STABILITY OF CONCORDANCE EMBEDDINGS

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Abstract. We supply a proof of a stability theorem for spaces of smooth concordance embeddings, based on disjunction results of Goodwillie. As a first application, we combine this stability result with work of Krannich and Randal-Williams to determine the optimal rational concordance stable range for all high-dimensional 1-connected spin manifolds.

Contents

Let $P \subset M$ be a compact submanifold of a smooth d-dimensional manifold M such that P meets ∂M transversally. Writing $I := [0,1]$, the space $\mathrm{CE}(P,M)$ of *concordance embeddings* of P into M is the space of smooth embeddings $e\colon P \times I \hookrightarrow M \times I$ that

(i) satisfy $e^{-1}(M \times \{i\}) = P \times \{i\}$ for $i = 0,1$ and
(ii) agree with the inclusion on a neighbourhood of $P \times \{0\} \cup (\partial M \cap P) \times I \subset M \times I$,

equipped with the smooth topology. There is a stabilisation map

$$\mathrm{CE}(P,M) \longrightarrow \mathrm{CE}(P \times J, M \times J)$$

given by taking products with $J := [-1,1]$ followed by bending and shrinking the result appropriately to make it satisfy the boundary condition (compare Figure 1 and see Section 1.2 for a precise definition). The purpose of this work is to use disjunction results of Goodwillie [Goo90] to establish a connectivity estimate for this map under the assumption that the handle dimension p of the inclusion $\partial M \cap P \subset P$ satisfies $p \le d-3$, that is, P can be built from a closed collar on $\partial M \cap P$ by attaching handles of index at most $d-3$.

Theorem A. *If $p \le d-3$, then the stabilisation map*

$$\mathrm{CE}(P,M) \longrightarrow \mathrm{CE}(P \times J, M \times J)$$

is $(2d-p-5)$-connected.

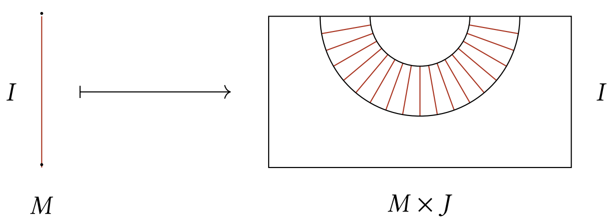

Figure 1. The stabilisation map.

Historical remark. This result might not come as a surprise to experts in the field. It first was announced by Goodwillie in his Ph.D. thesis [Goo82, p. 13], and then stated again in the thesis of Meng [Men93, Theorem 0.0.1] which contains a proof in the case $P = *$ (see Theorem 3.4.1 loc.cit.). The general case is referred to at various places in the literature such as in [Igu88, p. 6] or [WW01, p. 210], but a proof has never appeared.

Concordance diffeomorphisms. Our interest in spaces of concordance embeddings stems mainly from the fact that they provide information about maps between spaces of *concordance diffeomorphisms* of smooth d-manifolds M

$$\mathrm{C}(M) \coloneqq \big\{\phi\colon M\times I \xrightarrow{\cong} M\times I \mid \phi = \mathrm{id} \text{ in a neighbourhood of } M\times\{0\}\cup \partial M\times I\big\},$$

equipped with the smooth topology. More precisely, for a submanifold $P\subset M$ as above, a variant of the parametrised isotopy extension theorem yields a fibre sequence

$$\mathrm{C}(M\backslash \nu(P)) \longrightarrow \mathrm{C}(M) \longrightarrow \mathrm{CE}(P,M) \tag{1}$$

where $\nu(P)\subset M$ is an open tubular neighbourhood of P. If M is compact then $\mathrm{C}(M) = \mathrm{CE}(M,M)$, so the stabilisation map for concordance embeddings specialises to a stabilisation map $\mathrm{C}(M)\to\mathrm{C}(M\times J)$ for concordance diffeomorphisms. Building on ideas of Hatcher [Hat75], Igusa [Igu88, p. 6] proved that this map is approximately $\frac{d}{3}$-connected, so if the relative handle dimension of the inclusion $\partial M\cap P\subset P$ is at most $d-3$, then Theorem A shows that the stability range for the base space of the fibre sequence (1) is significantly better than the stability ranges for the total space and fibre resulting from Igusa's work. This can be used to transfer potential improvements of the stability range for concordance diffeomorphisms of specific manifolds to other manifolds. To exemplify this principle, we derive the following from Theorem A.

Corollary B. *For a compact* 1*-connected spin manifold* M *of dimension* $d\geq 6$ *and* $D^d\subset \mathrm{int}(M)$ *an embedded disc, the map induced by extension by the identity*

$$\pi_k\big(\mathrm{C}(D^d\times J),\mathrm{C}(D^d)\big)\longrightarrow \pi_k\big(\mathrm{C}(M\times J),\mathrm{C}(M)\big)$$

is an isomorphism for $k<d-2$*. The same conclusion holds for* $d=5$ *as long as* $\partial M=\varnothing$*.*

Remark. The range in Corollary B can be improved under additional assumptions on the manifold M (see Proposition 3.1 and Remark 3.2).

Rationally, the relative homotopy groups $\pi_k(C(D^d\times J),\mathrm{C}(D^d))$ of the stabilisation map for discs have been computed by Krannich and Randal-Williams [KRW21, Corollary B] in degrees up to approximately $\frac{3}{2}d$. Combined with Corollary B, this gives:

Corollary C. *For a compact* 1*-connected spin* d*-manifold* M *with* $d\geq 6$*, there is a morphism*

$$\pi_k\big(\mathrm{C}(M\times J),\mathrm{C}(M)\big)\otimes\mathbf{Q}\longrightarrow\begin{cases}\mathbf{Q} & \text{if } k=d-3,\\ 0 & \text{otherwise,}\end{cases}$$

which is an isomorphism in degrees $k<\min(d-2,\lfloor\frac{3}{2}d\rfloor-8)$ *and an epimorphism in degrees* $k<\min(d-2,\lfloor\frac{3}{2}d\rfloor-7)$*.*

Remark. The terms $\pi_k(\mathrm{C}(M\times J),\mathrm{C}(M))$ are abelian groups for $d>5$ (see [HW73, Lemma 1.1]), so the rationalisation in Corollary C is unambiguous.

In particular, denoting by

$$\phi(M)_{\mathbf{Q}}\coloneqq\min\big\{k\in\mathbf{Z}\;\big|\;\pi_i\big(\mathrm{C}(M\times J^{m+1}),\mathrm{C}(M\times J^m)\big)\otimes\mathbf{Q}=0 \text{ for } i\leq k \text{ and } m\geq 0\big\}$$

the *rational concordance stable range* of M (which is the main limiting factor in the classical approach to the rational homotopy type of $\mathrm{Diff}(M)$ through surgery and pseudoisotopy theory, see e.g. [WW01]), Corollary C implies that $\phi(M^d)_{\mathbf{Q}}=d-4$ for all 1-connected spin manifolds M of dimension $d>9$. For such manifolds, this confirms speculations of Hatcher [Hat78, p. 4] and Igusa [Igu88, p. 6] rationally and improves the ranges of various results in the literature that rely on the rational concordance stable range.

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1. The multi-relative stability theorem

Theorem A is proved as the case $r = 0$ of a more general multi-relative stability theorem that concerns $(r+1)$-cubes of spaces of concordance embeddings; see Theorem 1.6. We begin with a quick recollection on such cubical diagrams.

1.1. **Cubical diagrams.** An *r-cube* for $r \geq 0$ is a space-valued functor X on the poset category $\mathcal{P}(S)$ of (possibly empty) subsets of a finite set S of cardinality r, ordered by inclusion. To emphasise the particular finite set S, we sometimes also call X an *S-cube*. A 0-cube is simply a space, a 1-cube a map between two spaces, a 2-cube is a commutative square of spaces, and so on. The unique inclusion of $\varnothing$ into any subset of S induces a map

$$X(\varnothing) \longrightarrow \operatorname*{holim}_{\varnothing \neq T \subseteq S} X(T)$$

and X is called *k-cartesian* for an integer k if this map is k-connected. For instance, a 0-cube X is k-cartesian if the space $X(\varnothing)$ is $(k-1)$-connected and a 1-cube is k-cartesian if the map $X(\varnothing) \to X(S)$ is k-connected. A *map of S-cubes* $X \to Y$ is a natural transformation. Such a map can also be considered as a $(S \sqcup \{*\})$-cube via

$$\mathcal{P}(S \sqcup \{*\}) \ni T \longmapsto \begin{cases} X(T) & \text{if } * \notin T, \\ Y(T \backslash \{*\}) & \text{otherwise.} \end{cases}$$

Conversely, an S-cube X determines a map of $S\backslash\{*\}$-cubes for each $* \in S$, and the induced S-cube of each of these is isomorphic to X. A choice of basepoint $* \in X(\varnothing)$ induces compatible basepoints in $X(T)$ for all $T \in \mathcal{P}(S)$. Given a map of r-cubes $X \to Y$ and $* \in Y(\varnothing)$, we obtain an r-cube $\operatorname{hofib}_*(X \to Y)$ by taking homotopy fibres at the induced basepoints. Many facts on the connectivity behaviour of maps of spaces generalise to r-cubes, such as the following [Goo92, Propositions 1.6, 1.8, 1.18].

Lemma 1.1. *For a map $X \to Y$ of r-cubes, considered as an $(r+1)$-cube, we have:*

(i) If Y and $X \to Y$ are k-cartesian, then X is k-cartesian.
(ii) If X is k-cartesian and Y is $(k+1)$-cartesian, then $X \to Y$ is k-cartesian.
(iii) $X \to Y$ is k-cartesian if and only if $\operatorname{hofib}_*(X \to Y)$ is k-cartesian for all $* \in Y(\varnothing)$.

Given a further map $Y \to Z$ of r-cubes, considered as an $(r+1)$-cube, we have:

(iv) If $X \to Y$ and $Y \to Z$ are k-cartesian, then $X \to Z$ is k-cartesian.
(v) If $X \to Z$ is k-cartesian and $Y \to Z$ is $(k+1)$-cartesian, then $X \to Y$ is k-cartesian.

The later sections will also feature cubical diagrams whose values are themselves cubes. These are obtained from space-valued functors X on $\mathcal{P}(S \sqcup S')$ for finite sets S and S' by considering for each subset $U \subseteq S'$ the S-cube X_U given by $\mathcal{P}(S) \ni T \mapsto X(T \sqcup U)$. Varying U the X_U assemble to a functor from $\mathcal{P}(S')$ to the category of S-cubes.

Lemma 1.2. *Let S and S' be non-empty finite sets, and X be an $(S \sqcup S')$-cube.*

(i) If X is k-cartesian and the S-cube X_U is $(k+|U|-1)$-cartesian for all $\varnothing \neq U \subseteq S'$, then the S-cube $X_\varnothing$ is k-cartesian.
(ii) If the S-cube X_U is ∞-cartesian for all $U \subseteq S'$, then X is ∞-cartesian.

Proof. Part (i) is [Goo92, Proposition 1.20]. For part (ii) we use the following commutative square induced by the functoriality of X

$$\begin{array}{ccc} \operatorname{holim}_{\varnothing\neq T\sqcup T'\subseteq S\sqcup S'} X(T\sqcup T') & \longrightarrow & \operatorname{holim}_{T\subseteq S,\varnothing\neq T'\subseteq S'} X(T\sqcup T') \\ \downarrow & & \downarrow \\ \operatorname{holim}_{\varnothing\neq T\subseteq S, T'\subseteq S'} X(T\sqcup T') & \longrightarrow & \operatorname{holim}_{\varnothing\neq T\subseteq S,\varnothing\neq T'\subseteq S'} X(T\sqcup T') \end{array}$$

which is homotopy cartesian (see e.g. [Goo92, Proposition 0.2]). By the Fubini formula for homotopy limits and the fact that $\varnothing \in \mathcal{P}(S)$ is an initial object, we may replace the upper right corner up to equivalence with $\operatorname{holim}_{\varnothing\neq T'\subseteq S'} X(T')$ and the right vertical map with the map $\operatorname{holim}_{\varnothing\neq T'\subseteq S'} X(T') \to \operatorname{holim}_{\varnothing\neq T'\subseteq S'}\big(\operatorname{holim}_{\varnothing\neq T\subseteq S} X(T\sqcup T')\big)$ induced by the inclusions $T' \subset T\sqcup T'$. This is an equivalence because it is the homotopy limit of the maps $X_{T'}(\varnothing) \to \operatorname{holim}_{\varnothing\neq T\subseteq S} X_{T'}(T)$ that are equivalences by assumption. By cartesianness of the square the left vertical map is an equivalence as well. Using that the bottom left corner agrees with $\lim_{\varnothing\neq T\subseteq S} X(T)$ since $\varnothing \in \mathcal{P}(S')$ is an initial object, it follows that the map $X(\varnothing) \to \operatorname{holim}_{\varnothing\neq T\sqcup T'\subseteq S\sqcup S'} X(T\sqcup T')$ agrees up postcomposition with an equivalence with the map $X(\varnothing) \to \operatorname{holim}_{\varnothing\neq T\subseteq S} X(T)$. The assumption that $X_\varnothing$ is ∞-cartesian says that the latter map is an equivalence, so X is ∞-cartesian as claimed. □

Corollary 1.3. *Let S be a non-empty finite set.*
(i) The constant S-cube with value a fixed space X is ∞-cartesian as long as $|S| \geq 1$.
(ii) Given spaces X_s for $s \in S$, the S-cube

$$\mathcal{P}(S) \ni T \mapsto \prod_{s\in S\backslash T} X_s$$

defined by the canonical projections is ∞-cartesian as long as $|S| \geq 2$.

Proof. If $X = *$, then part (i) follows from the fact that the nerve of the category $\mathcal{P}(S\backslash\{*\})\backslash\varnothing$ is contractible since it has a terminal object. For a general space X, we fix $* \in S$ and view the constant S-cube on X as a map of $(S\backslash\{*\})$-cubes. By Lemma 1.1 Item (iii) it suffices to show that the $(S\backslash\{*\})$-cubes obtained by taking homotopy fibres are ∞-cartesian. These homotopy fibres are trivial by assumption, so ∞-cartesian by the initial case.

