan matrices to describe the representation theory of the given group.

Definition 5.16. The *canonical representation theoretic* realisation of $\mathfrak{q}$ is the triple $(\mathbb{Z}^N, (K_i)_{1\le i\le N}, (\chi_i)_{1\le i\le N})$, where the K_i for the standard basis of $\mathbb{Z}^N$ and the χ_i are uniquely determined by

$$h_{ij} = \chi_j(K_i), \qquad \text{for } 1 \le i, j \le N. \tag{6}$$

Before we continue we wish to make our lives easier and impose further conditions on the matrix of the braiding $\mathfrak{q}$.

(1) We assume that $\mathfrak{q}$ is in block-diagonal form. Where each block is indecomposable (To be defined in analogy to the definition for Cartan algebras.)
(2) The matrix $\mathfrak{q}$ is symmetric. By [AS02, Prop. 3.9] we might restrict ourselves to this setting.(Every matrix of a braiding is symmetric up to twist-equivalence.)
(3) We assume that $\mathfrak{q}$ is of Cartan type. That is there exists a generalised Cartan matrix $A = (a_{ij})_{1\le i,j\le N}$ such that

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}$$

(4) The diagonal entries of $\mathfrak{q}$ are different from 1.

Note that by no means the GCM A can be assumed to be unique. Rather we fix a choice of A by the conditions that for $i \neq j$

$$-N_i \le a_{ij} \le 0 \text{ where } N_i = \operatorname{ord}(q_{ii}) \tag{7}$$

Remark. Let us stress that under these assumptions we have the following identity:

$$q_{ij} = \pm h_{ii}^{a_{ij}} \tag{8}$$

It is not clear whether we can choose the h_{ii} in such a fashion that all signs are positive. However every matrix of a braiding is twist-equivalent to one where all signs can be chosen positive. Thus we will restrict ourselves to this case.

From [AA17, Theorem 2.16] we know that the generalised Cartan matrix A is of finite type if and only if the Nichols algebra $\mathcal{B}(V)$ of V is finite-dimensional.

We want to restrict ourselves to this setting and assume A to be a proper Cartan matrix in the following [1].

Before we enter the representation-theoretic realm of this work we need to introduce the Drinfeld double of a Nichols algebra as defined in [ARS10].

5.3. A version of the Drinfeld double and Verma modules.

Definition 5.17. [ARS10, Definition 3.1] A *reduced YD datum* of size N is a quadruple $D := (\Gamma, (K_i)_{1\le i\le N}, (L_i)_{1\le i\le N}, (\xi_i)_{1\le i\le N})$ comprising a group Γ, two collections of elements $K_i, L_i \in \Gamma$ and a collection of characteres $\xi_i \in \widehat{\Gamma}$ such that

$$K_iL_i \neq 1, \qquad \xi_j(K_i) = \xi_i(L_j) \tag{9}$$

We call D symmetric if $\xi_i(K_j) = \xi_j(K_i)$ for all $1 \le i, j \le N$.

Note that we ignore the *linking parameter*.

[1] Probably its fine if the Cartan matrix is affine (i.e. arithmetic root system in the language of Nichols algebras). What we really want and need is not a inner product but the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$, which also exists for the affine type, see[Kum02]

Remark. Let $D = (\Gamma, (K_i)_{1\leq i\leq N}, (L_i)_{1\leq i\leq N}, (\xi_i)_{1\leq i\leq N})$ be a reduced YD datum. We obtain two Γ Yetter-Drinfeld modules by

$$V := \oplus_{1\leq i\leq N} kx_i, \qquad kx_i = V_{K_i}^{\xi_i} \tag{10}$$

$$W := \oplus_{1\leq i\leq N} ky_i, \qquad ky_i = V_{L_i}^{\xi_i^{-1}} \tag{11}$$

Definition 5.18. [ARS10, Definition 3.3] The Drinfeld $D(V)$ double associated to a reduced YD datum $D = (\Gamma, (K_i)_{1\leq i\leq N}, (L_i)_{1\leq i\leq N}, (\xi_i)_{1\leq i\leq N})$ is the quotient of $T(V \oplus W)\#k\Gamma$ by the two-sided ideal coideal spanned by

(1) the relations of the Nichols algebra of V,
(2) the relations of the Nichols algebra of W,
(3) $x_i y_j - \xi_j^{-1}(K_i) y_j x_i = \delta_{i=j}(K_i L_i - 1)$.

