TA 4: Dynamic Discrete Choice Models. Full-Solution Approaches

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Structural Econometrics for Labor Economics
and Industrial Organization

IDEA PhD Program
Winter 2023

Model

Outline

Let's analyze the **maintenance** decisions made by a biking company in Barcelona.

- **Time is discrete**: $t = 1, ..., T$ (with T finite or infinite).
- **Choice variable**: bikes can be either **maintained** or **replaced**,

$$d_t = \{j : j \in \mathcal{D} = \{0, 1\}\}$$

  with $d_{jt} = \mathbb{1}\{d_t = j\}$ and $d_{0t} + d_{1t} = 1$.

- **State variable**: a_t is the age of the bike. It is observed by the econometrician and evolves according to

$$a_{t+1} = \begin{cases} 1 & \text{if} \quad d_t = 1, \\ a_t + 1 & \text{if} \quad d_t = 0. \end{cases}$$

- If the bike is A years old, it is replaced for sure.

Utility function

Trade-off: replacing $\Rightarrow$ replacement cost, lower maintenance cost; keeping $\Rightarrow$ saves the replacement cost, larger maintenance cost.

The one-period utility function is

$$U(d_t, a_t, \varepsilon_t) = \begin{cases} -\theta_R + \varepsilon_{1t} & \text{if} \quad d_t = 1, \\ -\theta_{M1} a_t - \theta_{M2} a_t^2 + \varepsilon_{0t} & \text{if} \quad d_t = 0. \end{cases}$$

where ε_{0t} and ε_{1t} are unobserved by the econometrician, i.i.d. distributed as Type-I extreme value, and meet all Rust's assumptions.

Conditional value function

Given our assumptions, we can write the **conditional value function**:

$$v_{jt}(a_t) = u_{jt}(a_t) + \beta V_{t+1}(a_{t+1}).$$

The Type-I extreme value assumption implies that the **Emax** can be written as a function of $v_{jt}(a_t)$:

$$V_{t+1}(a_{t+1}) = \ln \sum_{h \in \mathcal{D}} \exp\{v_{ht+1}(a_{t+1})\} + \gamma$$

$$v_{jt}(a_t) = u_{jt}(a_t) + \beta \ln \left(\sum_{h \in \mathcal{D}} \exp \left\{ v_{ht+1}(a) \right\} \right),$$

where the Euler-Mascheroni constant γ is irrelevant for utility maximization.

Conditional value function

In our model, given five possible states ($A = 5$), we have

$$\begin{aligned}
V_t(1) &= \ln\left(\exp\{v_{1t}\} + \exp\{v_{0t}(1)\}\right) \\
V_t(2) &= \ln\left(\exp\{v_{1t}\} + \exp\{v_{0t}(2)\}\right) \\
V_t(3) &= \ln\left(\exp\{v_{1t}\} + \exp\{v_{0t}(3)\}\right) \\
V_t(4) &= \ln\left(\exp\{v_{1t}\} + \exp\{v_{0t}(4)\}\right) \\
V_t(5) &= -\theta_R + \beta V_{t+1}(1),
\end{aligned}$$

where

$$\begin{aligned}
v_{1t} &= -\theta_R + \beta \ln\left(\exp\{v_{0t+1}(1)\} + \exp\{v_{1t+1}\}\right) \\
v_{0t}(1) &= -\theta_{M1} - \theta_{M2} + \beta \ln\left(\exp\{v_{0t+1}(2)\} + \exp\{v_{1t+1}\}\right) \\
v_{0t}(2) &= -2\theta_{M1} - 4\theta_{M2} + \beta \ln\left(\exp\{v_{0t+1}(3)\} + \exp\{v_{1t+1}\}\right) \\
v_{0t}(3) &= -3\theta_{M1} - 9\theta_{M2} + \beta \ln\left(\exp\{v_{0t+1}(4)\} + \exp\{v_{1t+1}\}\right) \\
v_{0t}(4) &= -4\theta_{M1} - 16\theta_{M2} + \beta \ln\left(\exp\{v_{1t+1}\}\right).
\end{aligned}$$

Full-Solution Estimation

Log-likelihood

We have **longitudinal** data $\{d_{it}, a_{it}\}_{i=1,\ldots,N}^{t=1,2,\ldots,T_i}$.

