A NOTE ON HOMOTOPY AND PSEUDOISOTOPY OF DIFFEOMORPHISMS OF 4-MANIFOLDS

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Abstract. This note serves to record examples of diffeomorphisms of closed smooth 4-manifolds X that are homotopic but not pseudoisotopic to the identity, and to explain why there are no such examples if X is orientable and its fundamental group is a free group.

Recall that two diffeomorphisms φ_0 and φ_1 of a smooth manifold X are called *pseudoisotopic* if there exists a diffeomorphism ϕ of $X \times [0,1]$ that restricts to φ_i on $X \times \{i\}$ for $i = 0, 1$.

As part of the K3 project[1], we were asked the following question:

Question. *Is any diffeomorphism of a connected closed smooth* 4*-manifold* X *that is homotopic to the identity also pseudoisotopic to the identity?*

Equivalently, the question is whether homotopy implies pseudoisotopy for diffeomorphisms of connected closed smooth 4-manifolds X. If the fundamental group $\pi_1(X)$ vanishes or if X is orientable and $\pi_1(X) \cong \mathbf{Z}$, the answer to this question and its analogue in the topological category is known to be positive (see [Kre79, Theorem 1], [Qui86, Proposition 2.2], and [SW00, p. 51]).

The purpose of this note is twofold: firstly, we illustrate how a relatively routine use of classical surgery theory allows one to answer the general form of the question in the negative.

Theorem A. *There exists a diffeomorphism of a smooth closed* 4*-manifold* X *that is homotopic but neither smoothly nor topologically pseudoisotopic to the identity.*

More concretely, we explain why a diffeomorphism as in Theorem A exists for any connected closed smooth stably parallelisable 4-manifolds X whose fundamental group $\pi \coloneqq \pi_1(X)$ satisfies:

(1) $\mathrm{H}_1(\pi)/(2\text{-torsion})$ is not annihilated by multiplication by 3 and
(2) the 5th simple L-group $L_5^s(\mathbf{Z}[\pi])$ of $\mathbf{Z}[\pi]$ with the standard involution vanishes.

There are many 4-manifolds with these properties: any finitely presented group π arises as $\pi_1(X)$ of a connected closed smooth stably parallelisable 4-manifold X [Ker69, Proof of Theorem 1] and there are many choices for π that satisfy (2), such as finite groups π of odd order [Bak75] or more generally products $\pi = \pi_{\mathrm{odd}} \times \mathbf{Z}/2^k$ where π_{odd} has odd order [HT00, p. 227,12.1,12.2], e.g. π can be any finite cyclic group. If π_{odd} has a nontrivial element of order $\neq 3$, then π also satisfies (1).

Example. The simplest example that satisfies the above conditions is the result of a surgery along an embedding $e\colon S^1 \times D^3 \hookrightarrow S^1 \times S^3$ such that (a) the class $[e] \in \pi_1(S^1 \times S^3) \cong \mathbf{Z}$ is ± 5 and (b) the result of the surgery is stably parallelisable (which is always possible; see [Mil61, Theorem 2]).

The second purpose of this note is to observe that a combination of the surgery exact exact sequence with arguments in work of Shaneson allows one to widen the class of examples for which the answer to the Question is positive from orientable 4-manifolds X such that $\pi_1(X)$ is trivial or free of rank 1 to those for which $\pi_1(X)$ is a free group F_n of arbitrary finite rank $n \geq 0$.

Theorem B. *For diffeomorphisms of connected closed smooth orientable* 4*-manifolds with free fundamental group, homotopy implies pseudoisotopy. The same holds in the topological category.*

Remark. Combined with recent work of Gabai [Gab22, Theorem 2.5, Remark 2.10], this implies that diffeomorphisms of 4-manifolds X as in Theorem B that are homotopic are also stably isotopic, i.e. are isotopic when extended by the identity to diffeomorphisms of $X \sharp^g (S^2 \times S^2)$ for some $g \geq 0$ after isotoping them to fix an embedded disc to form the connected sum.

[1] See https://aimath.org/workshops/upcoming/kirbylist/.

