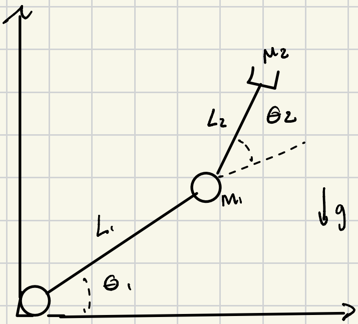


Point at $M_1 \to (x_1, y_1)$

$x_1 = L_1 \cos\theta_1$

$y_1 = L_1 \sin\theta_1$

Calculating Kinetic Energy, KE

Point at $M_2 \to (x_2, y_2)$

$x_2 = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2)$

$y_2 = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)$

$\dot{x}_1 = -L_1 \dot{\theta}_1 \sin(\theta_1)$

$\dot{y}_1 = L_2 \dot{\theta}_1 \cos(\theta_1)$

$\dot{x}_2 = [-L_1 \sin\theta_1 - L_2 \sin(\theta_1+\theta_2)]\dot{\theta}_1 - L_2 \sin(\theta_1+\theta_2)\dot{\theta}_2$

$\dot{y}_2 = L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2)\dot{\theta}_1 + L_2 \cos(\theta_1+\theta_2)\dot{\theta}_2$

$v_1 \to$ velocity M_1 $\quad v_1^2 = \dot{x}_1^2 + \dot{y}_1^2$

$v_2 \to$ velocity M_2 $\quad v_2^2 = \dot{x}_2^2 + \dot{y}_2^2$

$$\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1+\theta_2) & -L_2 \sin(\theta_1+\theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) & L_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$KE = \frac{1}{2} m v^2 \to$ Total KE of the System $= \underbrace{KE_{M_1} + KE_{M_2}}_{\text{Kinetic Energy of } M_1 \,\&\, M_2}$

$$KE = \frac{1}{2} M_1 v_1^2 + \frac{1}{2} M_2 v_2^2$$

$$KE_1 = \frac{1}{2} M_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} M_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$KE_1 = \frac{1}{2} M_1 \left[(-L_1 \dot{\theta}_1 \sin(\theta_1))^2 + (L_1 \dot{\theta}_1 \cos(\theta_1))^2\right] + \underbrace{\frac{1}{2} M_2 [\dot{x}_2^2 + \dot{y}_2^2]}_{\text{KE of } M_2}$$

$$\underbrace{KE_1 = \frac{1}{2} M_1 L_1^2 \dot{\theta}_1^2}_{\text{KE of } M_1}$$

Simplified we get the Kinetic Energy of the System.

Re derive this equation

$$KE = \frac{1}{2}(M_1 + M_2) L_1^2 \dot{\theta}_1^2 + \frac{1}{2} M_2 L_2^2 \dot{\theta}_2^2 + M_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\tau_1 = m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 C_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1+m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 S_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g C_{12}$$
$$+ (m_1+m_2) l_1 g C_1$$

$$\tau_2 = m_2 l_1 l_2 C_2 \ddot{\theta}_1 + m_2 l_1 l_2 S_2 \dot{\theta}_1^2 + m_2 l_2 g C_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2)$$

o General Structure of Manipulator Dynamic Equations

Creating the State Space equation for n-link robot

$$\tau = M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + G(\theta)$$

$$\begin{bmatrix} (m_1+m_2) l_1^2 + 2 m_2 l_1 l_2 \cos(\theta_2) + m_2 l_2^2 & m_2 (l_1 l_2 \cos\theta_2 + l_2^2) \\ m_2 (l_1 l_2 \cos\theta_2 + l_2^2) & m_2 l_2^2 \end{bmatrix} = M(\theta)$$

$$\begin{bmatrix} -m_2 l_1 l_2 \dot{\theta}_2 \sin\theta_2 & -2\dot{\theta}_1 m_2 l_1 l_2 \sin\theta_2 \\ m_2 l_1 l_2 \dot{\theta}_1 \sin\theta_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = C(\theta)[\dot{\theta}, \dot{\theta}_1] \quad \dot{\theta} \rightarrow \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Coriolis & Centrifugal forces