The **log-likelihood** of the sample is:

$$\mathcal{L}_N(\boldsymbol{\theta}) = \sum_{i=1}^{N} \ln \Pr\left(d_{it}, ..., d_{iT_i}, a_{it}, ..., a_{iT_i}; \boldsymbol{\theta}\right) = \sum_{i=1}^{N} \ell_i(\boldsymbol{\theta}).$$

Given Markovian structure and conditional independence, we can **factorize**:

$$\ell_i(\boldsymbol{\theta}) = \sum_{t=1}^{T_i} \ln \Pr\left(d_{it} | a_{it}; \boldsymbol{\theta}\right) + \ln \Pr\left(a_{it} | a_{it-1}, d_{it-1}; \boldsymbol{\theta}\right) + \ln \Pr\left(a_{i1}; \boldsymbol{\theta}\right).$$

In our model, the log-likelihood function is

$$\mathcal{L}_N(\boldsymbol{\theta}) = \sum_{t=1}^{T} \sum_{i=1}^{N} d_{it} \ln p_{1t}(a_t) + (1 - d_{it}) \ln p_{0t}(a_t).$$

Conditional choice probabilities

Given the CLOGIT assumption, the conditional choice probabilities $p_{jt}(a_t)$ are conditional logit type:

$$p_{jt}(a_t) = \mathbb{E}\left[d^*_{jt}|a_t\right] = \frac{e^{v_{jt}(a_t)}}{\sum_{h\in\mathcal{D}} e^{v_{ht}(a_t)}}.$$

In our model, in particular:

$$\begin{aligned}
p_{0t}(a_t) =& \frac{e^{v_{0t}(a_t)}}{e^{v_{1t}(a_t)} + e^{v_{0t}(a_t)}} = \frac{1}{1 + e^{v_{1t} - v_{0t}(a_t)}} \\
p_{1t}(a_t) =& \frac{e^{v_{1t}(a_t)}}{e^{v_{1t}(a_t)} + e^{v_{0t}(a_t)}} = \frac{e^{v_{1t} - v_{0t}(a_t)}}{1 + e^{v_{1t} - v_{0t}(a_t)}} = 1 - \frac{1}{1 + e^{v_{1t} - v_{0t}(a_t)}}.
\end{aligned}$$

Sequential partial likelihood estimation

Parameters to estimate: $\boldsymbol{\theta} = (\boldsymbol{\theta}_U, \boldsymbol{\theta}_a)$.

- $\boldsymbol{\theta}_a$: vector of parameters determining the law of motion of a_t.
- $\boldsymbol{\theta}_U$: remaining (payoff) parameters of the model.

Steps:

1. Estimate $\boldsymbol{\theta}_a$: $\hat{\boldsymbol{\theta}}_{\boldsymbol{a}} = \arg\max_{\boldsymbol{\theta}_a} \sum_{i=1}^{N} \sum_{t=2}^{T_i} \ln \Pr(a_{it} | a_{it-1}, d_{it-1}; \boldsymbol{\theta}_a)$ (*not required in our case*).
2. Estimate $\boldsymbol{\theta}_U$: $\hat{\boldsymbol{\theta}}_{\boldsymbol{U}} = \arg\max_{\boldsymbol{\theta}_u} \sum_{i=1}^{N} \sum_{t=1}^{T_i} \ln \Pr(d_{it} | a_{it}; \boldsymbol{\theta}_U, \hat{\boldsymbol{\theta}}_a)$.
3. *Optional*: single iteration for the full likelihood optimization (Newton-Raphson or BHHH) using $\hat{\boldsymbol{\theta}} = (\hat{\boldsymbol{\theta}}_U, \hat{\boldsymbol{\theta}}_{\boldsymbol{a}})$ as starting values to obtain a consistent, efficient estimator.

Nested fixed-point algorithm

Rust's nested fixed-point algorithm can be applied both to the partial and the full likelihood estimation.

The algorithm is composed by an inner loop and an outer loop:

- **Inner loop**: for each evaluation of $\boldsymbol{\theta}_U$, solve the dynamic programming problem.

- **Outer loop**: iterate over $\boldsymbol{\theta}_U$ to maximize the log-likelihood function.

Nested fixed-point algorithm

Inner loop

Given a guess of $\boldsymbol{\theta}_U$, find the fixed point of the dynamic problem.

Steps:

1. Make an initial guess for EV.
2. Given $\boldsymbol{\theta}_U$ and EV, compute the conditional value function $v_j(x)$ at all possible states.
3. Update your guess of EV according to:

$$EV(a) = \ln \sum_{j \in \mathcal{D}} \exp\{v_j(a)\}.$$

4. Iterate until convergence.

Nested fixed-point algorithm

How to code the algorithm

We are interested in writing a function that:

1. Takes $\boldsymbol{\theta}$ and the data as inputs.
2. Solves the fixed-point problem for each value of $\boldsymbol{\theta}_U$ in the inner loop.
3. Computes the conditional choice probabilities.
4. Returns the value of the likelihood function.