To prove part (ii), we pick $* \in S$ and view the S-cube in question as the map of $(S\backslash\{*\})$-cubes $\prod_{s\in S\backslash\bullet} X_s \to \prod_{s\in S\backslash(\bullet\cup\{*\})} X_s$. By Lemma 1.1 (iii), it suffices to show that the cubes of homotopy fibres at all basepoints are ∞-cartesian. These are constant cubes, so the claim follows from the first part. □

Another general principle regarding cubical diagrams we will make use of is the multi-relative form of the Blakers–Massey theorem. To state it, recall that an S-cube X is *strongly cocartesian* if for every subset $T \subset S$ and distinct elements $s_1 \neq s_2 \in S\backslash T$, the square

$$\begin{array}{ccc} X(T) & \longrightarrow & X(T\cup\{s_1\}) \\ \downarrow & & \downarrow \\ X(T\cup\{s_2\}) & \longrightarrow & X(T\cup\{s_1,s_2\}) \end{array}$$

is a homotopy pushout. The multi-relative Blakers–Massey theorem gives the following bound on the cartesianness of strongly cocartesian cubes (see [Goo92, Theorem 2.3]).

Theorem 1.4. *Let S be a non-empty finite set. If X is a strongly cocartesian S-cube such that the map $X(\varnothing) \to X(\{s\})$ is k_s-connected for all $s \in S$, then X is $(1-|S|+\sum_{s\in S} k_s)$-cartesian.*

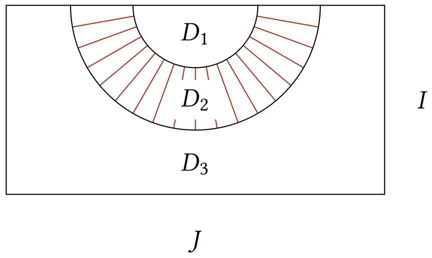

FIGURE 2. The decomposition $J \times I = D_1 \cup D_2 \cup D_3$. The red intervals indicate the parametrisation of D_2: the semi-circle is parametrised by $\theta \in [0, \pi]$ starting with $\theta = 0$ on the left, and the intervals are parametrised by $r \in [0, 1]$ starting with $r = 0$ at the outer semicircle.

1.2. **The stabilisation map.** We continue by spelling out a precise definition of the stabilisation map for spaces of concordance embeddings. As in the introduction, this involves a smooth manifold M and a compact submanifold $P \subset M$ such that P and ∂M are transverse. We first pass to the equivalent subspace

$$\mathrm{CE}'(P, N) \subset \mathrm{CE}(P, N)$$

of those concordance embeddings e that satisfy the property that $e(p, t) = ((\mathrm{pr}_1 \circ e)(p, 1), t)$ on a neighbourhood of $P \times \{1\}$. Writing $I \coloneqq [0, 1]$ and $J \coloneqq [-1, 1]$, we decompose the rectangle $J \times I$ into the three codimension zero submanifolds with corners

$$\begin{aligned} D_1 &\coloneqq \{(x, y) \in J \times I \mid x^2 + (y-1)^2 \leq \tfrac{1}{3}\} \\ D_2 &\coloneqq \{(x, y) \in J \times I \mid \tfrac{1}{3} \leq x^2 + (y-1)^2 \leq \tfrac{2}{3}\} \\ D_3 &\coloneqq \{(x, y) \in J \times I \mid \tfrac{2}{3} \leq x^2 + (y-1)^2\}; \end{aligned}$$

see Figure 2. The second of these can be parametrised by polar coordinates via

$$\begin{aligned} \Lambda\colon [0, 1] \times [0, \pi] &\longrightarrow D_2 \\ (r, \theta) &\longmapsto (\tfrac{1}{3}(2-r)\cos(\theta + \pi), \tfrac{1}{3}(2-r)\sin(\theta+\pi) + 1). \end{aligned} \tag{2}$$

Writing

$$e_M \coloneqq (\mathrm{pr}_1 \circ e)\colon P \times I \to M \quad \text{and} \quad e_I \coloneqq (\mathrm{pr}_2 \circ e)\colon P \times I \to I$$

for a map $e\colon P \times I \to M \times I$ (such as a concordance embedding), the stabilisation map

$$\sigma\colon \mathrm{CE}'(P, M) \longrightarrow \mathrm{CE}(P \times J, M \times J)$$

is defined by sending $e \in \mathrm{CE}'(P, M)$ to the concordance embedding

$$\begin{aligned} \sigma(e)\colon P \times J \times I &\longrightarrow M \times J \times I \\ (p, s, t) &\longmapsto \begin{cases} (e_M(p, 1), s, t) & \text{if } (s, t) \in D_1, \\ (e_M(p, r), \Lambda(e_I(p, r), \theta)) & \text{if } (s, t) = \Lambda(r, \theta) \in D_2, \\ (p, s, t) & \text{if } (s, t) \in D_3. \end{cases} \end{aligned}$$

The purpose of restricting to $\mathrm{CE}'(P, N)$ is to ensure that the map $\sigma(e)$ is smooth.

Convention 1.5. We do not distinguish between $\mathrm{CE}(P, M)$ and its homotopy equivalent subspace $\mathrm{CE}'(P, M) \subset \mathrm{CE}(P, M)$ in what follows. In particular, we write $\mathrm{CE}(P, M)$ for the domain of the stabilisation map, even though it should strictly speaking be $\mathrm{CE}'(P, M)$.

1.3. **Statement of the main theorem.** To state the multi-relative stability theorem we aim for, we fix a smooth d-manifold M and compact disjoint submanifolds $P, Q_1, \ldots, Q_r \subset M$ for $r \geq 0$, all transverse to ∂M. We abbreviate $\{1, \ldots, r\}$ by $\underline{r}$ and the r-cube

$$\underline{r} \supseteq S \longmapsto \mathrm{CE}(P, M \backslash \cup_{i \notin S} Q_i)$$

by $\mathrm{CE}(P, M \backslash Q_\bullet)$. As the construction of the stabilisation map from Section 1.2 is natural in codimension zero embeddings of the target, it extends to a map of r-cubes

$$\mathrm{CE}(P, M \backslash Q_\bullet) \longrightarrow \mathrm{CE}(P \times J, (M \backslash Q_\bullet) \times J) \tag{3}$$

which we view as an $(r+1)$-cube, denoted by $\mathrm{sCE}(P, M \backslash Q_\bullet)$. The main result of this work concerns this $(r+1)$-cube, and it includes Theorem A as the case $r = 0$.

Theorem 1.6. *If $d - p \geq 3$ and $d - q_i \geq 3$ for all i, then the $(r+1)$-cube*

$$\mathrm{sCE}(P, M \backslash Q_\bullet) = \Big(\mathrm{CE}(P, M \backslash Q_\bullet) \to \mathrm{CE}(P \times J, (M \backslash Q_\bullet) \times J)\Big)$$

is $(2d - p - 5 + \sum_{i=1}^r (d - q_i - 2))$-cartesian. Here p and q_i are the relative handle dimensions of the inclusions $\partial M \cap P \subset P$ and $\partial M \cap Q_i \subset Q_i$, respectively.

In the following subsections, we first discuss a few ingredients for the proof of the theorem (Sections 1.4–1.6) and then give the proof assuming the case $P = *$ (see Section 1.7). The latter is the most laborious part and will be dealt with in the separate Section 2.

Convention 1.7. In this and the following sections, all manifolds are assumed to be smooth and may have boundary. All submanifolds are implicitly assumed to be transverse to the boundary of the surrounding manifold. M always denotes a smooth d-manifold and $P, Q_1, \ldots, Q_r \subset M$ are always disjoint compact submanifolds that are transverse to the boundary. We write $p, q_1, \ldots, q_r \geq 0$ for the relative handle dimensions of the inclusions $\partial M \cap P$ and $\partial M \cap Q_1, \ldots, \partial M \cap Q_r \subset \partial M$. It will be convenient to abbreviate

$$\Sigma \coloneqq \textstyle\sum_{i=1}^r (d - q_i - 2)$$

since this quantity will be ubiquitous in what follows.

1.4. **Collars and disc-bundles.** The following two elementary lemmas are used at several points in the argument.

Lemma 1.8. *Let $\pi\colon D(P) \to P$ be a closed disc-bundle with an embedding $D(P) \subset M$ that extends the inclusion $P \subset M$ and satisfies $D(P) \cap \partial M = \pi^{-1}(P \cap \partial M)$. The map $\mathrm{CE}(D(P), M) \to \mathrm{CE}(P, M)$ induced by restriction along the 0-section is a weak equivalence.*

Proof. It suffices to prove that the homotopy fibre of the map $\mathrm{CE}(D(P), M) \to \mathrm{CE}(P, M)$ over any concordance embedding $e\colon P \times I \hookrightarrow M \times I$ is contractible. As a result of the isotopy extension theorem, this homotopy fibre is equivalent to the space of concordance embeddings $D(P) \times I \hookrightarrow M \times I$ extending e. Taking derivatives at the 0-section gives a homotopy equivalence from this space to the space of maps $P \times I \to V_n(\mathbf{R}^{d-p})$ that agree with the identity on $P \times \{0\} \cup (\partial M \cap P) \times I$; here n is the dimension of the fibre of π and $V_n(\mathbf{R}^{d-p})$ is the space of n-frames in $\mathbf{R}^{d-p}$. As the inclusion $P \times \{0\} \cup (\partial M \cap P) \times I \subset P \times I$ is a cofibration as well as a homotopy equivalence, this mapping space is contractible, so the claim follows. □

Lemma 1.9. *If P is a closed collar on $P \cap \partial M \subset P$, then $\mathrm{CE}(P, M)$ is contractible.*

Proof. This is a standard argument, similar to the contractibility of the space of collars, so we only sketch the proof. Firstly, by "pushing embeddings away from a collar", one shows that there exists a neighbourhood U of $P \times \{0\} \cup (\partial M \cap P) \times I$ such that the inclusion $\mathrm{CE}_U(P, M) \subset \mathrm{CE}(P, M)$ of the subspace concordance embeddings e that agree with the inclusion on U is a homotopy equivalence. Next, one shows that the inclusion $\mathrm{CE}_U(P, M) \subset \mathrm{CE}(P, M)$ is nullhomotopic (this works for any choice of U) by choosing a smooth family of self-embeddings $\eta_t\colon P \times I \hookrightarrow P \times I$ with parameter $t \in [0, 1]$ such that

(a) $\eta_t^{-1}(P \times \{i\}) = P \times \{i\}$ for $i = 0, 1$,
(b) η_t agrees with the inclusion on a neighbourhood of $P \times \{0\} \cup (\partial M \cap P) \times I$,
(c) $\eta_0 = \mathrm{id}_{P \times I}$, and
(d) $\eta_1(P \times I) \subset U$.

Precomposition with η_t defines a homotopy $[0,1] \times \mathrm{CE}_U(P, M) \to \mathrm{CE}(P, M)$ from the inclusion to the constant map at $\mathrm{inc} \circ \eta_1$. Concatenating this with the homotopy $t \mapsto (\mathrm{inc} \circ \eta_{1-t})$ yields a homotopy from the inclusion $\mathrm{CE}_U(P, M) \subset \mathrm{CE}(P, M)$ to the constant map at the inclusion. □

1.5. **Goodwillie's multiple disjunction lemma.** The main ingredient in the proof of the multi-relative stabilisation result (see Theorem 1.6) is Goodwillie's multi-relative generalisation of Morlet's lemma of disjunction [Goo90].

Theorem 1.10 (Goodwillie). *If $d - p \geq 3$ and $d - q_i \geq 3$ for all i, then the r-cube*

$$\mathrm{CE}(P, M \backslash Q_\bullet)$$

is $(d - p - 2 + \Sigma)$-cartesian.

Proof. The case $\partial M \cap P = \partial P$ and $\partial M \cap Q_i = \partial Q_i$ for all i is treated in [Goo90, Theorem D]. As pointed out in [GK15, p. 670], the general case can easily be reduced to this. □

For $r = 0$, Goodwillie's result implies Hudson's concordance-implies-isotopy theorem for concordance embeddings.

Theorem 1.11 (Hudson). *The space $\mathrm{CE}(P, M)$ is connected if $d - p \geq 3$.*

1.6. **The delooping trick.** There is a convenient trick to analyse concordance embeddings of discs of different dimension by means of a *scanning map*

$$\tau \colon \mathrm{CE}(D^p, M \backslash Q_\bullet) \longrightarrow \Omega \mathrm{CE}(D^{p-1}, M \backslash Q_\bullet). \tag{4}$$

To construct τ, we decompose D^p into three codimension 0 submanifolds (see Figure 3)

$$D^p_- \coloneqq \{x \in D^p \mid x_1 \leq -\tfrac{1}{2}\} \quad D^p_0 \coloneqq \{x \in D^p \mid -\tfrac{1}{2} \leq x_1 \leq \tfrac{1}{2}\} \quad D^p_+ \coloneqq \{x \in D^p \mid x_1 \geq \tfrac{1}{2}\}.$$

Restriction of concordance embeddings gives a commutative diagram of r-cubes

$$\begin{tikzcd} \mathrm{CE}(D^p, M \backslash Q_\bullet) \arrow[r] \arrow[d] & \mathrm{CE}(D^p_+ \cup D^p_0, M \backslash Q_\bullet) \arrow[d] \\ \mathrm{CE}(D^p_- \cup D^p_0, M \backslash Q_\bullet) \arrow[r] & \mathrm{CE}(D^p_0, M \backslash Q_\bullet) \end{tikzcd} \tag{5}$$

which induces a map from $\mathrm{CE}(D^p, M \backslash Q_\bullet)$ to the levelwise homotopy pullback of the other terms. Note that (i) the bottom-left and top-right terms are levelwise contractible by Lemma 1.9 and (ii) restriction to $\{0\} \times D^{p-1} \subset D^p_0$ induces a levelwise weak equivalence from the bottom right corner to $\mathrm{CE}(D^{p-1}, M \backslash Q_\bullet)$ by Lemma 1.8, so the levelwise homotopy pullback is weakly equivalent to the cube of loop spaces $\Omega \mathrm{CE}(D^{p-1}, M \backslash Q_\bullet)$ based at the inclusion. This defines the scanning map (4), up to contractible choices.

Lemma 1.12. *If $d - p \geq 3$ and $d - q_i \geq 3$ for all i, then the scanning map of r-cubes*

$$\tau \colon \mathrm{CE}(D^p, M \backslash Q_\bullet) \longrightarrow \Omega \mathrm{CE}(D^{p-1}, M \backslash Q_\bullet)$$

is $(2(d - p - 2) + \Sigma)$-cartesian when considered as an $(r + 1)$-cube.