Lemma 5.19. *Given our pair $(V, \mathfrak{q})$ considered in the beginning we can build a symmetric YD datum by letting $\Gamma := \mathbb{Z}^N$ be the free abelian group of dimension N, $K_1, \ldots K_N$ the standard basis of $\mathbb{Z}^N$, $L_i = K_i$ for $1 \leq i \leq N$ and $\chi_i = \xi_i^2$ where $\xi_j(K_i) = h_{ij} = h_{ji} = \xi_i(L_j)$.*

Convention. From now onwards we write $U := D(V)$ for the Drinfeld double associated to this particular YD datum.

Theorem 5.20. *The Drinfeld double U has a triangular decomposition. That is, the multiplication map*

$$\mathcal{B}(W) \otimes k\Gamma \otimes \mathcal{B}(V) \to D(W), w \otimes g \otimes v \mapsto wgv$$

is bijective.

We write $U_0 := k\Gamma$, $U^+ = \mathcal{B}(V)$, $U^- := \mathcal{B}(W)$ and $U_{\geq 0} = \mathcal{B}(V)\#k\Gamma$ to stress the similarities between the Drinfeld double and the universal enveloping algebras.

We will need the following relation:

Theorem 5.21. *[ARS10, 3.2] Let D be our fixed YD datum with underlying group Γ, V, W the Yetter-Drinfeld modules associated to D with the bases $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ as before. Write*

$$E_i = x_i, \ldots F_i = y_i L_i^{-1}.$$

Then U^+ is generated by the E_i's, U^- is generated by the F_i's and

$$gF_i = \xi_i^{-2}(g) F_i g \text{ for all } g \in \Gamma \tag{12}$$

$$E_i F_j - F_j E_i = \delta_{i=j}(K_i - L_i^{-1}) \tag{13}$$

Lemma 5.22. *[Jos95, 4.3] For $n > 0$ we have*

$$E_i F_i^n - F_i^n E_i = [n]_{h_{ii}} F_i^{n-1} \left(h_{ii}^{-(n-1)} K_i - h_{ii}^{n-1} L_i^{-1} \right), \tag{14}$$

where $[n]_h := \frac{h^n - h^{-n}}{h - h^{-1}}$.

Note that for each $1 \leq i \leq N$ we have $\mathrm{ord} h_{ii} \geq 2$ thus $[n]_{h_{ii}}$ is well-defined.

Proof. First let us recall that a q-natural is defined as

$$(n)_q := q^0 + q^1 + \ldots + q^{n-1} = \frac{q^n - 1}{q - 1} \text{ for } n > 0$$

Now using a telescope sum argument and the relation (13) we see that

$$\begin{aligned} E_iF_i^n - F_i^nE_i &= \sum_{s=0}^{n-1} F_i^s(E_iF_i - F_iE_i)F_i^{n-1-s} = \sum_{s=0}^{n-1} F_i^s(K_i - L_i^{-1})F_i^{n-1-s} \\ &= F_i^{n-1}\sum_{s=0}^{n-1} \xi_i^{-2(n-1-s)}(K_i)K_i - \xi_i^{-2(n-1-s)}(L_i^{-1})L_i^{-1} \\ &= F_i^{n-1}\sum_{s=0}^{n-1} h_{ii}^{-2s}K_i - h_{ii}^{2s}L_i^{-1} = F_i^{n-1}\left((n)_{h_{ii}^{-2}}K_i - (n)_{h_{ii}^2}L_i^{-1}\right) \\ &= [n]_{h_{ii}}F_i^{n-1}\left(h_{ii}^{-(n-1)}K_i - h_{ii}^{n-1}L_i^{-1}\right). \end{aligned}$$

In the last line we used that

$$\frac{h_{ii}^n - h_{ii}^{-n}}{h_{ii} - h_{ii}^{-1}}h_{ii}^{-(n-1)} = \frac{h_{ii} - h_{ii}^{-2n+1}}{h_{ii} - h_{ii}^{-1}} = \frac{h_{ii}^{-2n} - 1}{h_{ii}^{-2} - 1}$$

and

$$\frac{h_{ii}^n - h_{ii}^{-n}}{h_{ii} - h_{ii}^{-1}}h_{ii}^{(n-1)} = \frac{h_{ii}^{2n-1} - h_{ii}^{-1}}{h_{ii} - h_{ii}^{-1}} = \frac{h_{ii}^{2n} - 1}{h_{ii}^{2} - 1}.$$

