Kinematics, Dynamics and Control of a Two Link Planar Manipulator

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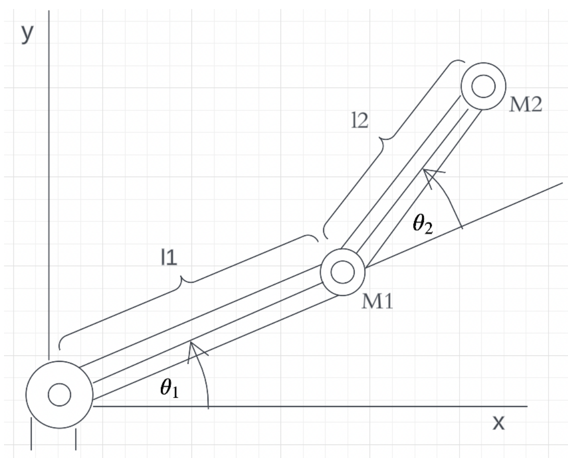
***Abstract*—Abstract—A robotic arm having two links is considered in this report for modeling and control. First the dynamic model of the robot is obtained using the so-called Lagrange equation, then a robust control strategy based on the conventional sliding mode control is introduced to control the motion of the robot at specific position for pick and place tasks along with PID control scheme. The results of the two controllers are compared. From the simulation results the SMC is found to be superior to the PID controller in term of fast and robust response yet with higher control input. High join speeds are observed in case of SMC which are related to the high control signals.**

***Index Terms*—Index Terms—dynamic model, robot manipulator, PID control, SMC control**

I. Dynamic Model



Two Link Manipulator

II. The Kinematics

Below we will introduce the kinematics for each link in which we label as l_1 and l_2

$$\begin{aligned} x_1 &= l_1 \sin\theta_1 \\ y_1 &= l_1 \cos\theta_1 \\ x_2 &= l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \\ y_2 &= l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) \end{aligned} \tag{1}$$

Note: $s_1 = \sin\theta_1$, $s_2 = \sin\theta_2$, $c_1 = \cos\theta_1$ and $c_2 = \cos\theta_2$

A. Inverse Kinematics

The elbow-up configuration determines the appropriate joint angles given the end-effector coordinates. Eq.(2) represents only the elbow-down configuration, as described in Section 1.

$$\begin{aligned} \theta_2 &= \frac{x_2^2 + y_2^2 - l_1^2 - l_2^2}{-2l_1 l_2}, \\ \theta_1 &= \tan^{-1}\left(\frac{y_2}{x_2}\right) - \tan^{-1}\left(\frac{l_2 \sin(\theta_2)}{l_1 + l_2\cos(\theta_2)}\right) \end{aligned} \tag{2}$$

These equations are used to determine the manipulator's initial position relative to a specific point in 2D space. From this point, we model the Jacobian inverse to calculate both the velocity kinematics and the next end-effector position the manipulator should reach.

III. The Jacobian

A key point in using the Jacobian is to analyze the manipulator in motion. Starting with Eq.(1) where we take the derivative with respect to time for x_2 and y_2.

The Generic form

$$J = \begin{bmatrix} J_v \\ J_w \end{bmatrix} \tag{3}$$

Where J_v maps linear velocities and J_w maps angular velocity for both our joints.

$$\dot{x}_2 = -\dot{\theta}_1 l_1 \sin\theta_1 - \dot{\theta}_1 l_2 \sin(\theta_1+\theta_2) - \dot{\theta}_2 l_2 \sin(\theta_1+\theta_2), \tag{4}$$

$$\dot{y}_2 = \dot{\theta}_1 l_1 \cos\theta_1 + \dot{\theta}_1 l_2 \cos(\theta_1+\theta_2) + \dot{\theta}_2 l_2 \cos(\theta_1+\theta_2) \tag{5}$$

The velocity kinematics can be expressed as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}, \tag{6}$$

where the Jacobian matrix J is given by:

$$J = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \tag{7}$$

Which then happens to be in the form:

$$J = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \end{bmatrix}$$

Given Eq(6), we can solve for the new joint angles ($\dot{\theta_1}$,$\dot{\theta_1}$) iteratively at each time step, allowing us to update the manipulator's configuration and calculate the new end-effector position.