Proof. By construction the map in question agrees with map on vertical homotopy fibres of (5). The latter agrees as a result of the isotopy extension theorem with the map

$$\mathrm{CE}\big(D^p_+, M' \backslash (Q_\bullet \cup D^p_-)\big) \longrightarrow \mathrm{CE}\big(D^p_+, M' \backslash Q_\bullet\big) \tag{6}$$

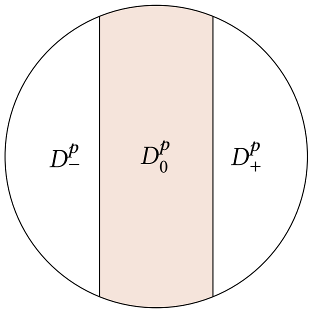

FIGURE 3. The decomposition $D^p = D^p_- \cup D^p_0 \cup D^p_+$.

given by postcomposition with the inclusion; here $M' \subset M$ is the complement of an open tubular neighbourhood of $D^p_0 \subset M$ with $D^p_+ \cap \partial M' = \partial D^p_+$. Considered as an $(r+1)$-cube, the map (6) agrees with the $(r+1)$-cube $\mathrm{CE}(D^p_+, M'\backslash Q_\bullet)$ with $Q_{r+1} \coloneqq D^p_-$, so it is $(d-p-2+\Sigma+(d-p-2))$-cartesian by an instance of Theorem 1.10. □

1.7. **Proof of Theorem 1.6 assuming the case $P = *$.** As mentioned previously, the main work of Theorem 1.6 is the case $P = *$. We state it again for future reference:

Theorem 1.13. *If $d \geq 3$ and $d - q_i \geq 3$ for all i, then the $(r+1)$-cube*

$$\mathrm{sCE}(*, M\backslash Q_\bullet) = \big(\mathrm{CE}(*, M\backslash Q_\bullet) \to \mathrm{CE}(J, (M\backslash Q_\bullet) \times J)\big)$$

is $(2d-5+\Sigma)$-cartesian.

We postpone the proof of this case to Section 2. Assuming it for now, we proceed by deriving the general case of Theorem 1.6 from it. The strategy will be to first deduce the case $P = D^p$ of discs, followed by the general case. For the former, it is convenient to consider the following statement for $p \geq 0$ and $k \in \mathbf{Z}$.

$(\mathbf{H}_{p,k})$ For $P = D^p \subset M$ with $D^p \cap M = \partial D^p$, the $(r+1)$-cube $\mathrm{sCE}(D^p, M\backslash Q_\bullet)$ is $(k+d-p-3+\Sigma)$-cartesian.

Theorem 1.6 for discs corresponds to $(\mathbf{H}_{p,k})$ for $p \leq d-3$ and $k \leq d-2$ which we will prove by a double induction on p and k. The induction starts with:

Lemma 1.14. $(\mathrm{H}_{0,k})$ *holds for all $k \leq d-2$ and* $(\mathbf{H}_{p,0})$ *holds for all $p \leq d-3$.*

Proof. The first part is precisely Theorem 1.13. The second part says that the $(r+1)$-cube

$$\mathrm{sCE}(D^p, M\backslash Q_\bullet) = \big(\mathrm{CE}(D^p, M\backslash Q_\bullet) \to \mathrm{CE}(D^p \times J, (M\backslash Q_\bullet) \times J)\big)$$

is $(d-p-3+\Sigma)$-cartesian. By Lemma 1.1 (ii) it suffices to show that the two individual r-cubes $\mathrm{CE}(D^p, M\backslash Q_\bullet)$ and $\mathrm{CE}(D^p \times J, (M\backslash Q_\bullet) \times J)$ are $(d-p-2+\Sigma)$-cartesian. This holds by Goodwillie's multiple disjunction lemma (see Theorem 1.10). □

The induction step will be provided by:

Lemma 1.15. $(\mathbf{H}_{p-1,k})$ *and* $(\mathbf{H}_{p,k-(d-p-2)})$ *imply* $(\mathbf{H}_{p,k})$ *for all $p \geq 1$ and $k \in \mathbf{Z}$.*

Proof. This follows from a variation of the delooping trick discussed in Section 1.6: since the square (5) is compatible with stabilisation, we have a square of $(r+1)$-cubes

$$\begin{tikzcd}
\mathrm{sCE}(D^p, M\backslash Q_\bullet) \arrow[r] \arrow[d] & \mathrm{sCE}(D^p_+ \cup D^p_0, M\backslash Q_\bullet) \arrow[d] \\
\mathrm{sCE}(D^p_- \cup D^p_0, M\backslash Q_\bullet) \arrow[r] & \mathrm{sCE}(D^p_0, M\backslash Q_\bullet).
\end{tikzcd}$$

For the same reason as for (5), the bottom left and top right corner are levelwise contractible, so in particular ∞-cartesian. By Lemma 1.1 (i) it thus suffices to show that the left vertical map considered as a $(r+2)$-cube is $(k+d-p-3+\Sigma)$-cartesian, assuming $(\mathbf{H}_{p-1,k})$ and $(\mathbf{H}_{p,k-(d-p-2)})$. Note that this property holds for the *right* vertical map as a result of Lemma 1.1 (ii), since the top right cube is ∞-cartesian and the bottom right cube is $(k+d-(p-1)-3+\Sigma)$-cartesian by $(\mathbf{H}_{p-1,k})$ since the restriction map $\mathrm{sCE}(D^p_0, M\backslash Q_\bullet) \to \mathrm{sCE}(D^{p-1}, M\backslash Q_\bullet)$ is a levelwise weak equivalence as a result of Lemma 1.8. To finish the proof, it thus suffices by Lemma 1.1 (i) to show that the square considered as an $(r+3)$-cube is $(k+d-p-3+\Sigma)$-cartesian. By part (iii) of the same lemma, we may check this by showing that the $(r+2)$-cube obtained by taking vertical homotopy fibres at the inclusion is as highly cartesian (it suffices to consider fibres over the inclusion, since $\mathrm{CE}(D^p_- \cup D^p_0, M\backslash Q_\bullet)$ is levelwise contractible). By the same argument as in the proof of Lemma 1.12, this map is equivalent to the map induced by inclusion

$$\mathrm{sCE}(D^p_+, M'\backslash(Q_\bullet \cup D^p_-)) \longrightarrow \mathrm{sCE}(D^p_+, M'\backslash Q_\bullet).$$

As an $(r+2)$-cube, this map agrees with $\mathrm{sCE}(D^p_+, M'\backslash Q_\bullet)$ if one sets $Q_{r+1} \coloneqq D^p_-$, so it is $(k-(d-p-2)+d-p-3+\Sigma+(d-p-2))$-cartesian by $(\mathbf{H}_{p,k-(d-p-2)})$. Cancelling the two terms $(d-p-2)$, we conclude the claim. □

Proof of Theorem 1.6. We first prove the case $P = D^p$ with $\partial M \cap D^p = \partial D^p$. As pointed out above, this is $(\mathbf{H}_{p,k})$ for $p \leq d-3$ and $k \leq d-2$. It follows from a double induction on p and k: the base cases are Lemma 1.15, and the induction step is provided by Lemma 1.14 which in particular says that $(\mathbf{H}_{p-1,k})$ and $(\mathbf{H}_{p,k-1})$ imply $(\mathbf{H}_{p,k})$ as long as $p \leq d-3$.

Using the case of discs, the general case follows by another induction, this time on the number h of handles of a relative handle decomposition of $\partial M \cap P \subset P$ consisting of handles of index at most $p \leq d-3$. If $h = 0$, then P is a closed collar on $P \cap \partial M$ so the cube $\mathrm{sCE}(P, M\backslash Q_\bullet)$ is levelwise contractible by Lemma 1.9, so it is in particular ∞-cartesian. If $P = P' \cup H$ where $H = D^i \times D^{\dim(P)-i}$ is a handle of index $i \leq p$ disjoint from $\partial M \cap P$ and we assume that the claim holds for P', then we may consider the restriction map $\mathrm{sCE}(P, M\backslash Q_\bullet) \to \mathrm{sCE}(P', M\backslash Q_\bullet)$. The base satisfies the cartesianness bound of the claim, so using Lemma 1.1 (i) it suffices to show that the map considered as an $(r+2)$-cube has the same cartesianness. As $\mathrm{CE}(P', M\backslash \cup_i Q_i)$ is connected as a result of Theorem 1.11, it suffices by Lemma 1.1 (iii) to show that the $(r+1)$-cube obtained by taking homotopy fibres at the inclusion has the same connectivity. The isotopy extension theorem shows that this cube of homotopy fibres is equivalent to $\mathrm{sCE}(H, M'\backslash Q_\bullet)$ where $M' \subset M$ is the complement of an open tubular neighbourhood of P' disjoint from the Q_i's. We may choose M' so that $H \cap \partial M' = \partial D^i \times D^{\dim(P)-i}$, so Lemma 1.8 implies that the restriction map $\mathrm{sCE}(H, M'\backslash Q_\bullet) \to \mathrm{sCE}(D^i, M'\backslash Q_\bullet)$ is a levelwise equivalence which reduces the claim to showing that $\mathrm{sCE}(D^i, M'\backslash Q_\bullet)$ has the claimed cartesianness. As $i \leq p \leq d-3$ this follows from the case of discs we already considered. □

2. Concordance embeddings of a point

This section serve to prove Theorem 1.13—the missing ingredient for Theorem 1.6.

2.1. **Strategy.** Concretely, the task is to show that the stabilisation map of r-cubes

$$\sigma\colon \mathrm{CE}(*, M\backslash Q_\bullet) \longrightarrow \mathrm{CE}(J, (M\backslash Q_\bullet) \times J)$$

is $(2d-5+\Sigma)$-cartesian as long as $d \geq 3$ and $d - q_i \geq 3$. Without loss of generality we may assume that M is path-connected by restricting to the component containing $*$ and replacing the Q_i's by their intersection with that component if necessary. The strategy of proof will be as follows.

① First we argue that it suffices to show that the composition

$$\mathrm{CE}(*, M\backslash Q_\bullet) \xrightarrow{\sigma} \mathrm{CE}(J, (M\backslash Q_\bullet) \times J) \xrightarrow{\tau} \Omega\mathrm{CE}(*, (M\backslash Q_\bullet) \times J)$$

of σ with the scanning map τ from Section 1.6 is $(2d-5+\Sigma)$-cartesian. We also construct a simplified map of r-cubes

$$\rho\colon \mathrm{CE}(*, M\backslash Q_\bullet) \longrightarrow \Omega\mathrm{CE}(*, (M\backslash Q_\bullet)\times J)$$

which is homotopic to the composition $(\tau\circ\sigma)$ and which we use instead.

② Next we argue that instead of ρ we may consider the induced map of r-cubes

$$\overline{\rho}\colon \overline{\mathrm{CE}}(*, M\backslash Q_\bullet) \longrightarrow \Omega\overline{\mathrm{CE}}(*, (M\backslash Q_\bullet)\times J)$$

between the homotopy fibres of the forgetful maps from the space of concordance embeddings to the space of concordance immersions. Moreover, we construct a map $\overline{\upsilon}$ that is homotopic to $\overline{\rho}$, but has better naturality properties.

③ More specifically, viewing $\mathrm{CE}(*, M) = \mathrm{Emb}_{\partial_0}(I, M) \subset \mathrm{Emb}(I, M)$ as the subspace of the space of all embeddings $I \hookrightarrow M$ with certain boundary conditions, the advantage of $\overline{\upsilon}$ over $\overline{\rho}$ is that it restricts to a map

$$\overline{\upsilon}\colon \overline{\mathrm{Emb}}_{\partial_0}(U, M\backslash Q_\bullet) \longrightarrow \Omega\overline{\mathrm{Emb}}_{\partial_0}(U, (M\backslash Q_\bullet)\times J) \tag{7}$$

for any submanifold $U\subset I$ containing ∂I. Using multiple disjunction for embeddings this allows us to reduce the claim to proving a bound on the cartesianness of (7) when $U\subset I$ is the complement of three disjoint intervals in $\mathrm{int}(I)$.

④ Finally, for such U, we establish the required bound by an application of the multi-relative Blakers–Massey theorem.

2.2. **Step ①: stabilisation and scanning.** We begin by considering the composition

$$\mathrm{CE}(*, M\backslash Q_\bullet) \xrightarrow{\sigma} \mathrm{CE}(J, (M\backslash Q_\bullet)\times J) \xrightarrow{\tau} \Omega\mathrm{CE}(*, (M\backslash Q_\bullet)\times J)$$

where the basepoint $*$ used in the definition of $\mathrm{CE}(*, (M\backslash Q_\bullet)\times J)$ is the image of the basepoint $*\in M$ under the inclusion

$$j\colon M\times I = M\times\{0\}\times I \xrightarrow{\subset} M\times J\times I. \tag{8}$$

The second map in the composition is $(2d-4+\Sigma)$-cartesian by Lemma 1.12, so to show that the first map is $(2d-5+\Sigma)$-cartesian, it suffices by Lemma 1.1 (v) to prove that the composition $(\tau\circ\sigma)$ is $(2d-5+\Sigma)$-cartesian. In fact, we will prove that $(\tau\circ\sigma)$ is $(2d-4+\Sigma)$-cartesian, so the limiting factor for the cartesianness of ρ comes from τ.

Instead of approaching $(\tau\circ\sigma)$ directly, we will study a homotopic map ρ that we construct in terms of two maps of the form

$$\rho_-, \rho_+\colon \mathrm{CE}(*, M\backslash Q_\bullet) \longrightarrow P_{\mathrm{inc}}\mathrm{CE}(*, (M\backslash Q_\bullet)\times J). \tag{9}$$

Here $P_{\mathrm{inc}}\mathrm{CE}(*, (M\backslash Q_\bullet)\times J)$ is the space of paths $[0,1]\to \mathrm{CE}(*, (M\backslash Q_\bullet)\times J)$ that *end* at the basepoint given by the inclusion $*\times\{0\}\times I\subset M\times J\times I$.

Notation 2.1. We use the following notation throughout this and the subsequent sections.

(i) Given a map $e\colon I\to M\times I$ we write $e_M\colon I\to M$ and $e_I\colon I\to I$ for the postcomposition of e with the two projections.

(ii) We use the letter t for an element in the domain I of a concordance embedding, and the letter s for an element in the domain $[0,1]$ of a path of embeddings.