Proof of Theorem A. The proof centres around the diagram of groups

$$
\begin{tikzcd}
\pi_0\mathrm{hAut}^s_\partial(X\times I) \arrow[r, "①"] & S^{s,\mathrm{triv}}_\partial(X\times I) \arrow[r, "②"] \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] & \pi_0\widetilde{\mathrm{Diff}}(X) \arrow[r, "③"] & \pi_0\mathrm{hAut}^s(X) \\
& S^s_\partial(X\times I) \arrow[d, "④"] & & \\
& {[\Sigma(X_+),\mathrm{G/O}]_*} \arrow[r, "⑥"] \arrow[d, "⑤"] & {[\Sigma(X_+),\mathrm{BO}]_*} & \\
& L^s_5(\mathbf{Z}[\pi_1(X)],w_1) & &
\end{tikzcd}
\tag{1}
$$

for any connected closed smooth 4-manifold X. The terms and maps involved are:

i) $\pi_0\widetilde{\mathrm{Diff}}(X)$ is the group of smooth pseudoisotopy classes of diffeomorphisms.
ii) $\pi_0\mathrm{hAut}^s(X)$ and $\pi_0\mathrm{hAut}^s_\partial(X\times I)$ are the groups of homotopy classes of simple homotopy automorphisms of X and $X\times I$ (fixing the boundary pointwise in the latter case),
iii) $S^s_\partial(X\times I)$ is the *simple structure set* of the pair $(X\times I,\partial(X\times I))$ in the sense of surgery theory [Wal99, Chapter 10] consisting of equivalence classes of pairs (W,φ) of a compact 5-manifold W and a simple homotopy equivalence $\varphi\colon W\to X\times I$ that restricts to a diffeomorphism $\varphi|_{\partial W}\colon\partial W\to\partial(X\times I)$ between the respective boundaries. The group structure is induced by "stacking". Note that since W is 5-dimensional, even though W is an s-cobordism, it need not be trivial. Restricting to those classes of pairs (W,φ) for which W is a trivial s-cobordism (i.e. diffeomorphic to $M\times I$, not necessarily preserving the identification of both ends) defines a subgroup $S^{s,\mathrm{triv}}_\partial(X\times I)\subseteq S^s_\partial(X\times I)$.
iv) The groups $[\Sigma(X_+),\mathrm{BO}]_*$ and $[\Sigma(X_+),\mathrm{G/O}]_*$ consist of pointed homotopy classes of maps from the suspension of $X_+\coloneqq X\sqcup\{*\}$ to the classifying space BO for stable vector bundles, and to the classifying space G/O for stable vector bundles together with a trivialisation of the underlying stable spherical fibration,
v) $L^s_5(\mathbf{Z}[\pi_1(X)],w_1)$ is the 5th simple L-group of the group ring $\mathbf{Z}[\pi_1(X)]$ with the standard involution, in the sense of surgery theory (see loc.cit.).
vi) The map ① sends $[\varphi]$ to $[X\times I,\varphi]$, ② sends $[X\times I,\varphi]$ to $[\varphi|_{X\times\{1\}}\circ\varphi|^{-1}_{X\times\{0\}}]$, ③ sends $[\varphi]$ to $[\varphi]$, ④ and ⑤ take normal invariants and surgery obstructions respectively (see loc.cit.), and ⑥ is induced by forgetting the trivialisation of the spherical fibration.

We will use the following facts about this diagram:

a) The top row and middle column are exact; the former by inspection and the latter by surgery theory (see e.g. Chapter 10 loc.cit.).
b) We have $[\Sigma(X_+),\mathrm{G/O}]\cong\mathrm{H}^1(X;\mathbf{Z}/2)\oplus\mathrm{H}^3(X;\mathbf{Z})$ (see e.g. [KT01, p. 398]).
c) The square of every element in $S^s_\partial(X\times I)$ lies in the subgroup $S^{s,\mathrm{triv}}_\partial(X\times I)$. This follows from the following trick (c.f. [Tho70]): given a 5-dimensional s-cobordism $W\colon M\rightsquigarrow N$ between 4-manifolds, view $W\times[0,1]$ as an s-cobordism of manifolds with boundary $M\times I\rightsquigarrow W\cup_N W$. This is trivial by the s-cobordism theorem, so $W\cup_N W\cong M\times I$.
d) The composition $S^s_\partial(X\times I)\to[\Sigma(M_+),\mathrm{BO}]_*$ has the following description (cf. [Wal99, p. 113–114]): for $[W,\varphi]\in S^s_\partial(X\times I)$, choose a homotopy inverse $\widetilde{\varphi}$ of φ with $\widetilde{\varphi}|_{\partial(X\times I)}=\varphi|^{-1}_{\partial W}$, consider the stable vector bundle $\widetilde{\varphi}^*(\nu_W)\oplus\tau_{X\times I}$ over $X\times I$ where $\nu_{(-)}$ and $\tau_{(-)}$ is the stable normal respectively tangent bundle. Together with the trivialisation of $(\widetilde{\varphi}^*(\nu_W)\oplus\tau_{X\times I})|_{\partial(X\times I)}$ induced by the derivative of $\widetilde{\varphi}_\partial$, this defines a class in $[X/\partial X,\mathrm{BO}]_*$.
e) After localisation away from 2 and Postnikov 7-truncation G/O and BO are both equivalent to $K(\mathbf{Z},4)$ and the map $\mathrm{G/O}\to\mathrm{BO}$ is induced by multiplication by 3. This follows from the computation of the stable homotopy groups of spheres in small degrees and the surjectivity of the stable J-homomorphism in degree 3.

If X is stably parallelisable and $\pi_1(X)$ satisfies the conditions (1) and (2), we can add to this list:

(f) The map $S^s_\partial(X\times I)\to[\Sigma(X_+),\mathrm{G/O}]_*$ is surjective since $L^s_5(\mathbf{Z}[\pi_1(X)],w_1)=L^s_5(\mathbf{Z}[\pi_1(X)])=0$ by (2) as X is orientable since it is stably parallelisable.

(g) The composition $\pi_0\mathrm{hAut}^s_\partial(X\times I) \to [\Sigma(X_+), \mathrm{BO}]_*$ is trivial. Factoring this map over $S^{s,\mathrm{triv}}_\partial(X \times I)$ and using the descriptions of the two maps involved given in vi) and d), this follows from the assumption that X (and thus $X \times I$) is stably parallelisable.

(h) By b) and Poincaré duality we have $[\Sigma(X_+), \mathrm{G/O}]_* \cong \mathrm{H}_1(X) \oplus \mathrm{H}^1(X; \mathbf{Z}/2)$. As a result of (1) there exists an element $x \in \mathrm{H}_1(X)$ with $6 \cdot x \neq 0 \in \mathrm{H}_1(X)/(2\text{-torsion})$. By c) and (f) the class $2 \cdot x$ lifts to $S^{s,\mathrm{triv}}_\partial(X \times I)$ and since $[\Sigma(X_+), \mathrm{G/O}] \to [\Sigma(X_+), \mathrm{BO}]_*$ is after inverting 2 given by multiplication by 3 in view of e), it follows that $2 \cdot x$ maps nontrivially to $[\Sigma(X_+), \mathrm{BO}]_*$. In particular the composition $S^{s,\mathrm{triv}}_\partial(X \times I) \to [\Sigma(X_+), \mathrm{BO}]_*$ is nontrivial.

Combining all this, the proof goes as follows: the claim is that there is a class in the kernel of ③ that maps nontrivially to the group $\pi_0\widetilde{\mathrm{Homeo}}(X)$ of topological pseudoisotopy classes of homeomorphisms. We will first show that there is even any a nontrivial element in the kernel of ③. By a) this corresponds to a nontrivial class in $S^{s,\mathrm{triv}}_\partial(X \times I)$ not hit by ①. Since $\pi_0\mathrm{hAut}^s_\partial(X \times I) \to [\Sigma(X_+), \mathrm{BO}]$ is trivial by (g), any class $S^{s,\mathrm{triv}}_\partial(X \times I)$ that maps nontrivially to $[\Sigma(X_+), \mathrm{BO}]$ has this property. But there are such classes by the final part of (h).