□

5.4. **Representation theory of Nichols algebras by geometric means.**

Definition 5.23. Let $D(V)$ be the Drinfeld double of V and γ a character of Γ. The *Verma module of weight* γ is the induced module

$$M(\gamma) := D(V) \otimes_{U_{\geq 0}} k_\gamma, \tag{15}$$

where the action of $U_{\geq 0}$ on k is given by

$$g \otimes v \triangleright 1 = \epsilon(v)\gamma(g).$$

A vector $0 \neq v \in M(\gamma)$ is called highest weight vector if $x \cdot v = 0$ for every $x \in U^+$.

Here something ‘weird’ happens which feels somewhat analogous to the situation in Galois - theory. Verma modules are generated by essentially unique highest weight vectors. However given a Verma module M we find n isomorphic submodule $M' \subset M$ such that $M/M' \neq \{0\}$. How does this fit with the claim that the generating highest weight vector is unique? Simply put the isomorphism $M' \to M$ does not extend to an isomorphism $M \to M$, i.e. is not contained in the ”Galois group”. This is something we might want to look into.

Lemma 5.24. *Every Verma module $V(\gamma)$ of U is generate by a highest weight vector $\nu(\gamma)$ and this vector is unique up to scalar multiplication.*

Proof. Generation is clear. To see uniqueness suppose that ν and ν' are highest weight vectors of weight γ. Now using the fact that we have a triangular composition we observe that there are elements $wgv, w'g'v' \in U$ with $w, w' \in U^-$, $g, g' \in U_0$ and $v, v' \in U^+$ such that

$$wgv \triangleright \nu = \nu' \text{ and } w'g'v' \triangleright \nu' = \nu.$$

As ν and ν' are highest weight vectors we have $v = v' = 1$ It follows from (12) that there exists a scalar λ_g' such that $\nu = w'g'wg\nu = \lambda g'w'wgg'\nu = \lambda g'\gamma(gg')w'w\nu.$

Since $V(\gamma)$ is free as a U^- module we observe that $w'w = \lambda' 1$ for some scalar λ'. As U^- is a graded, connected algebra w and w' needed to be scalar multiples of the unit. Thus we have a non-zero scalar λ'' such that $\nu = \lambda'' \nu'$. □

Theorem 5.25 (Universal property of Verma modules)**.** *Let M be some module over U. Given a weight γ there exists a non-zero morphism $\psi : V(\gamma) \to M$ if and only if M contains a highest weight vector m of weight γ. Moreover ψ is uniquely determined by mapping the highest weight vector of $\nu(\gamma) \in V(\gamma)$ to m.*

Proof. If such a vector exists its a direct calculation to show that ψ exists and that it is uniquely determined by the image of $\nu(\gamma)$.

Conversely assume that such a morphism of modules $\psi : V(\gamma) \to M$ is given. The image of $m := \psi(\nu(\gamma))$ needs to be a heighest weight vector of some weight γ' or zero. If it were zero ψ would be zero. Thus $0 \neq m$. Now assume $g \in U_0$ to be a group-like element. Then $\gamma'(g)m = g \triangleright m = \psi(g \triangleright \nu(\gamma)) = \gamma(g)m$. Thus $\gamma' = \gamma$. □

Lemma 5.26. *Let $V(\gamma)$ be a Verma module of weight γ and ν a highest weight vector of weight γ. Then $F_i^n \nu$ is a highest weight vector of weight $\xi_i^{-2n}\gamma$ if and only if $\gamma(K_i) = \pm h_{ii}^{(n-1)}$ or $n = 0 \mod ord(h_{ii})/2$.*