Construction 2.2. We first construct $\rho_-\colon \mathrm{CE}(*, M)\to P_{\mathrm{inc}}\mathrm{CE}(*, M\times J)$ for which we fix once and for all a continuous function $\lambda\colon [0,1]\to[0,1)$ with $\lambda^{-1}(0)=\{0,1\}$; the precise choice of λ will not be relevant as the space of such functions is contractible. Using λ, we set the value of $\rho_-(e)\colon [0,1]\to \mathrm{CE}(*, M\times J)$ for $e\in\mathrm{CE}(*, M)$ at time $s\in[0,1]$ to be

$$I\ni t \xmapsto{\rho_-(e)_s} \begin{pmatrix} e_M((1-s)\cdot t) \\ \lambda(s)\cdot t \\ (1-s)\cdot e_I(t) + s\cdot t\end{pmatrix} \in M\times J\times I.$$

This agrees for $s = 0$ with $(j \circ e)$ and for $s = 1$ with the inclusion $\{*\} \times \{0\} \times I \subset M \times J \times I$. During the intermediate times $s \in (0,1)$ it is still a concordance embedding, since the composition with the projection to J is an embedding as $\lambda(s) > 0$. The map ρ_+ is defined by replacing λ in this construction by $-\lambda$.

The maps ρ_- and ρ_+ are natural in codimension zero embeddings of M, so give rise to maps of cubes as in (9). Their postcomposition with evaluation at 0 agrees with the map induced by postcomposition with j, so we can pointwise concatenate the reverse of ρ_- with ρ_+ to obtain a map

$$\rho\colon \mathrm{CE}(*, M\backslash Q_\bullet) \longrightarrow \Omega\mathrm{CE}(*, (M\backslash Q_\bullet) \times J) \tag{10}$$

which is homotopic to $(\tau \circ \sigma)$ by the following lemma. Its proof takes several pages, so we postpone it to the seperate Section 2.6 to not distract us from the main line of argument.

Lemma 2.3. *The maps of r-cubes $(\tau \circ \sigma)$ and ρ are homotopic.*

Combined with the discussion at the beginning of this step, this reduces the proof of Theorem 1.13 to showing that the $(r+1)$-cube ρ is $(2d - 5 + \Sigma)$-cartesian.

2.3. **Step ②: comparing to concordance immersions.** The space $\mathrm{CI}(P, M)$ of *concordance immersions* is the space of smooth immersions $\varphi\colon P \times I \looparrowright M \times I$ that

(i) satisfy $\varphi^{-1}(M \times \{i\}) = P \times \{i\}$ for $i = 0, 1$ and
(ii) agree with the inclusion on a neighbourhood of $P \times \{0\} \cup (\partial M \cap P) \times I \subset M \times I$.

There is an evident inclusion map $\mathrm{CE}(P, M) \to \mathrm{CI}(P, M)$ whose homotopy fibre at the inclusion $P \times I \subset M \times I$ we write as

$$\overline{\mathrm{CE}}(P, M) := \mathrm{hofib}_{\mathrm{inc}}\big(\mathrm{CE}(P, M) \to \mathrm{CI}(P, M)\big).$$

For $P = *$, the homotopy type of $\mathrm{CI}(P, M)$ is rather simple:

Lemma 2.4. *There is a weak equivalence* $\mathrm{CI}(*, M) \simeq \Omega S^d$.

Proof. We consider the derivative map $\mathrm{CI}(*, M) \to \mathrm{Bun}_{\partial_0}(TI, T(M \times I))$ to the space of bundle monomorphisms $TI \to T(M \times I)$ that agree with the derivative of the inclusion near 0 and whose value at 1 agrees with $(0_m, 1, x) \in TM \times I \times \mathbf{R} = T(M \times I)$ for some $m \in M$ and $x > 0$ where 0 is the zero-section. This map is a weak equivalence by the Smale–Hirsch theorem and its target fits into a fibration sequence $\Omega V_1(\mathbf{R}^{d+1}) \to \mathrm{Bun}_{\partial_0}(TI, T(M \times I)) \to \mathrm{Map}_{\partial_0}(I, M \times I)$ where $V_1(\mathbf{R}^{d+1}) \simeq S^d$ is the Stiefel manifold of 1-frames in $\mathbf{R}^{d+1}$ and $\mathrm{Map}_{\partial_0}(I, -)$ is defined similarly to $\mathrm{Bun}_{\partial_0}(TI, -)$. The base of this fibre sequence is easily seen to be contractible, so the claim follows. □

Observing that the formulas that were used to define ρ in Step ① make equal sense for concordance immersions, we have a commutative diagram of r-cubes of the form

$$\begin{tikzcd}
\mathrm{CE}(*, M\backslash Q_\bullet) \arrow[r, "\rho"] \arrow[d] & \Omega\mathrm{CE}(*, (M\backslash Q_\bullet) \times J) \arrow[d] \\
\mathrm{CI}(*, M\backslash Q_\bullet) \arrow[r, "\rho"] & \Omega\mathrm{CI}(*, (M\backslash Q_\bullet) \times J),
\end{tikzcd}$$

whose vertical homotopy fibres at the inclusion define a map of r-cubes

$$\overline{\rho}\colon \overline{\mathrm{CE}}(*, M\backslash Q_\bullet) \longrightarrow \Omega\overline{\mathrm{CE}}(*, (M\backslash Q_\bullet) \times J). \tag{11}$$

The following lemma shows that in the range of interest, we may consider $\overline{\rho}$ instead of ρ.

Lemma 2.5. *For $r \geq 1$, the $(r+1)$-cube ρ from (10) is k-cartesian if the $(r+1)$-cube $\overline{\rho}$ from (11) is k-cartesian. For $r = 0$, the same is true as long as $k \leq 2d - 2$.*

Proof. We first treat the case $r \geq 1$. The proof of Lemma 2.4 shows that the cubes $\mathrm{CI}(*, M\backslash Q_\bullet)$ and $\mathrm{CI}(*, (M\backslash Q_\bullet) \times J)$ are weakly equivalent to the constant cubes on ΩS^d and ΩS^{d+1} respectively, so they are ∞-cartesian as long as $r \geq 1$ by Corollary 1.3 (i). The claim then follows from Lemma 1.1 (iii) using that it suffices to consider the homotopy fibres over the inclusion since $\mathrm{CI}(*, M\backslash \cup_i Q_i) \simeq \Omega S^d$ is connected as $d \geq 3$.

In the case $r = 0$, we consider the commutative square

$$\begin{array}{ccc} \mathrm{CE}(*, M) & \xrightarrow{\rho} & \Omega\mathrm{CE}(*, M \times J) \\ \downarrow & & \downarrow \\ \Omega S^d \simeq \mathrm{CI}(*, M) & \xrightarrow{\rho} & \mathrm{CI}(*, M \times J) \simeq \Omega^2 S^{d+1}. \end{array} \tag{12}$$

To describe the bottom map up to homotopy in terms of ΩS^d and $\Omega^2 S^{d+1}$, note that the derivatives of the concordance embeddings in the paths defining ρ_- land in the contractible subspace of $\mathrm{Bun}_{\partial_0}(TI, T(M \times J \times I))$ where the tangent vector of the J-component is non-negative, and similarly for ρ_+ using the subspace where it is non-positive. Combining this with the identifications from Lemma 2.4, one sees that the bottom map in the above diagram is up to homotopy induced by looping the map from the top-left corner in

$$\begin{array}{ccc} S^d = D^{d+1}_- \cap D^{d+1}_+ & \xrightarrow{\subset} & D^{d+1}_+ \\ \subset\downarrow & & \downarrow\subset \\ D^{d+1}_- & \xrightarrow{\subset} & S^{d+1} \end{array}$$

to the homotopy pullback of the remaining entries. Here $D^{d+1}_-, D^{d+1}_+ \subset S^{d+1}$ are two hemispheres. As the bottom and right map of this square are d-connected and the square is homotopy cocartesian, it is $(2d-1)$-cartesian by the Blakers–Massey theorem (Theorem 1.4 for $|S| = 2$), so the bottom map in (12) is $(2d-2)$-cartesian and the claim follows again from Lemma 1.1, using once more that the space ΩS^d is connected. $\square$

As a next step, we introduce a variant $\overline{\upsilon}$ of $\overline{\rho}$ with better naturality properties. The construction of $\overline{\upsilon}$ is similar to that of $\overline{\rho}$ in Construction 2.2; roughly speaking the part of the role of the concordance embedding is replaced by the path of concordance immersions to the inclusion. We begin by defining variants

$$\overline{\upsilon}_-, \overline{\upsilon}_+ \colon \overline{\mathrm{CE}}(*, M\backslash Q_\bullet) \longrightarrow P_{\mathrm{inc}}\overline{\mathrm{CE}}(*, (M\backslash Q_\bullet) \times J) \tag{13}$$

of the maps ρ_- and ρ_+ from (9).

Notation 2.6.

(i) Recall from Notation 2.1 that we use the letter t for an element in the domain I of a concordance embedding, and the letter s for an element in the domain $[0, 1]$ of a path of concordance embeddings. In addition, we will use the letter t' for an element of the domain $[0, 1]$ of a path of concordance immersions.

(ii) We model homotopy fibres $\mathrm{hofib}_{y_0}(f \colon X \to Y)$ as spaces of pairs (x, γ) of an element $x \in X$ and a path $\gamma \colon [0, 1] \to Y$ *from* $f(x)$ *to* the basepoint y_0.

Construction 2.7. We first construct the map $\overline{\upsilon}_- \colon \overline{\mathrm{CE}}(*, M) \to P_{\mathrm{inc}}\overline{\mathrm{CE}}(*, M \times J)$, using the map $\lambda \colon [0, 1] \to [0, 1)$ from Construction 2.2. The image $\overline{\upsilon}_-(e, \gamma) \in P_{\mathrm{inc}}\overline{\mathrm{CE}}(*, M \times J)$ of an element $(e, \gamma) \in \overline{\mathrm{CE}}(*, M)$ is defined as the pair of the path (with time parameter s) of concordance embeddings which at time $s \in [0, 1]$ is

$$I \ni t \longmapsto \begin{pmatrix} \gamma_M(t, s) \\ \lambda(s) \cdot t \\ \gamma_I(t, s) \end{pmatrix} \in M \times J \times I$$

and the path (with time parameter s) of paths of concordance immersions to the basepoint (with parameter t') which at time $s \in [0,1]$ is

$$I \times [0,1] \ni (t,t') \longmapsto \begin{pmatrix} \gamma_M(t,(1-s)\cdot t' + s) \\ \lambda(s)\cdot t \\ \gamma_I(t,(1-s)\cdot t'+s) \end{pmatrix} \in M \times J \times I.$$

This is well-defined by the same reasoning as in Construction 2.2. The map $\overline{v}_+$ is defined by replacing λ in this construction by $-\lambda$.

Note that the construction is natural in codimension zero embeddings of M, so induces maps of cubes (13). Postcomposed with the evaluation at 0, both $\overline{v}_-$ and $\overline{v}_+$ agree with the map $\overline{\mathrm{CE}}(*,(M\backslash Q_\bullet)) \to \overline{\mathrm{CE}}(*,(M\backslash Q_\bullet)\times J)$ induced by postcomposition with the inclusion j, so the pointwise concatenation of the reverse of $\overline{v}_-$ with $\overline{v}_+$ defines

$$\overline{v}\colon \overline{\mathrm{CE}}(*,M\backslash Q_\bullet) \longrightarrow \Omega\overline{\mathrm{CE}}(*,(M\backslash Q_\bullet)\times J) \tag{14}$$

which is homotopic to $\overline{\rho}$ from (11) by the following lemma.

Lemma 2.8. *$\overline{\rho}$ and $\overline{v}$ are homotopic as maps of r-cubes.*

Proof. It suffices to construct a homotopy of maps $\overline{\mathrm{CE}}(*,M) \to P_{\mathrm{inc}}\overline{\mathrm{CE}}(*,M\times J)$ between $\overline{\rho}_-$ and $\overline{v}_-$, and a homotopy between $\overline{\rho}_+$ and $\overline{v}_+$ such that the two homotopies agree when postcomposed with the evaluation at 0. To this end, we consider the map

$$H\colon [0,1]\times[0,1]\times\overline{\mathrm{CE}}(*,M) \longrightarrow \overline{\mathrm{CE}}(*,M\times J)$$

whose image at a point $(s_1,s_2,(e,\gamma)) \in [0,1]\times[0,1]\times\overline{\mathrm{CE}}(*,M)$ is given by the pair of concordance embedding

$$I \ni t \longmapsto \begin{pmatrix} \gamma_M((1-s_1)\cdot t, s_2) \\ \lambda(\max(s_1,s_2))\cdot t \\ (1-s_1)\cdot\gamma_I(t,s_2)+s_1\cdot t \end{pmatrix} \in M\times J\times I$$

and the path (with time parameter t') of concordance immersions to the basepoint

$$I\times[0,1] \ni (t,t') \longmapsto \begin{pmatrix} \gamma_M((1-s_1)\cdot t,(1-s_2)\cdot t'+s_2) \\ \lambda(\max(s_1,s_2))\cdot t \\ (1-s_1)\cdot\gamma_I(t,(1-s_2)\cdot t'+s_2)+s_1\cdot t \end{pmatrix} \in M\times J\times I.$$

The reason why these are paths of concordance embeddings respectively immersions is the same as in Construction 2.2. Restricted to $s_1 = 0$, the map H agrees with the adjoint of $\overline{v}_-$ and restricted $s_2 = 0$ it agrees with the adjoint of $\overline{\rho}_-$, so we obtain a homotopy between these two maps by interpolating between the intervals $\{0\}\times[0,1] \subset [0,1]\times[0,1]$ and $[0,1]\times\{0\} \subset [0,1]\times[0,1]$. Replacing λ by $-\lambda$ yields an analogous homotopy between $\overline{v}_+$ and $\overline{\rho}_+$ (using the same interpolation). On the subspace $[0,1]\times\{1\}\cup\{1\}\times[0,1] \subset [0,1]\times[0,1]$ replacing λ by $-\lambda$ has no effect, which allows us to pointwise concatenate the two homotopies to obtain a homotopy between $\overline{v}$ and $\overline{\rho}$ as required. □

Combined with Lemma 2.5 and Step ①, Lemma 2.8 reduces the proof of Theorem 1.13 to showing that the $(r+1)$-cube $\overline{v}$ is $(2d-5+\Sigma)$-cartesian.