Infinite horizon

Keep in mind that the number of states is **finite**: $a_t = 1, ..., 5$. If the time horizon is **infinite** ($T = \infty$) we can estimate $\boldsymbol{\theta}$ by following the next steps:

1. **Formulate the dynamic programming problem**: conditional value functions $v_{jt}(a_t)$ and the Emax $V_t(a_t)$.

2. **Value function iteration**: find the fixed point of the dynamic problem.

3. Formulate the **conditional choice probabilities**.

4. Construct the **log-likelihood** function.

5. Solve the log-likelihood maximization problem with respect to $\boldsymbol{\theta}$.

Infinite horizon

Conditional choice probabilities

For each state a and decision $j \in \{0, 1\}$, formulate the conditional choice probabilities.

Take the first period $t = 1$:

$$\mathcal{L}_N(t=1) = \sum_{i=1}^{N} d_{i1} \ln p_{11}(1) + (1 - d_{i1}) \ln p_{01}(1),$$

where

$$p_{01}(1) = \frac{1}{1 + e^{v_{11} - v_{01}(1)}} \qquad p_{11}(1) = \frac{e^{v_{11} - v_{01}(1)}}{1 + e^{v_{11} - v_{01}(1)}} = 1 - \frac{1}{1 + e^{v_{11} - v_{01}(1)}}$$

$$v_{11} - v_{01}(1) = [-\theta_R + \beta V_2(1)] - [-\theta_{M1} - \theta_{M2} + \beta V_2(2)].$$

Do the same for $t = 2, \ldots, 5$ and compute $\mathcal{L}_N(\boldsymbol{\theta}) = \sum_{t=1}^{T} \sum_{i=1}^{N} \ell_{it}(\boldsymbol{\theta})$.

Finite horizon

We have a **finite number of states** $a_t = 1, 2, 3$ and a **finite time horizon** $(T = 3)$. We can follow the next steps:

1. **Formulate the dynamic programming problem**: conditional value functions $v_{jt}(a_t)$ and the Emax $V_t(a)$.

2. Solve the dynamic programming problem by **backwards induction**.

3. Formulate the **conditional choice probabilities**.

4. Construct the **log-likelihood** function.

5. Solve the log-likelihood maximization problem with respect to $\boldsymbol{\theta}$.

Finite horizon

Backwards induction

Given the three possible states and $t = 1, 2, 3$, when we solve by backward induction we have:

$$v_{13}(3) = v_{1t} = v_1 = -\theta_R$$

$$v_{02}(2) = -2\theta_{M1} - 4\theta_{M2} + \beta v_{13}(3) = -2\theta_{M1} - 4\theta_{M2} - \beta\theta_R$$

$$\begin{aligned} v_{01}(1) &= -\theta_{M1} - \theta_{M2} + \beta \ln\left(\exp\{v_{02}(2)\} + \exp\{v_{12}\}\right) \\ &= -\theta_{M1} - \theta_{M2} + \beta \ln\left(\exp\{-2\theta_{M1} - 4\theta_{M2} - \beta\theta_R)\} + \exp\{-\theta_R\}\right). \end{aligned}$$

Finite horizon

Conditional choice probabilities

For each state a and decision $j \in \{0, 1\}$, formulate the conditional choice probabilities.

Take the second period $t = 2$:

$$\mathcal{L}_N(t = 2) = \sum_{i=1}^{N} d_{i2} \ln p_{12}(1) + (1 - d_{i2}) \ln p_{02}(2),$$

where:

$$p_{02}(1) = \frac{1}{1 + e^{v_{12} - v_{02}(1)}} \qquad p_{12}(2) = \frac{e^{v_{12} - v_{02}(2)}}{1 + e^{v_{12} - v_{02}(2)}} = 1 - \frac{1}{1 + e^{v_{12} - v_{02}(2)}}$$

$$v_{12} - v_{02}(2) = -\theta_R - (-2\theta_{M1} - 4\theta_{M2} - \beta\theta_R)\,.$$

Do the same for $t = 1$ and compute $\mathcal{L}_N(\boldsymbol{\theta}) = \sum_{t=1}^{T} \sum_{i=1}^{N} \ell_{it}(\boldsymbol{\theta})$.