TA 6: Conditional Choice Probability (CCP) Estimation

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Motivation

Full-solution techniques are **computationally challenging**.

CCP estimation: avoid solving the DP in each interation of the estimation algorithm.

Advantages:

- Less efficient than full-solution methods, but faster. $\Rightarrow$ Robustness checks.
- More transparent about the sources of variation in the data that identify the model parameters.
- Expand the set of problems that can be handled.

In a nutshell

This methods build on the seminal work of **Hotz and Miller (1993)**.

Main idea: individual choices (reflected in CCPs) contain rich information on expectations about future outcomes.

There exists a **mapping** between conditional value functions $v_{jt}(\boldsymbol{x}_t)$ and CCPs $\boldsymbol{p}_t(\boldsymbol{x}_t)$ that, in general, can be inverted:

$$\psi_j(\boldsymbol{p}_t(\boldsymbol{x}_t)) \equiv V_t(\boldsymbol{x}_t) - v_{jt}(\boldsymbol{x}_t) \Leftrightarrow V_t(\boldsymbol{x}_t) \equiv v_{jt}(\boldsymbol{x}_t) + \psi_j(\boldsymbol{p}_t(\boldsymbol{x}_t)).$$

Representing the mapping as a function of the CCPs + (nonparametric) **estimates** of the CCPs $\Rightarrow$ **no need to solve** for the value functions.

Bike replacement problem

Model

- **Time is discrete**: $t = 1, ..., \infty$.

- **Choice variable**: bikes can be either **maintained** or **replaced**,

$$d_t = \{j : j \in \mathcal{D} = \{0, 1\}\}$$

  with $d_{jt} = \mathbb{1}\{d_t = j\}$ and $d_{0t} + d_{1t} = 1$.

- **State variable**:

$$a_{t+1} = \begin{cases} 1 & \text{if} \quad d_t = 1, \\ a_t + 1 & \text{if} \quad d_t = 0. \end{cases}$$

- If the bike is A years old, it is replaced for sure.

Bike replacement problem

Model

One-period utility function:

$$U(d_t, a_t, \varepsilon_t) = \begin{cases} -\theta_R + \varepsilon_{1t} & \text{if} \quad d_t = 1, \\ -\theta_{M1} a_t - \theta_{M2} a_t^2 + \varepsilon_{0t} & \text{if} \quad d_t = 0. \end{cases}$$

Conditional value functions:

$$v_{jt}(a_t) = u_{jt}(a_t) + \beta V_{t+1}(a_{t+1}).$$

Bike replacement problem

CCP representation

By the Type-I extreme value assumption, we can rewrite the conditional value functions:

$$
\begin{aligned}
v_{jt}(a_t) &= u_{jt}(a_t) + \beta \ln \left(\sum_{h \in \mathcal{D}} \exp\left(v_h(a)\right) \right) \\
&= u_{jt}(a_t) + \beta \ln \exp\left(v_1(a)\right) \left(\sum_{h \in \mathcal{D}} \exp\left(v_h(a) - v_1(a)\right) \right).
\end{aligned}
$$

The **conditional choice probabilities (CCPs)** are

$$
p_{jt}(a_t) = \frac{\exp\left(v_{jt}(a_t)\right)}{\sum_{h \in \mathcal{D}} \exp\left(v_{ht}(a_t)\right)}.
$$

Bike replacement problem

CCP representation

Taking replacement $(j = 1)$ as base category:

$$p_{1t}(a_t) = \frac{1}{1 + \exp(v_{0t}(a_t) - v_{1t}(a_t))} \qquad p_{0t}(a_t) = \frac{\exp(v_{0t}(a_t) - v_{1t}(a_t))}{1 + \exp(v_{0t}(a_t) - v_{1t}(a_t))}.$$

It is possible to **invert the mapping** between CCPs and conditional value functions:

$$\ln p_{1t}(a_t) = -\ln\left(1 + \exp(v_{0t}(a_t) - v_{1t}(a_t))\right)$$

$$-\ln p_{1t}(a_t) = \ln\left(1 + \exp(v_{0t}(a_t) - v_{1t}(a_t))\right) = \ln\left(\sum_{h \in \mathcal{D}} \exp\left(v_h(a_t) - v_1(a_t)\right)\right).$$

Bike replacement problem

CCP representation

Then the conditional value functions are now

$$\begin{aligned} v_{jt}(a_t) &= u_{jt}(a_t) + \beta \ln \left(\sum_{h \in \mathcal{D}} \exp\left(v_h(a)\right) \right) \\ &= u_{jt}(a_t) + \beta v_{1t+1} - \beta \ln p_{1t+1}(a_{t+1}). \end{aligned}$$