or

$$\begin{bmatrix} -m_2 l_1 l_2 \sin(\theta_2) \dot{\theta}_2^2 - 2 m_2 l_1 l_2 \sin(\theta_2) \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 \sin(\theta_2) \dot{\theta}_1^2 \end{bmatrix} = C(\theta, \dot{\theta})$$

$$\begin{bmatrix} (m_1+m_2) l_1 g \cos(\theta_1) + m_2 g L_2 \cos(\theta_1+\theta_2) \\ m_2 g L_2 \cos(\theta_1+\theta_2) \end{bmatrix} = g(\theta)$$

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) = F$$

Simulation

$$\ddot{\theta} = M^{-1}(\theta)\left[\tau - V(\theta,\dot{\theta}) - G(\theta) - F(\theta,\dot{\theta})\right]$$

$\theta(0) = \theta_0$

$\dot{\theta}(0) = 0$

$\ddot{\theta}(0) = 0$

Made with Goodnotes

PE of 2 link

$$P_1 = M_1 g L_1 \sin(\theta_1)$$
$$P_2 = M_2 g \left[L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)\right]$$
$$\Big\} \; P_1 + P_2 =$$

Introduce Langrangian Dynamics

$$L = KE - PE$$

$$\left[\tfrac{1}{2}(M_1+M_2) L_1^2 \dot\theta_1^2 + \tfrac{1}{2} M_2 L_2^2 \dot\theta_2^2 + M_2 L_1 L_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2)\right] - \left[g L_1 (M_1 + M_2)\sin(\theta_1) + M_2 g L_2 \sin(\theta_2)\right]$$

Simplifying we get

$$L = \tfrac{1}{2}(M_1+M_2) L_1^2 \dot\theta_1^2 + \tfrac{1}{2} M_2 L_2^2 \dot\theta_2^2 + M_2 L_1 L_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2) - (M_1 + M_2)$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot\theta_i}\right] - \frac{\partial L}{\partial \theta_i} = \tau \qquad i = 1, 2$$

$\cos(\theta_1 - \theta_2)$

$\hookrightarrow -\sin(\theta_1 - \theta_2)$

$$\frac{\partial L}{\partial \dot\theta_1} = (M_1 + M_2) L_1^2 \dot\theta_1 + M_2 L_1 L_2 \dot\theta_2 \cos(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \dot\theta_2} = M_2 L_2^2 \dot\theta_2 + M_2 L_1 L_2 \dot\theta_1 \cos(\theta_1 - \theta_2) \leftarrow \text{Trig Property}$$

$$\frac{\partial L}{\partial \theta_1} = -M_2 L_1 L_2 \dot\theta_1 \sin(\theta_1 - \theta_2) - (M_1 + M_2) g L_1 \cos(\theta_1)$$

$$\frac{\partial L}{\partial \theta_2} = M_2 L_1 L_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - M_2 g L_2 \cos(\theta_2)$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot\theta_1}\right] = (M_1 + M_2) L_1^2 \ddot\theta_1 + M_2 L_1 L_2 \ddot\theta_2 \cos(\theta_1 - \theta_2) - M_2 L_1 L_2 \dot\theta_2 (\dot\theta_1 - \dot\theta_2) \sin(\theta_1 - \theta_2).$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot\theta_2}\right] = M_2 L_2^2 \ddot\theta_2 + M_2 L_1 L_2 \ddot\theta_1 \cos(\theta_1 - \theta_2) - M_2 L_1 L_2 \dot\theta_1 (\dot\theta_1 - \dot\theta_2) \sin(\theta_1 - \theta_2).$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$(M_1+M_2) L_1^2 \ddot{\theta}_1 + M_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) - M_2 L_1 L_2 \dot{\theta}_1 (\theta_1-\theta_2) \sin\theta(\theta_1-\theta_2)$$
$$+ M_2 L_1 L_2 \dot{\theta}_1 \sin(\theta_1-\theta_2) + (M_1+M_2) g L_1 \cos(\theta_1) = \tau_1$$

$$\boxed{(M_1+M_2) L_1^2 \ddot{\theta}_1 + M_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2)} + \boxed{M_2 L_1 L_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2)} + (M_1+M_2) g L_1 \cos(\theta_1) = \tau_1$$

$$\boxed{M_2 L_2^2 \ddot{\theta}_2 + M_2 L_1 L_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2)} \boxed{- M_2 L_1 L_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2)} + M_2 g L_2 \cos(\theta_2) = \tau_2$$