Proof. First we compute the weight of $F_i^n \nu$. Let $g \in U_0$ be a group-like element. We have

$$gF_i^n \nu = \xi_i^{-2n}(g) F_i^n g\nu = (\xi_i^{-2n}\gamma)(g) F_i^n \nu.$$

Next we have to check that for every $1 \leq j \leq N$ the identity $E_j F_i^n \nu = 0$ holds. If $i \neq j$ this holds trivially. Otherwise we compute

$$\begin{aligned} E_i F_i^n \nu &= ([n]_{h_{ii}} F_i^{n-1} (h_{ii}^{-(n-1)} K_i - h_{ii}^{n-1} (L_i)^{-1}) + F_i^n E_i)\nu \\ &= [n]_{h_{ii}} F_i^{n-1} (h_{ii}^{-(n-1)} K_i - h_{ii}^{n-1} (L_i)^{-1})\nu \\ &= \left(h_{ii}^{-(n-1)} \gamma(K_i) - h_{ii}^{(n-1)} \gamma(L_i^{-1}) \right) [n]_{h_{ii}} F_i^{n-1} \nu \stackrel{!}{=} 0 \end{aligned}$$

There are two cases which might occur:

(1) $[n]_{h_{ii}} = 0$ This is the case if and only if $h_{ii}^n = h_{ii}^{-n}$ which holds if and only if $n = 0 \mod \mathrm{ord}(h_{ii})/2$.
(2) $\left(h_{ii}^{-(n-1)} \gamma(K_i) - h_{ii}^{(n-1)} \gamma(L_i^{-1}) \right) \stackrel{!}{=} 0$. Therefore

$$\gamma(K_i L_i) = \left(h_{ii}^{(n-1)} \right)^2.$$

In particular if $K_i = L_i$ we have

$$\gamma(K_i) = \pm h_{ii}^{(n-1)}$$

□

Next we define a abstract weight space. The idea is that elements in this space are supposed to govern the (real) powers of our fixed characters ξ_i.

Definition 5.27. The *abstract weight space* of U is $\mathbb{R}^N$. The simple roots correspond to the standard basis $\alpha_1, \dots \alpha_N$. The root system Δ of U is the root system obtained by the Cartan matrix A and the choice of simple roots.

Theorem 5.28. *Given any vector $w \in E$ we define the character*

$$\xi_w : \mathbb{Z}^N \to S_1, \qquad \xi_w(K_i) = h_{ii}^{\langle w, \alpha_i \rangle}. \tag{16}$$

Conversely given any character ξ there exists at least one vector $w' \in E$ such that $\xi_{w'} = \xi$.

Proof. The first part of the theorem is shown by a direct calculation. To proof the second part observe that ξ is uniquely determined by its values on the K_i. We thus have to construct the vector w'. Our approach resembles Grahm-Schmidt to some degree.

We begin by choosing w_1 such that $\xi(K_1) = h_{11}^{\langle w_1, \alpha_1 \rangle}$. Next let E' be the orthogonal complement of $\mathbb{R}w_1$. Since the α_i form a basis of E and the bilinear form is non-degenerate there needs to be a vector w_2 in E' such that $\xi(K_2) = h_{22}^{\langle w_1 + w_2, \alpha_2 \rangle}$. Set $w_2' := w_1 + w_2$. Continue this process until the desired vector w' is obtained. □

Lemma 5.29. *Let $w, w' \in E$ be weight vectors. Ten $\xi_w = \xi_{w'}$ if and only if for all $1 \leq i \leq N$ the identity*

$$\langle w - w', \alpha_i \rangle \in \mathit{ord}(h_{ii})\mathbb{Z} \tag{17}$$

holds.

Proof. Suppose the identity holds and fix any $1 \leq i \leq N$. Then there exists an $m \in \mathbb{Z}$ such that $\langle w - w', \alpha_i \rangle = m \operatorname{ord}(h_{ii})\alpha_i$ and

$$\xi_w(K_i) = h_{ii}^{\langle w, \alpha_i \rangle} = h_{ii}^{\langle w - w' + w', \alpha_i \rangle} = h_{ii}^{m \operatorname{ord}(h_{ii}) + \langle w', \alpha_i \rangle} = h_{ii}^{\langle w', \alpha_i \rangle} = \xi_{w'}(K_i).$$

Assume conversely that $\xi_w = \xi_{w'}$. In this case we have for any $1 \leq i \leq N$ that

$$\xi_w(K_i)\xi_{w'}(K_i)^{-1} = h_{ii}^{\langle w, \alpha_i \rangle - \langle w', \alpha_i \rangle} = h_{ii}^{\langle w - w', \alpha_i \rangle} = 1$$

This is equivalent to $\langle w - w', \alpha_i \rangle \in \operatorname{ord}(h_{ii})\mathbb{Z}$. □

Remark. The numbers h_{ii} are not transcendental. Thus the vectors w' are not unique. In this setting we have a projection from the (abstract) weight space E onto the simple representations of $\mathbb{Z}^N$. It might be the case that, by considering $\mathbb{Z}^N$-graded modules, we obtain a bijection.