2.4. **Step ③: cutting the source interval.** For a codimension 0 submanifold $U \subset I$ containing ∂I, we denote by $\mathrm{Emb}_{\partial_0}(U, M\times I)$ the space of embeddings $e\colon U \hookrightarrow M\times I$ that satisfy the same boundary conditions as in the definition of $\mathrm{CE}(*,M)$. There is a forgetful map $\mathrm{Emb}_{\partial_0}(U,M\times I) \to \mathrm{Imm}_{\partial_0}(U,M\times I)$ to the space of immersions $U \looparrowright M\times I$ satisfying the same boundary conditions. We abbreviate the homotopy fibre at the inclusion $U \subset I = \{*\}\times M \subset I\times M$ by

$$\overline{\mathrm{Emb}}_{\partial_0}(U,M\times I) \coloneqq \mathrm{hofib}_{\mathrm{inc}}\big(\mathrm{Emb}_{\partial_0}(U,M\times I) \to \mathrm{Imm}_{\partial_0}(U,M\times I)\big).$$

Since the formulas appearing in the definition of $\overline{v}$ in Construction 2.7 make sense for any such submanifold $U \subset I$, they define more generally a map of r-cubes

$$\overline{v}\colon \overline{\mathrm{Emb}}_{\partial_0}(U,(M\backslash Q_\bullet)\times I) \longrightarrow \Omega\overline{\mathrm{Emb}}_{\partial_0}(U,(M\backslash Q_\bullet)\times J\times I) \tag{15}$$

that is natural in restricting to submanifolds $U' \subset U$ and agrees with (14) for $U = I$. This additional naturality is the advantage of $\overline{v}$ over $\overline{\rho}$ and we shall make us of it in the following way: fixing three disjoint closed intervals $V_1, V_2, V_3 \subset \mathrm{int}(I)$ and writing $V_T = \cup_{t\in ST} V_t$ for subsets $T \subseteq \underline{3} = \{1,2,3\}$, the map (15) induces a map of $(r+3)$-cubes

$$\overline{v}\colon \overline{\mathrm{Emb}}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times I) \longrightarrow \Omega\overline{\mathrm{Emb}}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times J\times I) \tag{16}$$

where $\overline{\mathrm{Emb}}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times I)$ is the $(r+3)$-cube

$$\underline{3}\times\underline{r} \supseteq T\times S \longmapsto \overline{\mathrm{Emb}}_{\partial_0}(I\backslash V_T,(M\backslash Q_S)\times I);$$

similarly for $\overline{\mathrm{Emb}}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times J\times I)$. For $\blacksquare = \varnothing$, the map (16) recovers the map (14) we are interested in, so the plan to prove the required cartesianness is to apply Lemma 1.2 (i) to (16). The following two lemmas will serve as ingredients to carry this out.

Lemma 2.9. *The $(r+3)$-cubes*

$$\overline{\mathrm{Emb}}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times I) \quad \textit{respectively} \quad \overline{\mathrm{Emb}}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times J\times I)$$

are $(2d-4+\Sigma)$-cartesian respectively $(2d-2+\Sigma)$-cartesian. In particular, the map (16) considered as an $(r+4)$-cube is $(2d-4+\Sigma)$-cartesian.

Proof. The second claim follows from the first by setting $M' := M\times J$ and $Q_i' := Q_i\times J$. To prove the first claim, recall

$$\overline{\mathrm{Emb}}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times I) = \mathrm{hofib}_{\mathrm{inc}}\big(\mathrm{Emb}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times I)\to \mathrm{Imm}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times I)\big).$$

We first show that $\mathrm{Imm}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)$ is ∞-cartesian for which it suffices by Lemma 1.2 (ii) to show that for each $S\subseteq\underline{r}$ the 3-cube $\mathrm{Imm}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_S)\times I)$ is ∞-cartesian. To do so, we choose a compact isotopy equivalent submanifold $D\subset I\backslash\cup_{i=1}^3 V_i$ containing ∂I and consider the map of $(r+3)$-cubes induced by restriction

$$\mathrm{Imm}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times I) \longrightarrow \mathrm{Imm}_{\partial_0}(D,(M\backslash Q_\bullet)\times I).$$

The target is ∞-cartesian since it is constant in some of the directions (combine Lemma 1.2 (ii) with Corollary 1.3 (i)). It thus suffices by Lemma 1.1 (iii) to prove that the $(r+3)$-cube of homotopy fibres over each $f\in\mathrm{Imm}_{\partial_0}(D,(M\backslash Q_\varnothing))$ is ∞-cartesian. Using the Smale–Hirsch theorem, one sees that this cube agrees with the cube $\mathrm{Imm}_\partial\big(W_\blacksquare,(M\backslash Q_\bullet)\times I\big)$ where $W_T := \cup_{t\notin T} V_T$. The latter is ∞-cartesian by Corollary 1.3 (ii) since $\mathrm{Imm}_\partial(-,(M\backslash Q_S)\times I)$ sends disjoint unions to products.

Using Lemma 1.1 (iii), it thus suffices to prove that the $(r+3)$-cube $\mathrm{Emb}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times I)$ is $(2d-4+\Sigma)$-cartesian. To do so, we repeat the above reasoning for embeddings: we consider the map of $(r+3)$-cubes

$$\mathrm{Emb}_{\partial_0}(I\backslash V_\blacksquare,(M\backslash Q_\bullet)\times I) \longrightarrow \mathrm{Emb}_{\partial_0}(D,(M\backslash Q_\bullet)\times I)$$

whose target is ∞-cartesian since it is constant in some of the directions. It thus suffices to prove that the $(r+3)$-cube of homotopy fibres over each embedding $e\in\mathrm{Emb}_{\partial_0}(D,(M\backslash Q_\varnothing))$ is $(2d-4+\Sigma)$-cartesian. Combining the parametrised isotopy extension theorem and the contractibility of spaces of embeddings of collars, one sees that this cube of homotopy fibres agrees up to equivalence with the $(r+3)$-cube $\mathrm{Emb}_\partial\big(W_\blacksquare,((M\backslash Q_\bullet)\times I)\backslash e(D)\big)$. We may in turn identify this as the $(r+3)$-cube of homotopy fibres over the inclusions of the map of $(r+3)$-cubes given by restriction

$$\mathrm{Emb}_{\partial_0}\big(W_\blacksquare\sqcup Q_\bullet\times I,(M\times I)\backslash e(D)\big) \longrightarrow \mathrm{Emb}_{\partial_0}\big(Q_\bullet\times I,(M\times I)\backslash e(D)\big)$$

The target of this map is ∞-cartesian because it is constant in some directions (see Lemma 1.2 (ii)), so using Lemma 1.1 (ii) and (iii) it suffices to prove that the source is $(2d-4+\Sigma)$-cartesian. This follows from an application of multiple disjunction for spaces of embeddings [GK15, Theorem B] since $(3-(d+1)+3(d-2)+\Sigma) = (2d-4+\Sigma)$. □

Lemma 2.10. *Let U be an open subset $U \subset I$ with $\partial I \subset U$. If U is a disjoint unions of two or three intervals, then $\overline{\mathrm{Emb}}_{\partial_0}(U, M)$ is weakly contractible.*

Proof. As a result of the contractibility of the space of collars, if $U' \subset U$ denote those components not intersecting ∂I the map $\overline{\mathrm{Emb}}_{\partial_0}(U, M) \to \overline{\mathrm{Emb}}(U', M)$ induced by restriction is a weak equivalence. By assumption U' is either empty or an open interval, so $\overline{\mathrm{Emb}}(U', M)$ is weakly contractible; in the first case this is trivial and in the second case one uses that the forgetful map $\mathrm{Emb}(\mathbf{R}, M \times I) \to \mathrm{Imm}(\mathbf{R}, M \times I)$ is a weak equivalence. □

Recall that the conclusion of Step ② was that in order to prove Theorem 1.13 it suffices to show that the $(r+1)$-cube $\overline{v}$ from (14) is $(2d-5+\Sigma)$-cartesian. We will in fact show that it is $(2d-4+\Sigma)$-cartesian. To to do, it is helpful to think of this cube as obtained from the $(r+4)$-cube (16) by setting $\blacksquare = \varnothing$. The previous lemma (applied to M and $M \times J$) ensures that if $T \subseteq \underline{3}$ does not agree with $\varnothing$ or $\underline{3}$, then the $(r+1)$-cube obtained from the $(r+4)$-cube (16) by setting $\blacksquare = T$ is levelwise weakly contractible, so these $(r+1)$-cubes are ∞-cartesian. Since the $(r+4)$-cube (16) is $(2d-4+\Sigma)$-cartesian by Lemma 2.9, we may apply Lemma 1.2 (i) to reduce the claim to showing that the map of r-cubes

$$\overline{v}\colon \overline{\mathrm{Emb}}_{\partial_0}(I \backslash V_{\underline{3}}, (M \backslash Q_\bullet) \times I) \longrightarrow \Omega\overline{\mathrm{Emb}}_{\partial_0}(I \backslash V_{\underline{3}}, (M \backslash Q_\bullet) \times J \times I) \tag{17}$$

is $(2d-2+\Sigma)$-cartesian when considered as an $(r+1)$-cube.

2.5. **Step ④: An application of the multi-relative Blakers–Massey theorem.** The final step is to prove that $(r+1)$-cube (17) is $(2d-2+\Sigma)$-cartesian. We abbreviate the domain manifold by $V^c := I \backslash V_{\underline{3}} = I \backslash (\cup_{i=1}^3 V_i)$. This is a union of four intervals of which two intersect ∂I; we denote the other two by $U_1, U_2 \subset V^c$. As a result of the contractibility of the space of collars, the vertical restriction maps in the commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{Emb}}_{\partial_0}(V^c, (M \backslash Q_\bullet) \times I) & \xrightarrow{\overline{v}} & \Omega\overline{\mathrm{Emb}}_{\partial_0}(V^c, (M \backslash Q_\bullet) \times J \times I) \\ \downarrow & & \downarrow \\ \overline{\mathrm{Emb}}(U_1 \cup U_2, (M \backslash Q_\bullet) \times I) & \xrightarrow{\overline{v}} & \Omega\overline{\mathrm{Emb}}(U_1 \cup U_2, (M \backslash Q_\bullet) \times J \times I) \end{array}$$

are levelwise weak equivalences, so it suffices to show that the bottom map considered as a $(r+1)$-cube is $(2d-2+\Sigma)$-cartesian.

For this we choose $u_1 \in \mathrm{int}(U_1)$ and $u_2 \in \mathrm{int}(U_1)$ and note that both the definition of the spaces $\overline{\mathrm{Emb}}_{\partial_0}(-, M)$ and that of the map $\overline{v}$ in Step ② make equal sense if the domain is a collection of points in I instead of a subinterval.

Lemma 2.11. *The vertical maps in the commutative diagram induced by restriction*

$$\begin{array}{ccc} \overline{\mathrm{Emb}}(U_1 \cup U_2, (M \backslash Q_\bullet) \times I) & \xrightarrow{\overline{v}} & \Omega\overline{\mathrm{Emb}}(U_1 \cup U_2, (M \backslash Q_\bullet) \times J \times I) \\ \downarrow & & \downarrow \\ \overline{\mathrm{Emb}}(\{u_1, u_2\}, (M \backslash Q_\bullet) \times I) & \xrightarrow{\overline{v}} & \Omega\overline{\mathrm{Emb}}(\{u_1, u_2\}, (M \backslash Q_\bullet) \times J \times I) \end{array}$$

are levelwise weak equivalences.

Proof. We prove the claim for the left-hand vertical map; the proof for the right-hand map is analogous. By the Smale–Hirsch theorem, the homotopy fibre of the left vertical map is the total homotopy fibre of the square induced by taking derivatives

$$\begin{array}{ccc} \mathrm{Emb}(U_1 \cup U_2, (M \backslash Q_\bullet) \times I) & \longrightarrow & \mathrm{Bun}(T(U_1 \cup U_2), T((M \backslash Q_\bullet) \times I)) \\ \downarrow & & \downarrow \\ \mathrm{Emb}(\{u_1, u_2\}, (M \backslash Q_\bullet) \times I) & \longrightarrow & \mathrm{Map}(\{u_1, u_2\}, (M \backslash Q_\bullet) \times I). \end{array}$$

The homotopy fibres of both vertical maps diagram are compatibly equivalent to the space $\mathrm{Map}(\{u_1,u_2\},V_1(\mathbf{R}^{n+1}))$, so the square is homotopy cartesian and the claim follows. $\square$

We are thus left with showing that the bottom map in Lemma 2.11 is $(2d-2+\Sigma)$-cartesian as an $(r+1)$-cube. The next proposition shows that it is even $(2d-1+\Sigma)$-cartesian. This will finish the proof of Theorem 1.13, aside from the postponed proof of Lemma 2.3.

Proposition 2.12. *The map of r-cubes*

$$\overline{\mathrm{Emb}}(\{u_1,u_2\},(M\backslash Q_\bullet)\times I)\xrightarrow{\overline{\upsilon}}\Omega\overline{\mathrm{Emb}}(\{u_1,u_2\},(M\backslash Q_\bullet)\times J\times I)$$

is $(2d-1+\Sigma)$-cartesian when considered as an $(r+1)$-cube.