$\ddot{\theta}$ $M(\theta)$

$$\underbrace{\begin{bmatrix} (M_1+M_2)L_1^2 & M_2 L_1 L_2 \cos(\theta_1-\theta_2) \\ M_2 L_1 L_2 \cos(\theta_1-\theta_2) & M_2 L_2^2 \end{bmatrix}}_{M(\theta)}, \overbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}^{\ddot{\theta}} \quad \underbrace{\begin{bmatrix} M_2 L_1 L_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) \\ -M_2 L_1 L_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) \end{bmatrix}}_{C(\theta,\dot{\theta})}$$

$M(\theta) \rightarrow$ Mass or Inertia matrix

In the form $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$

mass-matrix

$C(\theta,\dot{\theta}) =$ Coriolis & centrifugal forces

∘ Forces that act outwardly due to the rotation of the links

$$\underbrace{\begin{bmatrix} (M_1+M_2) g L_1 \cos(\theta_1) \\ M_2 g L_2 \cos(\theta_2) \end{bmatrix}}_{G(\theta)}$$

↳ Gravitational Forces

$M_{22} \rightarrow$ Inertia of 2nd Link

$M_{11} \rightarrow$ Inertia of 1st Link includes both masses of M_1 & M_2

$$\boxed{M(\theta)\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + C(\theta,\dot{\theta}) + G(\theta) = F}$$

$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = F$ Torques applied to joints

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta}) + G(\theta) = F \longrightarrow \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}$$

Solving for $\ddot{\theta}$

Note $\rightarrow$ $M^{-1}(\theta)F = \hat{F}$

$$\ddot{\theta} = -M^{-1}(\theta)\left[C(\theta,\dot{\theta}) + G(\theta)\right] + \hat{F} \longrightarrow \begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

$$F = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = M(\theta) = \begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

Calculating error

$$e(\theta_1) = \theta_{1f} - \theta_1$$

actual position ↑ Theoretical position

$$e(\theta_2) = \theta_{2f} - \theta_2$$

Introducing PID controllers

- Common technique for controlling a system with input

$$f = k_p e + k_D \dot{e} + k_I \int e\,dt$$

Initial positions

$$\theta_0 = \begin{bmatrix}\theta_1(0)\\ \theta_2(0)\end{bmatrix}$$

$$f_1 = k_p e(\theta_1) + k_D \dot{e}(\theta_1) + k_{I1}\int e(\theta_1)\,dt$$

$$f_1 = k_p(\theta_{1f} - \theta_1) \underbrace{- k_D(\theta_{1f} - \theta_1)}_{-\dot{\theta}_1} + k_{I1}\int(\theta_{1f} - \theta_1)\,dt$$

$$f_2 = k_p e(\theta_2) + k_D \dot{e}(\theta_2) + k_{I1}\int e(\theta_2)\,dt$$

$$f_2 = k_p(\theta_{2f} - \theta_2) - k_D(\theta_{2f} - \theta_2) + k_{I2}\int(\theta_{2f} - \theta_2)\,dt$$

$$\begin{bmatrix}f_1\\ f_2\end{bmatrix} = \begin{bmatrix}k_p(\theta_{1f}-\theta_1) - k_D\dot{\theta}_1 + k_{I1}\int(\theta_{1f}-\theta_1)\,dt\\ k_p(\theta_{2f}-\theta_2) - k_D\dot{\theta}_2 + k_{I2}\int(\theta_{2f}-\theta_2)\,dt\end{bmatrix} = \hat{F}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = M(\theta_1)\begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