Lemma 5.30. *Let M be a U module and $m \in M$ a weight vector whose abstract weight is represented by a vector $w \in E$. Then $F_i^n m$ has the weight $w - n\alpha_i$.*

Proof. Let ξ_w be the character assigned to w. We compute

$$K_j F_i^n m = \xi_i^{-2n}(K_j) F_i^n K_j m = \xi_j^{-2n}(K_i)\xi_w(K_j) F_i^n m = h_{jj}^{-n a_{ji}} h_{jj}^{\langle w, \alpha_j \rangle} F_i^n m.$$

Conversely we have

$$\xi_{w - n\alpha_i}(K_j) = h_{jj}^{\langle w - n\alpha_i, \alpha_j \rangle} = h_{jj}^{\langle w, \alpha_j \rangle - n\langle \alpha_i, \alpha_j \rangle} = h_{jj}^{-n a_{ji}} h_{jj}^{\langle w, \alpha_j \rangle}$$

□

Definition 5.31. The affine action of W on the abstract weight space $\mathbb{R}^N$ is defined by

$$w \cdot \lambda = w(\lambda + \delta) - \delta$$

where $\delta := \omega_1 + \ldots + \omega_n$ is the sum of fundamental weights.

Lemma 5.32. *Let $s_i \in W$ be a simple reflection and $\gamma \in \mathbb{R}^N$ any weight. Then*

$$\gamma - s_i \cdot \gamma = n\alpha_i, \text{ with } n := \langle \gamma, \alpha_i \rangle + 1$$

where α_i is the simple root corresponding to the reflection s_i. If γ is integral either $\gamma \leq s_i \cdot \gamma$ or $s_i \cdot \gamma \leq \gamma$.

Proof. We compute

$$\begin{aligned}\gamma - s_i \cdot \gamma &= \gamma - (s_i(\gamma + \delta) - \delta) = \gamma + \delta - s_i(\gamma + \delta)\\ &= \gamma + \delta - (\gamma + \delta) + \alpha_i \langle \gamma + \delta, \alpha_i \rangle = \alpha_i \langle \gamma + \delta, \alpha_i \rangle\\ &= \alpha_i \left(\langle \gamma, \alpha_i \rangle + \langle \delta, \alpha_i \rangle \right) = \alpha_i \left(\langle \gamma, \alpha_i \rangle + 1 \right) = n\alpha_i.\end{aligned}$$

In the last step we used that δ is the sum of fundamental weights. □

Theorem 5.33 ([Jos95, 4.4.7])**.** *Let $\gamma \in \mathbb{R}^N$ be a positive integral weight and $w, v \in W$. Then there exists a (necessarily injective) morphism between Verma modules*

$$V(v \cdot \gamma) \to V(w \cdot \gamma)$$

if (and only if ?) $v \geq w$.

Proof. The proof works by an induction over the length $l(v)$ of v. If $l(v) = 0$ we have $w = v$ and nothing needs to be shown. Assume $l(v) \geq 1$. There exists a simple root α such that $s_\alpha v < v$. ASSUMPTION: BRUHAT YOGA IMPLIES THAT $v \cdot \gamma < s_\alpha v \cdot \gamma$.

Define $n - 1 := \langle v \cdot \lambda, \alpha \rangle$ and write ν for a highest weight vector generating $V(s_\alpha v \cdot \gamma)$. By Lemma 5.32 we have $v \cdot \gamma - s_\alpha v \cdot \gamma = n\alpha_i$ As $v \cdot \gamma < s_\alpha v \cdot \gamma$ we have $n < 0$. Using the fact that U^- is a domain we set $0 \neq \tilde{\nu} := F_i^{-n} \nu$. As a consequence of Lemma 5.30 the weight of $\tilde{\nu}$ is $s_\alpha v \cdot \gamma + n\alpha_i = v \cdot \gamma$. Moreover it follows from Lemma 5.26 that $\tilde{\nu}$ is a highest weight vector. The universal property of Verma modules now implies that the required injective morphism