Proof. Restriction to u_1 gives a commutative diagram of r-cubes

$$\begin{tikzcd}
\overline{\mathrm{Emb}}(\{u_1,u_2\},(M\backslash Q_\bullet)\times I) \arrow[r, "\overline{\upsilon}"] \arrow[d] & \Omega\overline{\mathrm{Emb}}(\{u_1,u_2\},(M\backslash Q_\bullet)\times J\times I) \arrow[d] \\
\overline{\mathrm{Emb}}(\{u_2\},(M\backslash Q_\bullet)\times I) \arrow[r, "\overline{\upsilon}"] & \Omega\overline{\mathrm{Emb}}(\{u_2\},(M\backslash Q_\bullet)\times J\times I)
\end{tikzcd}$$

whose bottom left and bottom right cubes are levelwise contractible, so it suffices to show that the $(r+1)$-cube given by the map on vertical homotopy fibres over the inclusion is $(2d-1+\Sigma)$-cartesian. This map has the form

$$\begin{array}{c}
\mathrm{hofib}_{(*,u_1)}\Big(\big((M\backslash Q_\bullet)\times I\big)\backslash\{(*,u_2)\}\overset{\subset}{\longrightarrow}(M\backslash Q_\bullet)\times I\Big)\\
\big\downarrow{\scriptstyle\overline{\upsilon}}\\
\Omega\mathrm{hofib}_{(*,0,u_1)}\Big(\big((M\backslash Q_\bullet)\times J\times I\big)\backslash\{(*,0,u_2)\}\overset{\subset}{\longrightarrow}(M\backslash Q_\bullet)\times J)\times I\Big)
\end{array}\tag{18}$$

Going through the definition of $\overline{\upsilon}$ in Construction 2.7, we see that an element in the domain given by a path $\gamma\colon[0,1]\to(M\backslash Q_\bullet)\times I$ with $\gamma(0)\neq(*,u_2)$ and $\gamma(1)=(*,u_1)$ is sent to the loop given by the concatenation of paths in the homotopy fibre $(-\overline{\upsilon}_-(\gamma))*(\overline{\upsilon}_+(\gamma))$ of the reverse of $\overline{\upsilon}_-(\gamma)$ with $\overline{\upsilon}_+(\gamma)$ where $\overline{\upsilon}_-(\gamma)$ is the path in the homotopy fibre which at time $s\in[0,1]$ is given by the path

$$[0,1]\ni t'\xmapsto{\overline{\upsilon}_-(\gamma)_s}\begin{pmatrix}\gamma_M((1-s)\cdot t'+s)\\ \lambda(s)\cdot u_1\\ \gamma_I((1-s)\cdot t'+s)\end{pmatrix}\in M\times J\times I$$

from $(*,0,\gamma_I(0))$ to $(*,0,u_1)$. The other path $\overline{\upsilon}_+(\gamma)$ is given by replacing λ with $-\lambda$.

Writing

$$J_2=\{*\}\times J\times\{u_2\},\qquad J_2^-=\{*\}\times[-1,0]\times\{u_2\},$$
$$J_2^+=\{*\}\times[0,1]\times\{u_2\},\quad J_2^0=\{(*,0,u_2)\},$$

the $(r+1)$-cube (18) may be viewed equivalently as obtained from the $(r+3)$-cube

$$\begin{tikzcd}
((M\backslash Q_\bullet)\times I\times J)\backslash J_2 \arrow[rr] \arrow[dr] \arrow[dd] & & ((M\backslash Q_\bullet)\times I\times J)\backslash J_2^+ \arrow[dr] \arrow[dd, "(*)"] & \\
& ((M\backslash Q_\bullet)\times I\times J)\backslash J_2^- \arrow[rr] \arrow[dd, "(*)"] & & ((M\backslash Q_\bullet)\times I\times J)\backslash J_2^0 \arrow[dd] \\
(M\backslash Q_\bullet)\times I\times J \arrow[rr] \arrow[dr] & & ((M\backslash Q_\bullet)\times I\times J \arrow[dr] & \\
& (M\backslash Q_\bullet)\times I\times J \arrow[rr] & & (M\backslash Q_\bullet)\times I\times J
\end{tikzcd}$$

given by inclusions by first taking vertical homotopy fibres at $(*, 0, u_1)$ and then using that the vertical maps labelled $(*)$ are homotopy equivalences. It thus suffices to show that this $(r+3)$-cube is $(2d-1+\Sigma)$-cartesian. As the bottom square is ∞-cartesian since it is constant in some directions (see Lemma 1.2 (ii)), it is enough to prove that the top square considered as an $(r+2)$-cube is $(2d-1+\Sigma)$-cartesian. This cube is strongly cocartesian, so we may estimate its cartesianness using the multi-relative Blakers–Massey theorem (see Theorem 1.4). Firstly, the inclusions

$$((M\backslash \cup_i Q_i)\times I\times J)\backslash J_2 \subset ((M\backslash \cup_i Q_i)\times I\times J)\backslash J_2^{\pm}$$

agree up to equivalence with the inclusion $N\times I\backslash\{(*, u_2)\}\subset N\times I$, for $N := M\backslash \cup_i Q_i$, so they are d-connected by general position. Secondly, the inclusions for $1\le j\le r$

$$((M\backslash \cup_i Q_i)\times I\times J)\backslash J_2 \subset ((M\backslash \cup_{i\neq j} Q_i)\times I\times J)\backslash J_2$$

are of the form $W\backslash R\subset W$ with W a $(d+2)$-manifold and R a submanifold of handle dimension q_i+2 relative to $\partial W\cap R$ (set $W := ((M\backslash\cup_{i\neq j} Q_i)\times I\times J)\backslash J_2$ and $R := Q_j\times I\times J$) so they are $(d-q_i-1)$-connected by general position. Theorem 1.4 thus gives that the $(r+2)$-cube in question is $(1-(r+2)+2d+\Sigma+r) = (2d-1+\Sigma)$-cartesian, as claimed. □

2.6. **Proof of Lemma 2.3.** We now make good for the postponed proof of Lemma 2.3, beginning with two prepratory constructions. A trivial but useful observation to keep in mind throughout this section is that a smooth map $I\longrightarrow M\times J\times I$ is an embedding if its projection to J or I is an embedding.

Notation 2.13. As in Notation 2.1, given a map $e\colon I\longrightarrow M\times J\times I$ we write $e_M\colon I\longrightarrow M$, $e_J\colon I\longrightarrow J$, and $e_I\colon I\longrightarrow I$ for the composition of e with the respective projections. We write t for elements in the domain I of maps such as e, and s for elements in the domain $[0,1]$ of a homotopy between such maps, thought of as a path in the mapping space.

Construction 2.14 (Vertical interpolation with shrinking). Given $e\colon I\longrightarrow M\times J\times I$, we consider the homotopy $\Upsilon(e)\colon [0,1]\times I\longrightarrow M\times J\times I$ starting at e which at time $s\in[0,1]$ is

$$t \xmapsto{\Upsilon_s(e)} \begin{pmatrix} e_M((1-s)\cdot t)\\ e_J(t)\\ (1-s)\cdot e_I(t)+s\cdot t\end{pmatrix}.$$

That is, the M-coordinate is homotoped to a constant map by increasingly restricting the domain, the J-coordinate is left unchanged, and the I-coordinate is linearly interpolated to the identity. If $e\colon I\hookrightarrow M\times J\times I$ is a concordance embedding such that e_J or e_I is an embedding then $s\mapsto \Upsilon_s(e)$ is a path of concordance embeddings.

Construction 2.15 (Insertion along an embedding). Given a closed interval $[a,b]\subset I$ of positive length, a map $e\colon I\longrightarrow M\times I$, and an embedding $\eta\colon I\hookrightarrow J\times I$ such that $e^{-1}(M\times\{i\})=\{i\}$ and $\eta^{-1}(J\times\{i\})=\{i\}$ for $i=0,1$, we use the affine-linear diffeomorphism $\tau_{a,b}\colon[a,b]\longrightarrow[0,1]$ with $a\mapsto 0$ and $b\mapsto 1$ to define the map

$$I\ni t\xmapsto{\Xi(e,\eta,[a,b])}\begin{cases}\big(e_M(0),\eta_J(t),\eta_I(t)\big) & \text{if } t\in[0,a]\\ \big((e_M\circ\tau_{a,b})(t),\eta_J(t),(\eta_I\circ\tau_{a,b}^{-1}\circ e_I\circ\tau_{a,b})(t)\big) & \text{if } t\in[a,b]\\ \big(e_M(1),\eta_J(t),\eta_I(t)\big) & \text{if } t\in[b,1]\end{cases}\in M\times J\times I.$$

That is, we insert e into $M\times J\times I$ at the interval $[a,b]$ in the image of η, and connect up the endpoints using the rest of η. If $e\colon I\longrightarrow M\times I$ is a concordance embedding and η has the properties that (i) η_I is an embedding and (ii) η_J is an embedding on $[a,b]$, then $\Xi(e,\eta,[a,b])$ is also a concordance embedding. Indeed, (i) implies that $\Xi(e,\eta,[a,b])$ is an embedding on $[0,a]\cup[b,1]$ and that the image of (a,b) is disjoint from the image of $[0,a]\cup[b,1]$, and (ii) implies that $\Xi(e,\eta,[a,b])$ is an embedding on (a,b). Moreover, similarly considering $[0,a]\cup[b,1]$ and (a,b) separately shows that under assumptions (i) and (ii), then $s\mapsto\Upsilon_s(\Xi(e,\eta,[a,b]))$ defines a path of concordance embeddings.

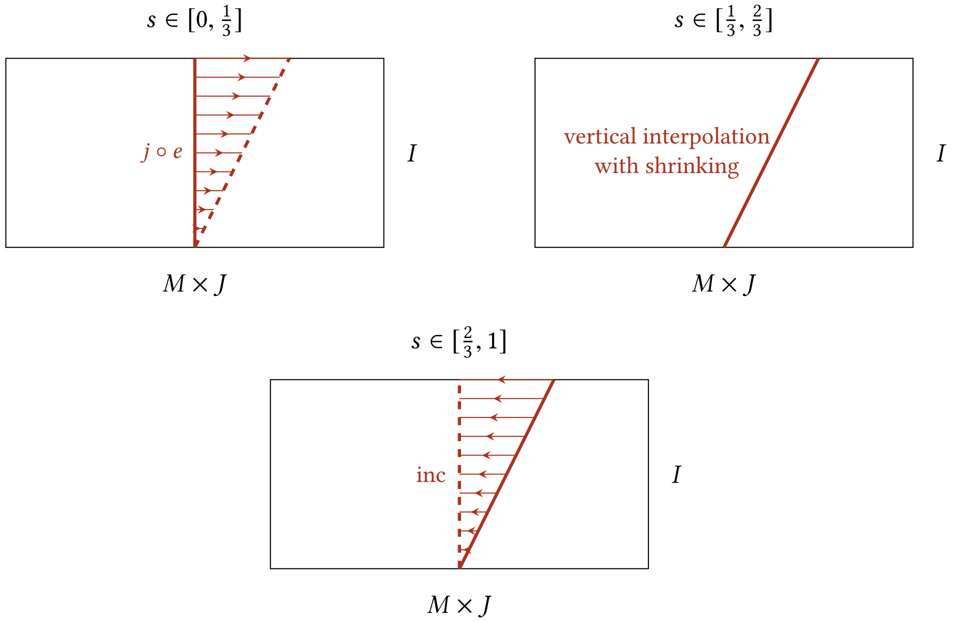

Figure 4. The path of concordance embeddings $\rho'_{-}(e)\colon [0,1] \to \mathrm{CE}(*, M \times J)$ from $j \circ e$ to inc.

The proof of Lemma 2.3 involves a variant ρ' of the map ρ from Step ①. Similarly to ρ, this variant will be obtained from two maps

$$\rho'_{-}, \rho'_{+}\colon \mathrm{CE}(*, M) \longrightarrow P_{\mathrm{inc}}\mathrm{CE}(*, M).$$

We fix a constant $0 < \epsilon < 1$.

Construction 2.16. The image $\rho'_{-}(e)$ for $e \in \mathrm{CE}(*, M)$ is defined as the concatenation of the following three paths (see Figure 4 for an illustration):

(1) We start with the path $s \mapsto \varpi_s(e)$ consisting of the concordance embeddings

$$I \ni t \xmapsto{\varpi_s(e)} \begin{pmatrix} e_M(t) \\ \epsilon \cdot s \cdot t \\ e_I(t) \end{pmatrix} \in M \times J \times I.$$

That is, we gradually tilt $j \circ e$ rightwards; here j is as in (8).
(2) Next we use the path $s \mapsto \Upsilon_s(\varpi_1(e))$ from Construction 2.14, which is a path of concordance embeddings because $\varpi_1(e)_J$ is an embedding.
(3) We finish with the path $s \mapsto \varpi_{1-s}(\mathrm{inc})$ using the inclusion $\mathrm{inc}\colon \{*\} \times \{0\} \times I \subset M \times J \times I$ and the fact that $\Upsilon_1(\varpi_1(e)) = \varpi_1(\mathrm{inc})$.

The path $\rho_{+}(e)$ is defined in the same way, but replacing ϵ by $-\epsilon$.

The maps ρ'_{-} and ρ'_{+} are natural in codimension zero embeddings of M and their postcomposition with evaluation at 0 agrees with the map induced by postcomposition with j, so we may pointwise concatenate the reverse of ρ'_{-} with ρ_{+} to obtain a map

$$\rho'\colon \mathrm{CE}(*, M\backslash Q_\bullet) \longrightarrow \Omega\mathrm{CE}(*, (M\backslash Q_\bullet) \times J). \tag{19}$$

Lemma 2.17. *The maps of r-cubes ρ of* (10) *and ρ' of* (19) *are homotopic.*

Proof. Spelling out the definitions, one sees that ρ' is obtained from ρ by choosing the function $\lambda\colon [0,1] \to [0,1)$ in Construction 2.2 to be

$$s \longmapsto \begin{cases} \epsilon \cdot 3 \cdot s & \text{if } s \in [0, \frac{1}{3}], \\ \epsilon & \text{if } s \in [\frac{1}{3}, \frac{2}{3}], \\ \epsilon \cdot 3 \cdot (1-s) & \text{if } s \in [\frac{2}{3}, 1], \end{cases}$$

and performing the vertical interpolation during the time interval $s \in [\frac{1}{3}, \frac{2}{3}]$ as opposed to during the time interval $s \in [0,1]$. The claimed homotopy is given by interpolating between these two time intervals. □

Lemma 2.17 reduces Lemma 2.3 to showing that the map of r-cubes ρ' is homotopic to the composition $(\tau \circ \sigma)$ from Step ①. This is the goal of the reminder of this section.