In particular:

$$\begin{aligned} v_{1t} &= -\theta_R + \beta v_{1t+1} - \beta \ln p_{1t+1}(1) \\ v_{0t}(1) &= -\theta_{M1} - \theta_{M2} + \beta v_{1t+1} - \beta \ln p_{1t+1}(2). \\ v_{0t}(2) &= -2\theta_{M1} - 4\theta_{M2} + \beta v_{1t+1} - \beta \ln p_{1t+1}(3). \\ v_{0t}(3) &= -3\theta_{M1} - 9\theta_{M2} + \beta v_{1t+1} - \beta \ln p_{1t+1}(4). \\ v_{0t}(4) &= -4\theta_{M1} - 16\theta_{M2} + \beta \ln v_{1t+1} - \beta \underbrace{\ln p_{1t+1}(5)}_{=0}. \end{aligned}$$

Bike replacement problem
CCP representation

Payoff differences:

$$v_{0t}(1) - v_{1t} = \theta_R - \theta_{M1} - \theta_{M2} + \beta \ln p_{1t+1}(1) - \beta \ln p_{1t+1}(2).$$
$$v_{0t}(2) - v_{1t} = \theta_R - 2\theta_{M1} - 4\theta_{M2} + \beta \ln p_{1t+1}(1) - \beta \ln p_{1t+1}(3).$$
$$v_{0t}(3) - v_{1t} = \theta_R - 3\theta_{M1} - 9\theta_{M2} + \beta \ln p_{1t+1}(1) - \beta \ln p_{1t+1}(4).$$
$$v_{0t}(4) - v_{1t} = \theta_R - 4\theta_{M1} - 16\theta_{M2} + \beta \ln p_{1t+1}(1).$$

We can obtain (nonparametric) estimates of p_1 to compute the payoff differences and substitute them in the **log-likelihood** function:

$$\mathcal{L}_N(\boldsymbol{\theta}) = \sum_{t=1}^{T} \sum_{i=1}^{N} d_{it} \ln \left(\frac{1}{1 + \exp(v_{0t}(a_t) - v_{1t})} \right) + (1 - d_{it}) \ln \left(\frac{\exp(v_{0t}(a_t) - v_{1t})}{1 + \exp(v_{0t}(a_t) - v_{1t})} \right).$$

Bike replacement problem

Hotz-Miller (CCP) Estimation

1. Formulate the **dynamic programming problem** (conditional value functions $v_{jt}(a_t)$ and CCPs $p_{jt}(a_t)$).

2. **Invert the mapping** between $v_{jt}(a_t)$ and $p_{jt}(a_t)$.

3. Obtain **(nonparametric) estimates** of $p_{jt}(a_t)$. A simple bin estimator is

$$\hat{p}(d_t = j | a_t = x) = \frac{\sum_{i=1}^{N} \sum_{t=1}^{T} \mathbb{1}\{d_{it} = j\} \mathbb{1}\{a_{it} = a\}}{\sum_{i=1}^{N} \sum_{t=1}^{T} \mathbb{1}\{a_{it} = a\}}.$$

4. Substitute p_{jt} by $\hat{p}_{jt}$ in the inverted mapping that goes into the **log-likelihood function**.

5. **Maximize the log-likelihood** to get $\hat{\boldsymbol{\theta}}_{\text{CCP}}$.

NPL algorithm

Aguirregabiria and Mira (2002) propose a nested pseudo-likelihood (NPL) algorithm that swaps Rust's NFXP, using the CCP representation:

- **Inner loop**: Hotz-Miller estimation of the model parameters, starting from consistent estimates of the CCPs, and later using the CCPs from the outer loop as input. $\Rightarrow \hat{\boldsymbol{\theta}}^{(0)}_{NPL} = \hat{\boldsymbol{\theta}}_{\text{CCP}}$.

- **Outer loop**: with $\hat{\boldsymbol{\theta}}^{(k)}_{NPL}$, update the CCPs.

Repeat K times or until reaching convergence in $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{p}}$. The estimates $\hat{\boldsymbol{\theta}}^{(k)}_{NPL}$ are consistent for any $k = 1, \ldots, K$.

Table 1: Bike Replacement Problem - Estimation Results

	NFXP	CCP	NPL
θ_R	5.718	10.151	11.364
	(0.596)	(0.438)	(0.421)
θ_{M1}	1.632	7.140	8.103
	(0.385)	(0.340)	(0.329)
θ_{M2}	-0.374	-1.469	-1.649
	(0.070)	(0.060)	(0.059)

Standard errors in parentheses.