Rolling cylinder on a horizontal plane

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Abstract
A cylinder on a horizontal plane, acted upon by a constant horizontal force, provides a variety of physical situations, which are helpful for understanding the behaviour and the role of friction in systems of particles. In our study, we consider rolling with and without slipping and establish the conditions for the frictional force to have the direction of the centre-of-mass velocity, contrary to the common idea that this force is always opposite to motion. In general, rolling without slipping requires static friction. However, under some conditions, rolling without slipping may take place even in the absence of a static frictional force. On the other hand, if the applied force and the kinetic frictional force fulfil certain relations, a pure translation of the cylinder may occur. The work done by friction is also discussed.

Introduction

Frictional forces are introduced in elementary Physics courses, even before high school. Usually one considers a body at rest or in sliding motion, lying on a table, acted upon by a horizontal external force of variable magnitude. Students are left with the idea that the direction of the frictional force is opposite to the direction of the external force, whether the body is at rest or in motion under the action of the force. If the body starts moving, the direction of its velocity is the direction of the force and, therefore, the frictional force is opposite to the velocity. These intuitive ideas, which are subsequently reinforced using other examples, are correct if the body may be considered as a particle (or if there is no rotation). Later on, when students study systems of particles, some of them get perplexed with respect to the direction of the frictional force since static or kinetic friction can be either opposite to or in the same direction as the centre-of-mass velocity.

On the other hand, the concept of work is introduced for a body, regarded as a particle, in sliding motion. It is clear that the kinetic frictional force, opposite to the velocity (i.e. to the displacement), always produces negative work, while the static frictional force, acting on a particle, does not produce any work simply because the particle does not move. In systems of particles the kinetic friction always produces (negative) work, and in rolling without slipping the static frictional force produces no work. The situation is, therefore, identical to the single particle case. However, some students again get surprised by the zero work of a force pointing in the direction of (or opposite to) the centre-of-mass motion, which, moreover, contributes to an increase (or a decrease) in the centre-of-mass velocity.

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These difficulties have merited special attention by quite a few authors. In this vein, we report here a systematic study of a rigid cylinder rolling on a horizontal surface, acted upon by a constant horizontal force. The system shows up various situations, which have been very useful in helping our students to understand the subtle differences between the behaviour of the frictional force in systems of particles and for just one particle, and this motivated this work. Let us briefly mention the main ideas of other papers on the issue of the direction of the frictional force and on the work done by this force.

Shaw (1979) proposed an experiment using a cylinder that rolls without slipping on a horizontal surface. The magnitude and the direction of the frictional force are determined experimentally. Our study contains the situation described by Shaw, but we also study the motion with slipping.

Salazar *et al* (1990) claimed that situations where the frictional force and the centre-of-mass velocity have a common direction are considered paradoxical by some students. They used two examples illustrating that friction may be opposed to or in the direction of the centre-of-mass. In particular, a sphere rolling without slipping is considered to discuss the 'apparent paradox' of the friction being in the same direction as the centre-of-mass velocity.

McClelland (1991) suggested that the word 'friction' should be applied only in situations where the contacting solid surfaces are in relative motion. If such motion does not occur, he proposed that such interaction be called 'shear adhesion'. These considerations are pertinent but they do not explain the direction of friction.

Regarding the work of the frictional force, a study by Carnero *et al* (1993) was based on the pseudowork–energy theorem. They showed that the frictional force produces translation and rolling work with opposite signs, which, therefore, cancel each other out, and the net work of the static friction vanishes.

Sousa and Pina (1997) discussed the dissipation effects in a rolling cylinder, in connection with the laws of thermodynamics, concluding that a better physical understanding is achieved by considering mechanics and thermodynamics together.

In this article we first present a systematic study of the motion of a cylinder on a horizontal plane, with emphasis on the direction of the frictional force, then we briefly comment on the work done by friction, and finally we present our conclusions.