IV. Manipulator Dynamic Equations

A. The Lagrangian

We can use the Lagrangian to form the state-space representation of a n-link manipulator by calculating both the potential and kinetic energy of our two-link system.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} = 0$$

$$\mathcal{L} = KE - PE$$

Using the Lagrangian formulation allows us to derive the equations of motion in state-space form and systematically compute the dynamic matrices, including the mass matrix, Coriolis and centrifugal terms, and gravitational forces. These matrices are essential for analyzing and controlling the manipulator's dynamics

The potential and kinetic energies of the system are given by:

$$KE = \frac{1}{2}m_1(\dot{x_1^2} + \dot{y_1^2}) + \frac{1}{2}m_2(\dot{x_2^2} + \dot{y_2^2}) \tag{8}$$

$$PE = \frac{1}{2}m_1 g l_1 s_1 + m_2 g(l_1 s_1 + \frac{1}{2}l_2 s_{12}) \tag{9}$$

For both l_2 and l_2 our joint torques come out to:

$$\begin{aligned}\tau_1 = {} & m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2(2\ddot{\theta}_1 + \ddot{\theta}_2) \\ & + (m_1 + m_2)l_1^2\ddot{\theta}_1 - m_2 l_1 l_2 s_2 \dot{\theta}_2^2 \\ & - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1.\end{aligned} \tag{10}$$

$$\tau_2 = m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2). \tag{11}$$

These equations, while derived for a relatively simple two-link manipulator, already exhibit significant complexity. This highlights the increasing mathematical and computational challenges that arise when extending such analyses to manipulators with six or more links.
Looking at Eq.(10) and Eq.(11) we get the dynamic equation written in the form

$$\tau = M(\Theta)\ddot{\Theta} + C(\Theta, \dot{\Theta}) + G(\Theta) \tag{12}$$

B. The state space equation

Let us define M as our $n \times n$ mass matrix:

$$M(\Theta) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

Where the elements of the mass matrix:

$$M_{11} = l_2^2 m_2 + 2l_1 l_2 m_2 c_2 + l_1^2(m_1 + m_2)$$
$$M_{12} = l_2^2 + l_1 l_2 m_2 c_2$$
$$M_{21} = M_{12} \qquad M_{22} = l_2^2 m_2$$

Let $C(\Theta, \dot{\Theta})$ be the Coriolis and Centripetal matrix

$$C(\Theta, \dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}$$

G represents the Gravity term or matrix of our two link arm:

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

and

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = F \tag{13}$$

Where F is torque exerted by the actuators in each joint.

V. Control

A. PID Controller

PID controller is show to be:

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau)\, d\tau + K_D \dot{e(t)} \tag{14}$$

In this context, K_P, K_D, and K_I represent the proportional, derivative, and integral gain matrices, respectively, which are symmetric positive-definite matrices. The term $e(\theta_i) = e_{id} - e_i$ denotes the deviation from the desired angle e_{id} where $i = 1, 2$. Here, the robot actuators are considered as ideal sources of forces and torques. Under this assumption, the dynamic model of an n-DOF robot is given by:

$$F = M(\Theta)\ddot{\Theta} + C(\Theta, \dot{\Theta}) + G(\Theta) \tag{15}$$

The main task now is to propose a PID controller, as shown in Eq. (14), to the dynamic model given in Eq.(15). An additional state variable is introduced to account for the integral action of the PID control law in Eq.(15). We denote the new state variable by ξ, such that $\dot{\xi} = \tilde{\theta}$, where $\dot{\xi}$ is the time derivative of ξ. The PID control law is then given by:

$$F = K_P e(t) + K_I \xi + K_D \dot{e(t)}, \tag{16}$$

where $\dot{\xi} = e(\theta)$
We can then rewrite our model

$$M(\Theta)\ddot{\Theta} + C(\Theta, \dot{\Theta}) + G(\Theta) = K_P e(t) + K_I \xi + K_D \dot{e(t)} \tag{17}$$