To Implement PID controller -> we must implement these States

$$X_1 = \int e(\theta_1)\,dt \qquad \dot{X}_1 = e(\theta_1) = \theta_{1f} - \theta_1 \,,\ \dot{X}_2 = e(\theta_2) = \theta_{2f} - \theta_2$$

$$\dot{X}_1 = \theta_1 f - \theta_1$$

$$X_2 = \int e(\theta_2)\,dt \qquad \dot{X}_2 = \theta_2 f - \theta_2$$

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = -M^{-1}(\theta)\left[C(\theta, \dot{\theta}) + G(\theta_1) \right] + \begin{bmatrix} K_{P_1}(\theta_{1f} - \theta_1) - K_{D_1}\dot{\theta}_1 + K_{I_1}X_1 \\ K_{P_2}(\theta_{2f} - \theta_2) - K_{D2}\dot{\theta}_2 + K_{I2}X_2 \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = M(\theta) \begin{bmatrix} K_{P_1}(\theta_{1f} - \theta_1) - K_{D_1}\dot{\theta}_1 + K_{I_1}X_1 \\ K_{P_2}(\theta_{2f} - \theta_2) - K_{D2}\dot{\theta}_2 + K_{I2}X_2 \end{bmatrix}$$

$$U_1 = X_1 \quad U_2 = X_2 \quad U_3 = \theta_1 \,,\ U_4 = \theta_2 \,,\ U_5 = \dot{\theta}_1 \,,\ U_6 = \dot{\theta}_2$$

$$\dot{U}_1 = \dot{X}_1$$

$$\dot{U}_2 = \dot{X}_2 \qquad \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = M(\theta) \begin{bmatrix} K_{P_1}(\theta_{1f} - U_3) - K_{D_1}U_5 + K_{I_1}U_1 \\ K_{P_2}(\theta_{2f} - U_4) - K_{D2}U_6 + K_{I2}U_2 \end{bmatrix}$$

$$\ddot{U}_5 = \ddot{\theta}_1 = \phi(t, U, U_1, U_2, U_3, U_4, U_5, U_6)$$

$$\ddot{U}_6 = \ddot{\theta}_2 = \varphi(t, U, U_1, U_2, U_3, U_4, U_5, U_6)$$

$$\begin{bmatrix} \phi \\ \varphi \end{bmatrix} = -M^{-1}(\theta)\left[C(\theta, \dot{\theta}) + G(\theta) \right] + \begin{bmatrix} K_{P_1}(\theta_{1f} - U_3) - K_{D_1}U_5 + K_{I_1}U_1 \\ K_{P_2}(\theta_{2f} - U_4) - K_{D2}U_6 + K_{I2}U_2 \end{bmatrix}$$

$$\theta = \begin{bmatrix} U_3 \\ U_4 \end{bmatrix}$$

$$(M_1 + M_2)L_2^2\ddot{\theta}_1 + M_2L_1L_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + M_2L_1L_2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + (M_1 + M_2)gL_1\cos(\theta_1) = \tau_1, \longrightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}_1}\right] - \frac{\partial L}{\partial \theta_1} = \tau_1$$

$$\underbrace{M_2L_2^2\ddot{\theta}_2 + M_2L_1L_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - M_2L_1L_2\dot{\theta}_1^2\sin(\theta_1 - \theta_2)}_{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}_2}\right]} + \underbrace{M_2gL_2\cos(\theta_2)}_{-\frac{\partial L}{\partial \theta}} = \tau_2, \longrightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \theta_2}\right] - \frac{\partial L}{\partial \theta_1} = \tau_2$$

$$\begin{bmatrix} (M_1+M_2)L_2^2 & M_2L_1L_2\cos(\theta_1-\theta_2) \\ M_2L_1L_2\cos(\theta_1-\theta_2) & M_2L_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$\delta = \frac{M_2}{M_1+M_2}$$

$$\ddot{\theta}\ M(\theta) + C(\theta, \dot{\theta}) + G(\theta) = F$$

$$\ddot{\theta} = M^{-1}(\theta)\left[F - C(\theta, \dot{\theta}) - G(\theta)\right] + F$$

$$\hat{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

PID controller $\quad f = K_p e(t) + K_D \dot{e}(t) + K_I \int e(t)\, dt$

Components:

- Proportional Gain

$$U_p(t) = K_p e(t)$$

↑ Proportional gain

○ Large error → Large correction

○ Small error → Small correction

○ (K_p) Proportional Gain determ sensitivity of the controller to current error

○ Temperature Control System → Current temperature below set point → proportional term increases heating power proportionally to temperature difference.