The direction of the frictional force

Let us consider a rigid, homogeneous cylinder with mass m and radius R, lying on a rigid and horizontal surface, as shown in figure 1. The velocity of the contact point, $\boldsymbol{v}_P$, and the velocity of the centre-of-mass, $\boldsymbol{v}_C$, are related by

$$\boldsymbol{v}_P = \boldsymbol{v}_C + \boldsymbol{\omega} \times \boldsymbol{R} \tag{1}$$

where $\boldsymbol{\omega}$ is the angular velocity with respect to the axis of the cylinder and $\boldsymbol{R}$ is the position vector of point P relative to the centre-of-mass. The angular velocity is $\boldsymbol{\omega} = \omega\hat{e}_z$ with $\omega > 0$ if the cylinder rotates clockwise ($\hat{e}_z$ points inward, relative to the plane of the figure). Projecting equation (1) along the direction of x, one gets

$$v_P = v_C - \omega R. \tag{2}$$

When the velocity of the contact point P is zero, $v_P = 0$, one has pure rolling and, from equation (2), one gets the condition for rolling without slipping:

$$v_C = \omega R \tag{3}$$

while the condition for rolling with slipping can be expressed either by $v_P \neq 0$ or by

$$v_C \neq \omega R. \tag{4}$$

We assume that a constant horizontal force, $\boldsymbol{F}$, is applied to the cylinder, at a certain height, h, as shown in figure 1. The cylinder moves on a horizontal plane acted upon by this force and by the frictional force, $\boldsymbol{f}$, generated at the contact

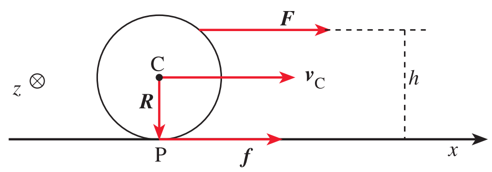

Figure 1. Cylinder on a horizontal plane acted upon by the force $\boldsymbol{F}$ and friction $\boldsymbol{f}$ (which is tentatively represented pointing to the right). The weight and the normal reaction, which are equal in magnitude and have opposite directions, are not shown.

points between the cylinder and the horizontal surface, which tentatively points as indicated in figure 1. Only a detailed analysis will allow us to determine whether the direction of that force is the indicated one or not. We also assume that friction (static or kinetic) is independent of the magnitude of the velocity, so that both the centre-of-mass acceleration, a_C, and the angular acceleration, α, are constants. Assuming that the cylinder is initially at rest, i.e. $\omega(t = 0) = 0$ and $v_C(t = 0) = 0$, in general, at any t,

$$\frac{v_C}{\omega R} = \frac{a_C}{\alpha R}. \tag{5}$$

The (projected) equations of motion are

$$F + f = m a_C \tag{6}$$

$$F(h - R) - f R = \tfrac{1}{2} m R^2 \alpha \tag{7}$$

where $\frac{1}{2} m R^2$ is the moment of inertia of the cylinder relative to its axis, $F > 0$ and f may be positive, negative or zero. These equations may still be written as

$$a_C = \frac{F + f}{m} \tag{8}$$

$$\alpha R = \frac{2}{m R}[F(h - R) - f R]. \tag{9}$$

Depending on the velocity of P, the friction may be static ($v_P = 0$) or kinetic ($v_P \neq 0$). Let us analyse these two cases.

Case 1: rolling without slipping

The coefficient of static friction is supposed to be large enough to ensure rolling without slipping, and condition (3) is fulfilled. Inserting that equation into equation (5) yields $a_C = \alpha R$. Using this expression in equation (8) and taking equation (9), one arrives at the following relation between the static friction, the applied force and the geometrical parameters h and R:

$$f = F\left(\frac{2h}{3R} - 1\right). \tag{10}$$

Figure 2 shows this function.

In the range $0 \leqslant h < \frac{3}{2}R$ the frictional force is negative, i.e. it points in the opposite direction to that shown in figure 1. It vanishes for $h = \frac{3}{2}R$ and it becomes positive, i.e. pointing as in figure 1, in the range $\frac{3}{2}R < h \leqslant 2R$. We consider this

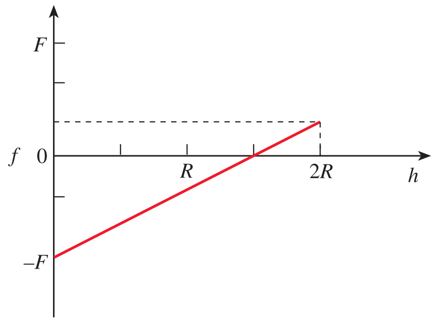

Figure 2. The value of the frictional force in the case of rolling without slipping. Depending on the height, h (see figure 1), f may be positive, negative or zero.