$$V(v \cdot \gamma) \to V(s_\alpha v \cdot \gamma)$$

exists. TODO:MORE BRUHAT YOGA. Either $s_\alpha v > w$ or $s_\alpha v \geq s_\alpha w$ and $s_\alpha w \geq w$. If the first case holds we're done by the induction hypothesis. Thus let $s_\alpha v \geq s_\alpha w$ be true. Again the induction hypothesis grants us the existence of a morphism $\psi : M(s_\alpha v \cdot \gamma) \to M(s_\alpha w \cdot \gamma)$ and a injective morphism $\iota : M(w \cdot \gamma) \to M(s_\alpha w \cdot \gamma)$. We identify $M(s_\alpha w \cdot \gamma)$ with its image in $M(s_\alpha w \cdot \gamma)$ under ι. Let $M := M(s_\alpha w \cdot \gamma)/M(w \cdot \gamma)$. We write $\mu := \psi(\tilde{\nu})$ and $[\mu]$ for its equivalence class in M. We claim that M is locally finite under the action F_i, see [Jos95, 4.3.5] [2]. Therefore it somehow follows (?) that an $r \in \mathbb{N}$ exists such that $[F_i^r \mu] = 0$. Now different cases might occur.

(1) $r = 0 \mod \mathrm{ord}(h_{ii})/2$ or
(2) $r \neq 0 \mod \mathrm{ord}(h_{ii})/2$.

Observe that by Lemma 5.30 and 5.29 we have that $\mu' := F_i^{m\mathrm{ord}(h_{ii})} \mu$ has the same weight as μ. Moreover by Lemma 5.26 it is a highest weight vector. Thus choose $m \in \mathbb{N}$ minimal such that $[F_i^{m\mathrm{ord}(h_{ii})} \mu] = 0$. □

[2]This means that for every $[X] \in M$ there is a non-negative integer r such that $[F_i^r X] = 0$.

Remark. Some questions: Can we find the "biggest" copy of $M(v\cdot\gamma)$ in $M(w\cdot\gamma)$? How do vectors of weight zero enter the picture/ what is their role?

Does the above theorem hold in the graded setting. This means that we need to be able to show that r can be choosen smaller than ord $h_{ii}/2$.

Example 5.34. We show with a minimal counterexample that the above claims do not work in the case of Nichols algebras. Fix the matrix $\mathfrak{q} := \begin{bmatrix} -1 & i \\ i & -1 \end{bmatrix}$ Its generalised Dynkin-diagram is

$$-1 \overset{-1}{\text{———}} -1 \tag{18}$$

The associated Nichols algebra has two generators F_1, F_2 and the PBW-basis $\{F_2^{j_2}F_{12}^{j_{12}}F_1^{j_1} \mid 0 \le j_1, j_{12}, j_2 \le 2\}$. Accordingly the set of roots is $\{\alpha_1, \alpha 1 + \alpha 2, \alpha 2\}$ and the fundamental weight is $\alpha_1+\alpha_2$. Its Weyl group is $\{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2\}$. Fix the weight 0 and the elements of the Weyl group $v := s_1s_2 > s_1 =: w$. We compute

$$w\cdot 0 = -\alpha_1, \qquad s_2\cdot 0 = -\alpha_2, \qquad v\cdot 0 = s_1\cdot(-\alpha_2) = -2\alpha_1 - \alpha_2. \tag{19}$$

Let us write x_w, x_v and x_{s_2} for the generating highest weight vectors of the Verma modules $V(w\cdot 0)$, $V(v\cdot 0)$ and $V(s_2\cdot 0)$ respectively. We have $s_1v = s_2 < v$ and $v\cdot 0 - s_2\cdot 0 = -2\alpha_1$. Now we claim that $F_1x_{s_2}$ is the highest weight vector implementing $V(v\cdot 0)\to V(s_1\cdot 0)$. Note that the above map is not injective!

Now we want to find a map going from $V(s_1\cdot 0)$ to $V(s_1w\cdot 0) = V(0)$. The highest weight vector is F_1x_0

(1) Rough strategy: Starting with an integrable weight λ we obtain a sequence of inclusions of Verma modules which is esentially unique up to scalar multiplication.
(2) This would allow us to build a BGG-styled complex.

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