- Integral Term

$$U_I(t) = K_i \int_0^t e(\tau)\, d\tau$$

○ Accumulates error over time which address residual error from the proportional gain K_p

○ Summing error over time eliminate persistent offset

K_i → determines how aggressively the controller reacts to accumulated error.

Example: If a car is not reaching target velocity because of an uphill, accumulator integrator component will increase throttle over time to overcome incline.

○ Overshooting is an issue

Derivative term:

$$U_D(t) = K_D \frac{de(t)}{dt}$$

○ Reacts to how quickly the error is changing, applies correction before the error becomes to large.

○ helps reduce overshoot provides a dampening effect.

Iteration example: (Extremely Simple)

Following PID Gains

- Set point (SP) = 10
- Current Value (PV) = 7

error = SP − PV = 3

$K_p = 1.0$ (proportional Gain)

$K_I = 0.1$ (Integral Gain)

$K_D = 0.05$ (Derivative Gain)

lets say previous error = 2

Time step (Δt) = 1

Integral Sum of errors over time = 1.5

accumulated error over time

$P = [3(1.0)]K_p = 3(1.0) = 3$

$I = (e(t)\Delta t + I)K_I = 0.1(4.5) = 0.45$

$D = \frac{e(t) - e_{prev}}{\Delta t} K_D = 0.05$

$P + I + D = 3.5$ System is adjusted 3.5

New PV = current PV + PID output = 10.5

So compared to our SP we would overshoot by 0.5

- error will decrease as current value is closer to 10
- Integral term will continue accumulating (smaller) error.
- Derivative term will adjust based on error potentially becoming smaller as the system stabilizes.

Combined affect of PID components

$$u(t) = K_p e(t) + K_D \dot{e}(t) + K_I \int_0^t e(\tau)\, d\tau$$

Code Construction Notes

$\Delta t = t_f - t_i$

Proportional term

error = Setpoint − Current Value

$\underline{P = K_p(\text{error})}$

Integral term

integral += error (Δt)

$\underline{I = K_I(\text{integral})}$

Derivative term

If $\Delta t > 0$:

derivative = (error − previous error) / Δt

else:

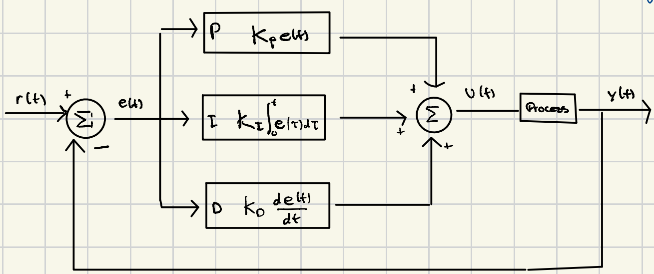
derivative = 0

$\underline{D = K_D(\text{derivative})}$

Updating error

Previous error = error

Output $y(t) = P + I + D$



$y(t)$ = output

$$G(s) = \frac{1}{(s+1)(s+2)} = G_{PID}(s) = k_p + \frac{k_i}{s} + k_d s$$

o Closed-loop Transfer function: $T(s) = \dfrac{\left(k_p + \frac{k_i}{s} + k_d s\right)\frac{1}{(s+1)(s+2)}}{\left(1 + k_p + \frac{k_i}{s} + k_d s\right)\frac{1}{(s+1)(s+2)}}$

Characteristic Equation:

$$1 + G_{PID}(s)\, G(s) = 0$$