example and this result to be of great pedagogical value since they show how subtle is the direction of the frictional force for a system of particles (a rigid body in our case). Somehow a common idea is that frictional forces are opposed to the motion. This is certainly true for a particle (or a body in pure translation, which may be regarded as a particle). In the example studied above, the direction of the frictional force behaves according to that idea only for $0 \leqslant h < \frac{3}{2}R$. It is worth noticing that there is a value h (which depends on the moment of inertia of the rolling body) for which the body *always* rolls without slipping, irrespective of the value of the friction coefficient. Actually, for $h = \frac{3}{2}R$, in the case of the cylinder, one has always $f = 0$ for any F. Hence, the existence of a static frictional force is not a necessary condition for the occurrence of rolling without slipping. For $h < \frac{3}{2}R$ the static frictional force prevents the body from having larger accelerations when $\boldsymbol{F}$ is applied (as for the single particle) but for $h > \frac{3}{2}R$ the static frictional force also contributes to increasing the velocity of the centre-of-mass.

The static frictional force has a maximal value: $|f| \leqslant \mu m g$, where μ is the coefficient of static friction. Therefore, in order to guarantee rolling without slipping, one should have

$$\mu \geqslant \frac{F}{mg}\left|\frac{2h}{3R} - 1\right| \tag{11}$$

where use has been made of equation (10). If

$$\mu < \frac{F}{mg}\left|\frac{2h}{3R} - 1\right| \tag{12}$$

the cylinder slips (and rolls) under the action of force $\boldsymbol{F}$.

Table 1. Qualitative values of the velocity of the contact point, v_P, and of the frictional force, f, for the rolling cylinder, originally at rest, acted upon by a constant horizontal force, F, applied at different heights, h.

Case 1: rolling without slipping	$0 \leqslant h < \frac{3}{2}R$	$v_P = 0$	$f < 0$
$\mu \geqslant \frac{F}{mg}\left\|\frac{2h}{3R} - 1\right\|$	$h = \frac{3}{2}R$	$v_P = 0$	$f = 0$
	$\frac{3}{2}R < h \leqslant 2R$	$v_P = 0$	$f > 0$
Case 2: rolling and slipping	$0 \leqslant h < \frac{3}{2}R$	$v_P > 0$	$f \leqslant 0$
$\mu < \frac{F}{mg}\left\|\frac{2h}{3R} - 1\right\|$	$h = \frac{3}{2}R$	$v_P = 0$	$f = 0$
	$\frac{3}{2}R < h \leqslant 2R$	$v_P < 0$	$f \geqslant 0$

Case 2: rolling and slipping

We turn now to the case $v_P \neq 0$, i.e. we suppose rolling with slipping. This happens when the static friction coefficient does not satisfy equation (11). We may have $a_C > \alpha R$, for $v_P > 0$, or $a_C < \alpha R$, for $v_P < 0$ (see equations (2), (4) and (5)).

In the first situation, $a_C > \alpha R$, and from the definition of the (kinetic) frictional force,

$$f \leqslant 0 \tag{13}$$

since the velocity of point P is positive, i.e. the point represented in figure 1 moves to the right (the equality in (13) holds for frictionless motion). On the other hand, using $a_C > \alpha R$ in equations (8) and (9), one finds

$$f > F\left(\frac{2h}{3R} - 1\right). \tag{14}$$

Inequalities (13) and (14) are met simultaneously for heights in the range

$$0 \leqslant h < \tfrac{3}{2}R. \tag{15}$$

In the second situation, $v_P < 0$, and therefore $f \geqslant 0$ (again the equality holds for frictionless motion). From $a_C < \alpha R$ and using the same reasoning that led to (15), one finds

$$\tfrac{3}{2}R < h \leqslant 2R. \tag{16}$$

For $h = \frac{3}{2}R$ one has frictionless motion ($f = 0$) with $v_P = 0$, and one has case 1.

Table 1 summarizes the results for the two cases.