2.6.1. *Detailed description of* $(\rho \circ \tau)$. In Section 1.6 we gave a description of the map $\tau\colon \mathrm{CE}(J, (M \backslash Q_\bullet) \times J) \to \Omega\mathrm{CE}(*, (M \backslash Q_\bullet) \times J)$ up to a contractible space of choices. To construct a convincing homotopy between $(\rho \circ \tau)$ and ρ' it is helpful to fix choices to make the composition $(\rho \circ \tau)$ more explicit. To do so, we recall from Section 1.6 the construction of $\tau\colon \mathrm{CE}(J, M \times J) \to \Omega\mathrm{CE}(*, M \times J)$. It involved decomposing $J = [-1, 1]$ into

$$J_- = [-1, -\tfrac{1}{2}], \qquad J_0 = [-\tfrac{1}{2}, \tfrac{1}{2}], \quad \text{and} \quad J_+ = [\tfrac{1}{2}, 1],$$

and the square induced by restriction maps

$$\begin{tikzcd} \mathrm{CE}(J, M \times J) \arrow[r, "\mathrm{res}_+"] \arrow[d, "\mathrm{res}_-"'] & \mathrm{CE}(J_0 \cup J_+, M \times J) \arrow[d] \\ \mathrm{CE}(J_- \cup J_0, M \times J) \arrow[r] & \mathrm{CE}(J_0, M \times J). \end{tikzcd}$$

Using that the bottom left and top right corners are levelwise contractible by Lemma 1.9, the square induces up to canonical equivalence a map $\mathrm{CE}(J, M \times J) \to \Omega\mathrm{CE}(J_0, M \times J)$ whose postcomposition with the map induced by the restriction map $\mathrm{CE}(J_0, M \times J) \to \mathrm{CE}(*, M \times J)$ to $\{*\} = \{0\} \subset J_0$ (this is a weak equivalence by Lemma 1.8) is the scanning map τ. To explain our preferred model for the composition $(\tau \circ \sigma)$, note that the composition $(\mathrm{res}_- \circ \sigma)$ lands in the subspace $\mathrm{CE}_{U_-}(J_- \cup J_0, M) \subset \mathrm{CE}(J_- \cup J_0, M)$ of those concordance embeddings that agree with the inclusion on the neighbourhood $U_- := D_3 \cap (J_- \cup J_0) \times I$ of $(J_- \cup J_0) \times \{0\} \cup \{-1\} \times I$ in $(J_- \cup J_0) \times I$; here D_3 is as in the definition of the stabilisation map in Section 1.2. Replacing J_- by J_+ the analogous statement holds for $(\mathrm{res}_- \circ \sigma)$. Mapping in $\mathrm{CE}(*, M)$ into the above square thus induces the inner square in the commutative diagram

$$\begin{tikzcd} \mathrm{CE}(*, M) \arrow[r, "\mathrm{res}_+ \circ \sigma"] \arrow[d, "\mathrm{res}_- \circ \sigma"'] & \mathrm{CE}_{U_+}(J_0 \cup J_+, M \times J) \arrow[r, "\tau_+", "\simeq"'] \arrow[d] & P_{\mathrm{inc}}\mathrm{CE}(*, M \times J) \arrow[dd, "\mathrm{ev}_0"] \\ \mathrm{CE}_{U_-}(J_- \cup J_0, M \times J) \arrow[r] \arrow[d, "\tau_-"', "\simeq"] & \mathrm{CE}(J_0, M \times J) \arrow[dr, "\simeq"] & \\ P_{\mathrm{inc}}\mathrm{CE}(*, M \times J) \arrow[rr, "\mathrm{ev}_0"] & & \mathrm{CE}(*, M \times J) \end{tikzcd}$$

where the diagonal arrow is the equivalence given by restriction to $\{*\} = \{0\} \subset J_0$. To define the outer equivalences τ_- and τ_+, we fix a choice of paths of concordance embeddings

$$\eta_-\colon [0,1] \longrightarrow \mathrm{CE}(*, J_- \cup J_0) \quad \text{and} \quad \eta_+\colon [0,1] \longrightarrow \mathrm{CE}(*, J_0 \cup J_+)$$

from the inclusion $\{0\} \times I \subset J \times I$ to an embedding with image in U_- and U_+ respectively. Using η_-, the map τ_- is defined by mapping $e \in \mathrm{CE}_{U_-}(J_- \cup J_0, (M \backslash Q_\bullet) \times J)$ to the path $\tau_-(e)$ from $e \circ j$ to the inclusion, defined as the concatenation of the two paths

1. $[0,1] \ni s \mapsto e \circ (\eta_-)_s$ for $s \in [0,1]$ and
2. $[0,1] \ni s \mapsto \mathrm{inc} \circ (\eta_-)_{1-s}$, noting that $e \circ (\eta_-)_1 = \mathrm{inc} \circ (\eta_-)_1$ since $\mathrm{im}((\eta_-)_1) \subset U_-$.

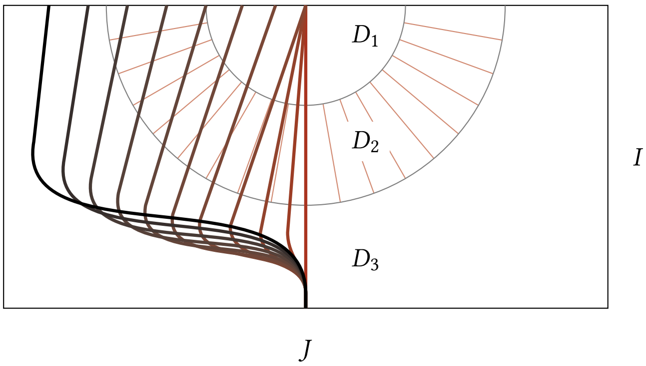

FIGURE 5. The path $\eta_-\colon [0,1] \longrightarrow \mathrm{CE}(*, J_- \cup J_0)$ from the inclusion to an embedding with image in U_-.

The map τ_+ is defined in the same way, but using η_+ instead of η_-. Note that the maps $\tau_\pm$ are equivalences since both their targets and sources are contractible; the latter by the proof of Lemma 1.9. The long horizontal and vertical compositions $(\tau_- \circ \mathrm{res}_- \circ \sigma)$ and $(\tau_+ \circ \mathrm{res}_+ \circ \sigma)$ in the diagram agree after postcomposition with ev_0, so we may pointwise concatenate the reverse of the former with the latter to arrive at a map $\mathrm{CE}(*, M) \longrightarrow \Omega\mathrm{CE}(*, M \times J)$. This construction is natural with respect to codimension 0 embeddings in the M-variable, so it induces a map of r-cubes

$$\mathrm{CE}(*, M\backslash Q_\bullet) \longrightarrow \Omega\mathrm{CE}(*, (M\backslash Q_\bullet) \times J).$$

This is our preferred model for the composition $(\tau \circ \sigma)$. However, to construct the homotopy to ρ we have in mind, we need to be more specific about the choice of the paths η_- and η_+. Namely, we choose η_+ as the mirror image of η_- along the $\{J = 0\}$-axis, i.e. $\eta_+ = ((\eta_-)_I, -(\eta_-)_J))$, and require that there is an $0 < \delta < 1$ such that $\eta_-\colon [0,1] \longrightarrow \mathrm{CE}(*, J_- \cup J_0)$ satisfies the following conditions (see Figure 5 for an illustrative example):

(a) At time $s = 0$, the embedding $(\eta_-)_s$ is the inclusion $\{0\} \times I \subset J \times I$.
(b) At times $s \in [0,1]$, the composition $\mathrm{pr}_I \circ (\eta_-)_s\colon I \longrightarrow I$ is a diffeomorphism.
(c) At times $s \in [0, 1-\delta]$ the embedding $(\eta_-)_s$ intersects the arc $D_2 \cap D_3$ in exactly one point, and at times $s \in (1-\delta, 1]$ the embedding $(\eta_-)_s$ is disjoint from $D_2 \cap D_3$.
(d) At times $s \in (0, 1-\delta)$, the restriction of the composition $(\mathrm{pr}_J \circ (\eta_-)_s)\colon I \longrightarrow J_- \cup J_0$ to $(\eta_-)_s^{-1}(D_1 \cup D_2) \subset I$ is an orientation-preserving embedding.
(e) At times $s \in [0, \delta]$ we have $(\eta_-)_s^{-1}(D_2) = [\frac{1}{3}, \frac{2}{3}]$ and $(\eta_-)_s(t) = \Lambda(3 \cdot t - 1, \frac{3}{2} \cdot \pi - s)$ for all $t \in [\frac{1}{3}, \frac{2}{3}]$ where Λ is the parametrisation (2).

Informally speaking, condition (e) in particular says that at times $s \in [0, \delta]$, the embedding $(\eta_-)_s$ maps $[\frac{1}{3}, \frac{2}{3}]$ to one of the red line segment within D_2 in Figure 5, starting with the vertical red line segments for $s = 0$ followed by red line segments in D_2 to the left of this.

2.6.2. *The homotopy between $\tau \circ \sigma$ and ρ'.* To construct a homotopy between $(\tau \circ \sigma)$ and ρ', it suffices to construct two homotopies, one between $(\tau_- \circ \mathrm{res}_- \circ \sigma)$ and ρ'_-, and one between $(\tau_+ \circ \mathrm{res}_+ \circ \sigma)$ and ρ'_+, such that these homotopies agree after postcomposition with the evaluation at 0. The construction uses the following lemma, which gives conditions under which Construction 2.14 yields a path of concordance embeddings:

Lemma 2.18. *Let $e\colon I \hookrightarrow M \times J \times I$ be a concordance embedding, such that*

(i) e_J is an embedding when restricted to $A_{12} := e^{-1}(M \times (D_1 \cup D_2))$,
(ii) e_I is an embedding when restricted to $A_3 := e^{-1}(M \times D_3)$, and
(iii) if $A_{12} \neq \varnothing$ then $A_{12} \cap A_3$ consists of a single point t_0, $e_I(A_{12})$ attains its minimum only at t_0, and $e_I(A_3)$ attains its maximum only at t_0.

Then $\Upsilon_s(e)$ of Construction 2.14 is a concordance embedding for all $s \in [0,1]$.

Proof. The map $\Upsilon_s(e)$ always satisfies the correct boundary conditions, so it suffices to prove that it is an embedding. If $A_{12} = \varnothing$ then $A_3 = I$, so e_I is an embedding by (ii) which implies that $\Upsilon_s(e)_I$ is an embedding. If $A_{12} \neq \varnothing$, then (iii) implies that $\Upsilon_s(e)_I(A_{12})$ intersects $\Upsilon_s(e)_I(A_3)$ in a single point. Thus it suffices to prove that $\Upsilon_s(e)$ is an embedding when restricted to A_{12} and when restricted to A_3. In the former case this follows from (i) and in the latter case from (ii). ◻

Using this lemma, the homotopy between $(\tau_- \circ \mathrm{res}_- \circ \sigma)$ and ρ'_- on a concordance embedding $e \in \mathrm{CE}(*, M)$ is given by the homotopy of paths relative to the endpoints from $(\tau_- \circ \mathrm{res}_- \circ \sigma)(e)$ to $\rho'_-(e)$ obtained as the concatenation of homotopies (1)–(3) below:

(1) Note that if $s > 0$ then $\mathrm{res}_-(\sigma(e)) \circ (\eta_-)_s$ satisfies the conditions in Lemma 2.18. With this in mind, we consider the two 2-parameter families of concordance embeddings

$$[\delta,1]\times[0,1] \longrightarrow \mathrm{CE}(*, M\times J) \qquad [\delta,1]\times[0,1] \longrightarrow \mathrm{CE}(*, M\times J)$$
$$(s,s') \longmapsto \Upsilon_{s'}\big(\mathrm{res}_-(\sigma(e))\circ(\eta_-)_s\big) \qquad (s,s') \longmapsto \Upsilon_{s'}\big(\mathrm{res}_-(\sigma(\mathrm{inc}))\circ(\eta_-)_s\big).$$

Here $0 < \delta < 1$ is as in the conditions on η_- above. Thinking of the path $\tau_-(\sigma)$ up to reparametrisation as the concatenation of four paths with time parameter $s \in [0,1]$

$$\big[\sigma(e)\circ(\eta_-)_{\delta\cdot s}\big] * \big[\sigma(e)\circ(\eta_-)_{\delta+(1-\delta)\cdot s}\big] * \big[\sigma(\mathrm{inc})\circ(\eta_-)_{\delta+(1-\delta)\cdot(1-s)}\big] * \big[\sigma(\mathrm{inc})\circ(\eta_-)_{\delta\cdot(1-s)}\big]$$

the two 2-parameter families above yield a homotopy rel endpoints from $\tau_-(\sigma)$ to

$$\big[\sigma(e)\circ(\eta_-)_{\delta\cdot s}\big] * \big[\Upsilon_s(\sigma(e)\circ(\eta_-)_\delta)\big] * \big[\Upsilon_{1-s}(\sigma(\mathrm{inc})\circ(\eta_-)_\delta)\big] * \big[\sigma(\mathrm{inc})\circ(\eta_-)_{\delta\cdot(1-s)}\big].$$

(2) Observe that $\sigma(e)\circ(\eta_-)_{\delta\cdot s}$ is equal to $\Xi(e, (\eta_-)_{\delta\cdot s}, [\frac{1}{3},\frac{2}{3}])$ as in Construction 2.15. We can now linearly interpolate between the family of embeddings $(\eta_-)_{\delta\cdot s}\colon I \hookrightarrow J\times I$ and the family of embeddings $\epsilon_s\colon I \hookrightarrow J\times I$, $t \mapsto (\epsilon\cdot s\cdot t, t)$ for $\epsilon > 0$ as in the definition of ρ', keeping the interval $[\frac{1}{3},\frac{2}{3}] \subset I$ fixed. Applying $\Xi(e, -, [\frac{1}{3},\frac{2}{3}])$ we obtain a path of concordance embeddings; for $s > 0$ by the remarks in Construction 2.15, and for $s = 0$ by inspection (it is a constant path). This yields a homotopy relative to the endpoints from the path the homotopy in (1) to the path

$$\big[\Xi(e,\epsilon_s,[\tfrac{1}{3},\tfrac{2}{3}])\big] * \big[\Upsilon_s(\Xi(e,\epsilon_1,[\tfrac{1}{3},\tfrac{2}{3}]))\big] * \big[\Upsilon_{1-s}(\Xi(\mathrm{inc},\epsilon_1,[\tfrac{1}{3},\tfrac{2}{3}]))\big] * \big[\Xi(\mathrm{inc},\epsilon_{1-s},[\tfrac{1}{3},\tfrac{2}{3}])\big].$$

(3) Finally, we note firstly that the path $\Upsilon_{1-s}(\Xi(\mathrm{inc},\epsilon_1,[\frac{1}{3},\frac{2}{3}]))$ is constant so we may homotope it away, and secondly that we may increase the size of the intervals linearly from $[\frac{1}{3},\frac{2}{3}]$ to $[0,1]$. This yields a homotopy relative to the endpoints from the path we arrived in (2) to $\rho'_-(e)$, since $\Xi(e,\epsilon_s,[0,1]) = \varpi_s(e)$ and $\Xi(\mathrm{inc},\epsilon_{1-s},[0,1]) = \varpi_{1-s}(\mathrm{inc})$.