To see that the elements in the kernel of ③ detected in this way are also nontrivial in $\pi_0\widetilde{\mathrm{Homeo}}(X)$, one argues as follows: by forgetting smoothness the diagram (1) maps compatibly to the corresponding diagram in the topological category, so it suffices to show that the elements in $S^{s,\mathrm{triv}}_\partial(X \times I)$ used above are, when mapped to the topological version of $S^s_\partial(X \times I)$, still not hit by the topological analogue of ①. By the way we detected these elements in $S^s_\partial(X \times I)$ above, it suffices to show that the map $[\Sigma(X_+), \mathrm{BO}]_* \to [\Sigma(X_+), \mathrm{BTop}]_*$ induced by the map $\mathrm{BO} \to \mathrm{BTop}$ classifying the underlying stable Euclidean space bundle of a stable vector bundle is injective after inverting 2. This holds because after Postnikov 7-truncation and localisation away from 2 the map $\mathrm{BO} \to \mathrm{BTop}$ is an equivalence (see e.g. [KS77, p. 246, 5.0, (5)]).

Proof of Theorem B. We will explain the proof in the smooth case. The topological case follows in the same way, simply by replacing smooth by topological surgery theory.

For a connected closed smooth orientable 4-manifold X with $\pi_1(X) \cong F_n$ for some $n \geq 0$, by exactness of the top row in (1), it suffices to show that $S^s_\partial(X \times I)$ vanishes. To see this, we use that—since they are part of the surgery exact sequence [Wal99, Chapter 10]—the maps ④ and ⑤ can be extended to the left to an exact sequence of groups (we suppress the orientation character in the L-groups since X is orientable, so the involution on $\mathbf{Z}[\pi_1(X)]$ is the standard one)

$$L^s_6(\mathbf{Z}[\pi_1(X)]) \xrightarrow{⑦} S^s_\partial(X \times I) \xrightarrow{④} [\Sigma(X_+), \mathrm{G/O}] \xrightarrow{⑤} L^s_5(\mathbf{Z}[\pi_1(X)]),$$

thus it suffices to show that ⑦ and ④ are trivial. Choosing an embedded disc $D^5 \subset \mathrm{int}(X \times I)$ and using that the surgery exact sequence is natural in codimension 0 embeddings, the fact that ⑦ is zero follows by combining that $L^s_6(\mathbf{Z}[1]) \to L^s_6(\mathbf{Z}[F_n])$ is an isomorphism [Cap71, Corollary 6] and that $S^s_\partial(D^5) = 0$ as a consequence of the solution of the 5-dimensional smooth Poincaré conjecture. If $n = 0$, so if $\pi_1(X)$ and thus $\mathrm{H}_1(X)$ vanish, the map ④ is trivially zero since $[\Sigma(X_+), \mathrm{G/O}]_*$ vanishes by b) and Poincaré duality, so the proof for $n = 0$ is finished. For $n > 0$, showing that ④ is zero is more subtle and follows by suitable adapting arguments of Shaneson in the case $n = 1$:

As a first step, one argues that the composition ⑥ ∘ ④ is trivial. We argue as in [Sha70, p. 349]: By "gluing the ends of $X \times I$" we can extend ⑥ ∘ ④ to a commutative diagram

$$\begin{tikzcd}
S^s_\partial(X \times I) \arrow[r, "④"] \arrow[d, "⑧"] & {[\Sigma(X_+), \mathrm{G/O}]_*} \arrow[r, "⑥"] \arrow[d] & {[\Sigma(X_+), \mathrm{BO}]_*} \arrow[d] \\
S^s(X \times S^1) \arrow[r] & {[(X \times S^1)_+, \mathrm{G/O}]_*} \arrow[r] & {[(X \times S^1)_+, \mathrm{BO}]_*}
\end{tikzcd}$$

whose middle and right vertical map are split injective, using that the quotient $(X \times S^1)_+ \to \Sigma(X_+)$ splits after suspension and that G/O and BO are loop spaces. By the description of ⑥ ∘ ④ from d) it thus suffices to show that any homotopy equivalence of $X \times S^1$ that induces the identity on $\pi_1(X \times S^1)$ (for example any in the image of ⑧) preserves the stable tangent bundle of $X \times S^1$. This follows by the argument in the proof of [Sha69, Theorem 6.1] (the statement assumes that the fundamental group is free abelian, but the proof goes through for $\pi_1(X \times S^1) \cong \mathbf{Z} \times F_n$, since the version of the Novikov conjecture proved in [FH73, Theorem 7] applies).