Then solving for $\ddot{\Theta}$

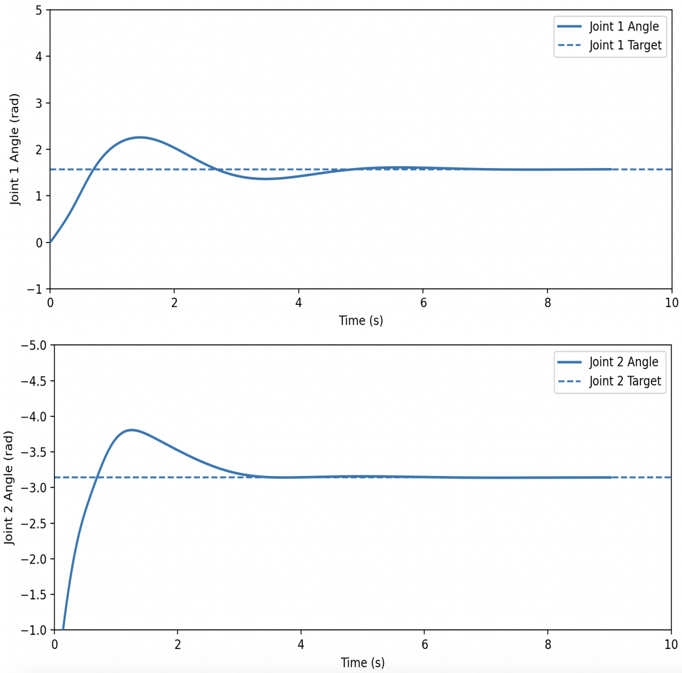
$$\ddot{\Theta} = M(\Theta)^{-1}[C(\Theta, \dot{\Theta}) - G(\Theta)] + \hat{F} \tag{18}$$

Where $\hat{F} = M(\Theta)^{-1}[F]$ and $\hat{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

VI. Simulation and Prototype Ideas

VII. Results and Discussion

This section would detail the outcomes of the simulations and prototype testing. Analysis should include:



VIII. Societal Impact of Research on Two-Link Planar Manipulators

The study and development of two-link planar manipulators hold significant potential to influence numerous aspects of society positively. While this research focuses on the fundamental dynamics and control of simple robotic systems, its implications extend far beyond academia. Below, we outline the key societal impacts:

A. Advancement in Robotic Automation

The foundational principles derived from two-link planar manipulator research contribute directly to the advancement of automation. These principles are instrumental in designing efficient robotic arms used in manufacturing, healthcare, and agriculture. For instance, assembly line robots rely on accurate modeling and control techniques like those developed in this research to perform precise, repetitive tasks, improving productivity and reducing costs.

B. Educational Impact

Research on two-link planar manipulators serves as an accessible entry point for students and researchers in robotics. By providing a simplified yet rigorous framework for studying kinematics, dynamics, and control, this research fosters the development of future engineers and scientists. These skills can later be applied to more complex robotic systems, such as six-degree-of-freedom manipulators or autonomous systems.

C. Applications in Healthcare

Insights from this research can influence the design of robotic prosthetics and rehabilitation devices. By leveraging precise control strategies and modeling techniques, these devices can provide smoother, more natural movements for individuals with physical disabilities, enhancing their quality of life.

D. Precision Agriculture and Sustainability

Simplified robotic systems, derived from the principles of planar manipulators, can be adapted for precision agriculture. Tasks such as planting, harvesting, and pesticide application require high levels of accuracy and repeatability, which this research supports. Furthermore, these applications contribute to sustainable farming practices by optimizing resource usage.

E. Cost-Effective Robotic Solutions

Developing and understanding simplified robotic systems, such as the two-link planar manipulator, makes robotics accessible to smaller industries and research institutions with limited resources. This democratization of robotics research can lead to innovations in sectors previously unable to adopt advanced automation.

F. Innovation in Human-Robot Interaction

The two-link planar manipulator serves as a fundamental model for improving human-robot interaction. Understanding how to precisely control robotic arms lays the groundwork for collaborative robots (cobots) that can safely and effectively work alongside humans in industrial and domestic environments.

G. Future Developments

This research provides a stepping stone for developing more complex robotic systems. By addressing challenges in control and dynamics for a two-link planar system, researchers can scale these insights to six-axis manipulators or multi-robot systems. Such advancements are crucial for the robotics industry's future, particularly in autonomous systems and artificial intelligence integration.

In summary, this research on two-link planar manipulators has the potential to transform various sectors by enabling innovation in robotics. Its societal impact spans improving quality of life, advancing education, and enabling sustainable development, highlighting its importance in addressing contemporary global challenges.

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