This homotopy is natural in codimension zero embeddings of M, so it gives rise to a homotopy of maps of r-cubes between $(\tau_- \circ \mathrm{res}_- \circ \sigma)$ and ρ'_- which is constant when postcomposed with the evaluation at 0. Replacing ϵ with $-\epsilon$ and η_- with η_+ defines a similar homotopy between $(\tau_+ \circ \mathrm{res}_- \circ \sigma)$ and ρ'_+. This completes the proof of Lemma 2.3.

3. Applications to concordance diffeomorphisms and homeomorphisms

This section serves to explain some applications of Theorem A to spaces of concordance diffeomorphisms (see Section 3.1) and of concordance homeomorphisms (see Section 3.2).

3.1. **Smooth concordances.**

Recall from the introduction that $\mathrm{C}(M)$ for a compact smooth d-manifold is the topological group of diffeomorphisms of $M\times I \to M\times I$ that agree with identity on a neighbourhood of $M\times\{0\}\cup\partial M\times I$, equipped with the smooth topology. Corollary B states that if M is 1-connected spin and $d \geq 6$, then the map

$$\pi_*\big(\mathrm{C}(D^d\times J), \mathrm{C}(D^d)\big) \longrightarrow \pi_*\big(\mathrm{C}(M\times J), \mathrm{C}(M)\big)$$

induced by extending concordances of an embedded disc $D^d \subset \mathrm{int}(M)$ via the identity is an isomorphism in degrees $k < d-2$ when $d \geq 6$, and when $d = 5$ as long as $\partial M = \varnothing$. The following proposition is an ingredient in the proof.

Proposition 3.1. *Let M and N be compact d-manifolds such that N is obtained from M by attaching finitely many handles of index $\geq k$. If $k \geq 3$ then the commutative square*

$$\begin{tikzcd} \mathrm{BC}(M) \arrow[r] \arrow[d] & \mathrm{BC}(N) \arrow[d] \\ \mathrm{BC}(M \times J) \arrow[r] & \mathrm{BC}(N \times J) \end{tikzcd}$$

induced by stabilisation and extension by the identity is $(d+k-5)$-cartesian.

Proof. By induction over the number of handles, it suffices to prove the case $N = M \cup H$ where H is a single k-handle with $k \geq 3$. In this case, as a result of the parametrised isotopy extension theorem together with Lemma 1.8 and Theorem 1.11, the map on horizontal homotopy fibres of the square in question agrees up to equivalence with the stabilisation map $\mathrm{CE}(D^{d-k}, M \cup H) \to \mathrm{CE}(D^{d-k} \times J, (M \cup H) \times J)$ where $D^{d-k} \subset H$ is a cocore of the k-handle. The latter map is $(d+k-5)$-connected by Theorem A, so the claim follows. □

Proof of Corollary B. We first assume that M is 2-connected. Under our assumption that $d \geq 6$, or that $d = 5$ and $\partial M = \varnothing$, the manifold can be obtained from a disc $D^d \subset M$ by attaching finitely many handles of index ≥ 3 (see e.g. [Wal71, Theorem 3] for $d \geq 6$ and [Bar65, Corollary 2.2.2] for $d = 5$). Then Proposition 3.1 says that the square

$$\begin{tikzcd} \mathrm{BC}(D^d) \arrow[r] \arrow[d] & \mathrm{BC}(M) \arrow[d] \\ \mathrm{BC}(D^d \times J) \arrow[r] & \mathrm{BC}(M \times J) \end{tikzcd}$$

is $(d-2)$-cartesian, which translates into the desired statement about relative homotopy groups by considering the long exact sequence of homotopy groups.

The next step is to reduce the case of a general 1-connected spin d-manifold M with $d \geq 5$ ($\partial M = \varnothing$ if $d = 5$) to the 2-connected situation. The group $\pi_2(M) \cong \mathrm{H}_2(M)$ is generated by finitely many elements, say $x_1, \ldots, x_m$, which we can represented by an embedding $x\colon \sqcup^m S^2 \hookrightarrow \mathrm{int}(M)$ by general position. As M is spin, the normal bundle of embedded 2-spheres in M is trivial, so x can be extended to an embedding $\overline{x}\colon \sqcup^m S^2 \times D^{d-2} \hookrightarrow \mathrm{int}(M)$. Abbreviating the complement by $M' := M \backslash \mathrm{int}(\sqcup^m S^2 \times D^{d-2})$, we consider the map of homotopy fibre sequences (this uses that the fibers are connected by Theorem 1.11)

$$\begin{tikzcd} \mathrm{CE}\big(\sqcup^m S^2 \times D^{d-2}, M\big) \arrow[r] \arrow[d] & \mathrm{BC}(M') \arrow[r] \arrow[d] & \mathrm{BC}(M) \arrow[d] \\ \mathrm{CE}\big((\sqcup^m S^2 \times D^{d-2}) \times J, M \times J\big) \arrow[r] & \mathrm{BC}(M' \times J) \arrow[r] & \mathrm{BC}(M \times J). \end{tikzcd}$$

As $\partial M \cap (\sqcup^m S^2 \times D^{d-2}) = \varnothing$ and $\sqcup^m S^2 \times D^{d-2}$ has handle dimension $2 \leq d-3$ the leftmost vertical map is $(2d-7)$-connected by Theorem A, so the right-hand square is $(2d-7)$-cartesian. This helps since, without loss of generality, we can assume that the disc $D^d \subset M$ is contained in the interior of M', so we may consider the commutative diagram

$$\begin{tikzcd} \mathrm{BC}(D^d) \arrow[r] \arrow[d] & \mathrm{BC}(M') \arrow[r] \arrow[d] & \mathrm{BC}(M) \arrow[d] \\ \mathrm{BC}(D^d \times J) \arrow[r] & \mathrm{BC}(M' \times J) \arrow[r] & \mathrm{BC}(M \times J) \end{tikzcd} \tag{20}$$

The right-hand square is $(d-2)$-cartesian since $(d-2) \leq (2d-7)$ for $d \geq 5$, so the result for M follows from that for M'. To prove the latter, we perform surgery to obtain $\chi(M) := M' \cup_{\sqcup^m S^2 \times S^{d-3}} (\sqcup^m D^3 \times S^{d-3})$ and consider the map of homotopy fibre sequences (again using that the fibres are connected by Theorem 1.11)

$$\begin{tikzcd}
\mathrm{CE}\big(\sqcup^m D^3 \times S^{d-3}, \chi(M)\big) \arrow[r] \arrow[d] & \mathrm{BC}(M') \arrow[r] \arrow[d] & \mathrm{BC}(\chi(M)) \arrow[d] \\
\mathrm{CE}\big((\sqcup^m D^3 \times S^{d-3}) \times J, \chi(M) \times J\big) \arrow[r] & \mathrm{BC}(M' \times J) \arrow[r] & \mathrm{BC}(\chi(M) \times J).
\end{tikzcd}$$

As $(\sqcup^m D^3 \times S^{d-3}) \cap \partial(\chi(M)) = \varnothing$ and the handle dimension of $\sqcup^m D^3 \times S^{d-3}$ is $d-3$, the leftmost vertical map is $(2d-(d-3)-5) = (d-2)$-connected by Theorem A, so the right-hand square is $(d-2)$-cartesian. Replacing M by $\chi(M)$ in (20), we get a diagram where the right-hand square is $(d-2)$-cartesian and hence the the result for M' follows from that for $\chi(M)$. But the claim for $\chi(M)$ holds by the first part since this manifold is 2-connected by construction. □

Remark 3.2. Note that if M is 2-connected then the previous proof gives not only the isomorphism stated in Corollary B but also an epimorphism in degree $d-2$.

In fact, if one assumes $d \geq 7$ in addition to the 2-connectivity, then the map in Corollary B is an isomorphism in degree $d-2$ as well. To see this one picks a handle decomposition with no handles of index 1 or 2 and considers the submanifold $M_{\leq 3}$ of ≤ 3-handles. By construction, M can be obtained from $M_{\leq 3}$ by attaching finitely many handles of index ≥ 4, so by Proposition 3.1 shows that the square

$$\begin{tikzcd}
\mathrm{BC}(M_{\leq 3}) \arrow[r] \arrow[d] & \mathrm{BC}(M) \arrow[d] \\
\mathrm{BC}(M_{\leq 3} \times J) \arrow[r] & \mathrm{BC}(M \times J)
\end{tikzcd}$$

is $(d-1)$-cartesian which reduces the claim to showing it for $M_{\leq 3}$. Now one observes that $M_{\leq 3}$ admits a codimension 0 embedding into D^d: first immerse the 3-handles using that $\pi_3(\mathrm{BO}(d)) = 0$ and then make them embedded by general position. The sequence of embeddings $D^d \subset M_{\leq 3} \subset D^d$ shows that the stabilisation map $\mathrm{BC}(M_{\leq 3}) \to \mathrm{BC}(M_{\leq 3} \times J)$ is $M_{\leq 3}$ is a homotopy retract of that for D^d, so the map $\pi_{d-2}(\mathrm{C}(D^d \times J), \mathrm{C}(D^d)) \to \pi_{d-2}(\mathrm{C}(M_{\leq 3} \times J), \mathrm{C}(M_{\leq 3}))$ is not only surjective but also injective.

Remark 3.3. The manifold M can also be built from a collar $\partial M \times [0,1] \subset M$ of its boundary instead of from an embedded disc. If M has dimension $d \geq 6$ and the inclusion $\partial M \subset M$ is 2-connected, then it suffices to use handles of index ≥ 3 (see e.g. [Wal71, Theorem 3]) so

$$\begin{tikzcd}
\mathrm{BC}(\partial M \times [0,1]) \arrow[r] \arrow[d] & \mathrm{BC}(M) \arrow[d] \\
\mathrm{BC}(\partial M \times [0,1] \times J) \arrow[r] & \mathrm{BC}(M \times J)
\end{tikzcd}$$

is $(d-2)$-cartesian by Proposition 3.1 (cf. [WW01, p. 210]).

3.2. **Topological concordances.** We write $\mathrm{C}^{\mathrm{Top}}(M)$ for the topological group of *topological* concordances by which we mean the space of homeomorphisms (in the compact-open topology) of $M \times I$ that are the identity in a neighbourhood of $M \times \{0\} \cup \partial M \times I$. The definition of the stabilisation map makes equal sense for topological concordances and it is compatible with the evident forgetful map $\mathrm{C}(M) \to \mathrm{C}^{\mathrm{Top}}(M)$.

Proposition 3.4. *Let M be a compact smooth d-manifold with $d \geq 5$. If*

$$\mathrm{BC}(D^d) \longrightarrow \mathrm{BC}(D^d \times J)$$

is k-connected for some $k \geq 0$, then for any smooth compact d-manifold M the square

$$\begin{tikzcd} \mathrm{BC}(M) \arrow[r] \arrow[d] & \mathrm{BC}^{\mathrm{Top}}(M) \arrow[d] \\ \mathrm{BC}(M \times J) \arrow[r] & \mathrm{BC}^{\mathrm{Top}}(M \times J) \end{tikzcd}$$

is k-cartesian. The same implication holds rationally and p-locally for any prime p.

Proof. As explained in [BL77, p. 453–458], it follows from smoothing theory that there is a map of homotopy fibre sequences

$$\begin{tikzcd} \mathrm{C}(M) \arrow[r] \arrow[d] & \mathrm{C}^{\mathrm{Top}}(M) \arrow[r] \arrow[d] & \mathrm{Sect}(M, F_d) \arrow[d] \\ \mathrm{C}(M \times J) \arrow[r] & \mathrm{C}^{\mathrm{Top}}(M \times J) \arrow[r] & \mathrm{Sect}(M, \Omega F_{d+1}) \end{tikzcd} \tag{21}$$

where $\mathrm{Sect}(M, F_d)$ and $\mathrm{Sect}(M, \Omega F_d)$ are the space of sections, fixed on the boundary, of the bundles $\mathrm{Fr}(M) \times_{\mathrm{O}(d)} F_d$ and $\mathrm{Fr}(M) \times_{\mathrm{O}(d)} \Omega F_{d+1}$ over M where $\mathrm{Fr}(M)$ is the frame bundle and $F_d \coloneqq \mathrm{hofib}(\mathrm{Top}(d)/\mathrm{O}(d) \to \mathrm{Top}(d+1)/\mathrm{O}(d+1))$ is the homotopy fibre of the map induced by taking products with the real line. The rightmost vertical map is induced by the stabilisation map $F_d \to \Omega F_{d+1}$ [BL77, p. 450], so its homotopy fibre is given by a similar space of sections $\mathrm{Sect}(M, G_d)$ over M with fibre $G_d \coloneqq \mathrm{hofib}(F_d \to \Omega F_{d+1})$.

For $M = D^d$ the middle terms in (21) are contractible by the Alexander trick, so since $\mathrm{C}(D^d) \to \mathrm{C}(D^d \times J)$ is $(k-1)$-connected by assumption it follows that $\Omega\mathrm{Sect}(D^d, G_d) \simeq \Omega^{d+1} G_d$ is $(k-2)$-connected. Moreover, as F_d is $(d+1)$-connected for $d \geq 5$ by [KS77, Essay V.5.2], the space G_d is d-connected and thus in fact $(d+k-1)$-connected. For a general M, this implies that the right map in (21) is k-connected by obstruction theory, so the left square is $(k-1)$-cartesian and the claim follows by taking classifying spaces.

The rational (or p-local) addendum follows by the same argument: the case of a disc shows that G_d is d-connected and rationally (or p-locally) $(d+k-1)$-connected, so the claim follows again from obstruction theory. □

Corollary 3.5. *For a smoothable* 1*-connected compact spin d-manifold M with $d \geq 6$,*

$$\mathrm{BC}^{\mathrm{Top}}(M) \longrightarrow \mathrm{BC}^{\mathrm{Top}}(M \times J)$$

is rationally $\min(d-4, \lfloor \frac{3}{2}d \rfloor - 9)$*-connected.*

Proof. This follows from Proposition 3.4 since the maps $\mathrm{BC}(D^d) \to \mathrm{BC}(D^{d+1})$ and $\mathrm{BC}(M) \to \mathrm{BC}(M \times J)$ are rationally $\min(d-3, \lfloor \frac{3}{2}d \rfloor - 8)$-connected by Corollary C. □

Remark 3.6. As the forgetful map $\mathrm{C}^{\mathrm{PL}}(M) \to \mathrm{C}^{\mathrm{Top}}(M)$ from the PL-version of the space of concordance homeomorphisms is a weak equivalence for all PL-manifolds M of dimension ≥ 5 [BL74, Theorem 6.2], the results of this section also apply to $\mathrm{C}^{\mathrm{PL}}(M)$.

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