Since ⑥ ∘ ④ is trivial, it follows from e) and the fact that $\mathrm{H}^3(X) \cong \mathrm{H}_1(X) \cong \mathbf{Z}^n$ is torsion free that ④ lands in the 2-torsion summand $\mathrm{H}^1(X;\mathbf{Z}/2)$ of $[\Sigma(X_+), \mathrm{G/O}]_*$. By exactness of the upper row in (1) it thus suffices to show ④ ∘ ①: $\pi_0\mathrm{hAut}^s_\partial(X \times I) \to [\Sigma(X_+), \mathrm{G/O}]_*$ hits this 2-torsion summand. This follows from constructing homotopy equivalences as in [Sha70, p. 349–350]: first show that the Hurewicz homomorphism $\pi_3(X) \to \mathrm{H}_3(X) \cong \mathbf{Z}^n$ is surjective, which can be done as in the proof of Lemma 6.2 loc.cit. using the 1-truncation $X \to B(F_n) \simeq \vee_n S^1$, then choose a basis (β_i) for $\mathrm{H}_3(X)$, lift each β_i to $\pi_3(X)$ and uses the lifts to construct elements $h_i \in \pi_0\mathrm{hAut}^s_\partial(X \times I)$ analogous to the construction of h on the top of p. 350 loc.cit. Since ④ is injective as ⑦ vanishes, if a sum $\sum_i \varepsilon_i h_i \in \pi_0\mathrm{hAut}^s_\partial(X \times I)$ with $\varepsilon_i \in \{0,1\}$ not all zero were sent under ④ ∘ ① to the trivial element of $\mathrm{H}^1(X;\mathbf{Z}/2)$, then its image under ⑧ would be homotopic to a diffeomorphism. But the argument on p. 350 loc.cit. shows that this is not the case, so the images of h_i in $\mathrm{H}^1(X;\mathbf{Z}/2)$ are linearly independent and hence ④ ∘ ① is surjective for dimension reasons.

Remark. Theorem B was proved for *closed* manifolds, but it does not seem unreasonable that the proof extends to allow nonempty boundary. For the boundary connected sums $Y_g := \natural^g(S^1 \times D^3)$ with $g \geq 0$ the statement can in fact be *deduced* from the closed case by a trick:

Given a diffeomorphism $\phi\colon Y_g \to Y_g$ that fixes ∂Y_g and is homotopic to id_{Y_g}, we will show that ϕ is also pseudoisotopic to id_{Y_g} relative to ∂Y_g. Extending ϕ by the identity on the second copy of Y_g in its double $D(Y_g) := Y_g \cup_\partial \overline{Y}_g \cong \sharp^g(S^1 \times S^3)$, we obtain a diffeomorphisms $(\phi \cup \mathrm{id})\colon D(Y_g) \to D(Y_g)$ that is homotopic to the identity. As $\pi_1(D(Y_g)) \cong F_g$, Theorem B ensures the existence of a pseudoisotopy H from $(\phi \cup \mathrm{id})$ to $\mathrm{id}_{D(Y_g)}$. Restricting H to $\overline{Y}_g$ gives a concordance embedding $H|_{I \times \overline{Y}_g}$ from $\overline{Y}_g$ into $D(X)$, and by isotopy extension it suffices to show that $H|_{I \times \overline{Y}_g}$ is isotopic, as a concordance embedding, to the inclusion $I \times \overline{Y}_g \subset I \times D(Y_g)$. As $\overline{Y}_g \subset D(Y_g)$ has handle codimension ≥ 3, this holds by [Hud70, Theorem 2.1, Addendum 2.1.2].

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