In cases 1 and 2, rolling may take place even with a vanishing frictional force (if $h = \frac{3}{2}R$). Curiously enough, friction may exist and, nevertheless, motion without rolling may occur, that is, we may have $v_C = v_P > 0$ and $\omega = 0$, which is a particular situation of case 2. Imposing $\omega = 0$ in equation (5) implies $\alpha = 0$. Using this result in equation (9), one arrives at

$$f = F\left(\frac{h}{R} - 1\right)$$

and, since $f \leqslant 0$, the height ought to be $h \leqslant R$. This is a necessary condition for the motion to be a pure translation. On the other hand, if the kinetic frictional force is written as $|f| = \mu_k mg$, where μ_k is the kinetic friction coefficient, then in order to have translation without rotation, one should have

$$\mu_k = \frac{F}{mg}\left(1 - \frac{h}{R}\right).$$

This value is compatible with (12), since it is always smaller than the maximal static friction coefficient.

One may wonder whether the force $\boldsymbol{F}$ may produce a pure rotation of the cylinder. From equation (8) this would imply $f = -F$ and a negative f would, in turn, imply $v_P \geqslant 0$. However, inserting $f = -F$ into equation (9) yields an angular acceleration $\alpha > 0$, meaning that the cylinder would rotate clockwise around its axis, i.e. $v_P < 0$. This contradiction in the sign of v_P means that such a motion can never happen.

The work of the frictional force on the rolling cylinder

The elementary work of a force $\boldsymbol{f}$ applied on a particle which is displaced by $\mathrm{d}\boldsymbol{r}$ is

$$\delta W = \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{r}. \tag{17}$$

In the case of rolling, one is dealing with a system of particles and a difficulty arises in applying equation (17) to the frictional force since $\boldsymbol{f}$ is not always applied to the same particle. The issue of the work done by the static force has motivated several interesting discussions in the pedagogical literature.

The 'centre-of-mass' equation (Benson 1996), which relates the pseudowork with the variation of kinetic energy (and the 'rotation centre-of-mass equation'), is, of course, an appropriate tool to study rigid bodies. Using it one concludes that the static friction produces zero net work, irrespective of its direction. Considering a cylinder rolling down an incline (without slipping), Carnero *et al* showed that the pseudowork done by the frictional force associated with the translation of the centre-of-mass is equal in magnitude but has the opposite sign to the pseudowork associated with the torque produced by friction relative to the centre-of-mass.

When the force is not applied to the same particle, Bruhat (1967) claims that the following definition of infinitesimal work

$$\delta W = \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}t \tag{18}$$

is more appropriate, $\boldsymbol{v}$ being the velocity of the particle acted upon by force $\boldsymbol{f}$. Considering the work of the frictional force in the example discussed earlier, in (18) $\boldsymbol{f}$ is the frictional force and one should replace $\boldsymbol{v}$ by $\boldsymbol{v}_\mathrm{P}$ in that equation. Whenever rolling occurs without slipping, $\boldsymbol{v}_\mathrm{P} = 0$ (see table 1) and (18) gives zero elementary work. Therefore the total work is also zero. If slippage occurs, the directions of $\boldsymbol{v}_\mathrm{P}$ and $\boldsymbol{f}$ (now a kinetic frictional force) are opposite (see table 1) and the elementary work given by (18) is always negative. The total work of the kinetic friction is negative, preventing the total kinetic energy of the cylinder from increasing as it would if $\boldsymbol{F}$ were the only horizontal force producing work. Associated with slipping motion, there is dissipative work, whose effect is to increase the temperature of the cylinder and/or plane. We refer the reader to the interesting discussion by Sousa and Pina (1997) on the work of the frictional force in rolling cylinders and its relation with the laws of thermodynamics.

Conclusions

The direction of the frictional force on a rigid body can be either the direction of the centre-of-mass velocity or the opposite one. Moreover, that force may not do any work, although the centre-of-mass of the system may move.

These results can only be observed in systems of particles. However, the concepts of friction and work are introduced for just a single particle and, in that case, these results never happen. Only to this extent can the mentioned results be considered surprising, and, in fact, to some students they are! Clearly, if a body has only sliding motion (no rotation) the study of its motion reduces to the study of its centre-of-mass motion. As long as rotations take place, the analysis of the frictional force and the work it does requires a detailed knowledge of the motion of the particle(s) where the frictional force(s) is (are) applied.

The rolling cylinder on a horizontal plane, acted upon by a constant horizontal force, shows a variety of situations, which we found most clarifying with regard to the subtleties of the direction of the friction and the work done by that force.

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