

# eSTA approach to Josephson Junction

## 1 Josephson Junction from the ground-up

Here I will summarize the paper [GO07] where the Josephson Junction is derived. We are considering  $N$  particles in a double well potential, the form of which is

$$V_{dw} = \frac{1}{2}m \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right) + \frac{V_0}{2} \left( 1 + \cos \frac{2\pi}{d_{sw}} x \right). \quad (1)$$

Usually this problem is really hard to solve at involves many body particles etc. However the single particle spectrum of the double well is almost degenerate for the first two levels, while there is a wide gap between the second and third eigentates of the double well Hamiltonian. This allows us to introduce the *two mode approximation* in which only the first two eigentates are considered. It has been shown how with this assumption the Hamiltonian of the system can be written as

$$H = \hat{H}_0 + \hat{H}_{int} \quad (2)$$

$$\hat{H}_0 = \int d\mathbf{r} \left( -\frac{\hbar^2}{2m} \hat{\Psi}^\dagger \nabla^2 \hat{\Psi} + \hat{\Psi}^\dagger V_{dw} \hat{\Psi} \right), \quad (3)$$

$$\hat{H}_{int} = \frac{g}{2} \int d\mathbf{r} \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \quad (4)$$

with  $\hat{\Psi}$  field operator that (loosely speaking) creates / annihilates a particle in the position  $\mathbf{r}$  and  $g$  is the coupling constant. Now we take into account only the mean-field ground and excited states (which I assume they are  $\Phi_g = |N, 0\rangle$  and  $\Phi_e = |0, N\rangle$ ) so we can rewrite the general wavefunction  $\hat{\Psi}$  as

$$\hat{\Psi} = \hat{c}_g \Phi_g + \hat{c}_e \Phi_e \quad (5)$$

with  $\hat{c}_g^\dagger$  and  $\hat{c}_e^\dagger$  the creation operator for the ground and excited states respectively. A more convenient choice in this case is usually to choose the left and right operators

$$\hat{c}_l^\dagger = \frac{1}{\sqrt{2}} \left( \hat{c}_g^\dagger + \hat{c}_e^\dagger \right) \quad \hat{c}_r^\dagger = \frac{1}{\sqrt{2}} \left( \hat{c}_g^\dagger - \hat{c}_e^\dagger \right). \quad (6)$$

With this choice, we can now rewrite  $\hat{\Psi}$

$$\hat{\Psi} = \frac{1}{\sqrt{2}} \left( \hat{c}_l (\Phi_g + \Phi_e) + \hat{c}_r (\Phi_g - \Phi_e) \right). \quad (7)$$

We are now in the position to insert eq. (7) into eq. (2) and by doing some calculations, we obtain the two-mode Hamiltonian

$$\hat{H}_{2M} = \frac{E_c}{8} \left( \hat{c}_r^\dagger \hat{c}_r - \hat{c}_l^\dagger \hat{c}_l \right)^2 - \frac{E_j}{N} \left( \hat{c}_l^\dagger \hat{c}_r - \hat{c}_r^\dagger \hat{c}_l \right) + \frac{\delta E}{4} \left( \hat{c}_l^\dagger \hat{c}_r - \hat{c}_r^\dagger \hat{c}_l \right)^2 \quad (8)$$

where

- $E_j$  describes the tunneling rate from one well to the other.
- $E_c$  corresponds to the local interaction within the two wells.
- $\delta E$  takes into account additional two-particles processes.

In many discussions, the last term  $\delta E$  is often neglected and thus the Hamiltonian eq. (8) can be further simplified and becomes

$$\hat{H}_{2M} = \frac{E_c}{2} \hat{n}^2 - \frac{2E_j}{N} \hat{\alpha} \quad (9)$$

with  $\hat{n}^2$  the population imbalance and  $\hat{\alpha}$  the tunneling operator. Their form is

$$\hat{n} = \frac{\hat{c}_r^\dagger \hat{c}_r - \hat{c}_l^\dagger \hat{c}_l}{2}, \quad \hat{\alpha} = \frac{\hat{c}_l^\dagger \hat{c}_r + \hat{c}_r^\dagger \hat{c}_l}{2} \quad (10)$$

and we will see that these two operators are basically the same thing as the ones in [JDTM<sup>+</sup>12]

- Write down the calculations that allow us to get from the textbook example to the two JJ Hamiltonians (the one with the angular momentum and the one with number operators)
- Explain where they got STA from
- Explain what we did with eSTA up to now
- Explain how we can improve on that with the full calculation

## 2 Josephson Junction Approximation

### 2.1 JJ Hamiltonian in continous variables

In this section we will outline the calculations that allow STA to be applied in these Josephson Junction settings. Starting from eq. (9) it can be shown that the operators  $\hat{n}$  and  $\hat{\alpha}$  behave as pseudoangular momentum operators if we set

$$\hat{J}_z = \frac{\hat{c}_r^\dagger \hat{c}_r - \hat{c}_l^\dagger \hat{c}_l}{2} = \hat{n}, \quad \hat{J}_x = \frac{\hat{c}_r^\dagger \hat{c}_l - \hat{c}_l^\dagger \hat{c}_r}{2} = \hat{\alpha}, \quad \hat{J}_y = \frac{\hat{c}_r^\dagger \hat{c}_l - \hat{c}_l^\dagger \hat{c}_r}{2i} \quad (11)$$

and we obtain the Bose Hubbard Hamiltonian

$$H_{BH} = U \hat{J}_z^2 - 2J \hat{J}_x \quad (12)$$

if the value  $U$  and  $J$  are chosen accordingly. It can also be shown that the operators defined in eq. (11) follow in fact the angular momentum relations.

By using the pseudoangular momentum approach, a system of  $N$  particles can be described as a single particle with spin  $N/2$  and the basis set is of the form  $\{|m\rangle\}$  with  $m = -N/2, \dots, N/2$  eigenstates of the  $\hat{J}_z$  operator.

The Hamiltonian of the system is then defined via eq. (12) and the general state  $|\Psi\rangle$  can be written as

$$|\Psi\rangle = \sum_{m=-N/2}^{N/2} c_m |m\rangle. \quad (13)$$

The Schrödinger equation is then written as

$$i\partial_t |\Psi\rangle = H_{BH} |\Psi\rangle \quad (14)$$

If we want to apply STA to eq. (14), we need to perform some approximations. In the following I will try to perform the same approximation they used in [JDMP10] in order to move from the discrete to the continuous variable. I will follow the calculations I found in [JDMP10] as they give a better idea on what is the Hamiltonian of the system and what are the steps and approximations we need to make in order to obtain an idealised version of the Hamiltonian where we can apply STA. My plan is to obtain the idealised version of the Hamiltonian they used in [JDTM<sup>+</sup>12]

The first thing to do is to define a new dimensionless Hamiltonian  $H_S = \frac{H_{BH}}{NJ}$  that reads

$$H_S = -\frac{2}{N} \hat{J}_x + \frac{U}{NJ} \hat{J}_z^2 = -\frac{2}{N} \hat{J} + \frac{2\Lambda}{N^2} \hat{J}_z^2 \quad (15)$$

where we defined  $\Lambda = NU/(2J)$ . The corresponding Schrödinger equation then becomes  $\frac{i}{NJ} \partial_t |\Psi\rangle = H_S |\Psi\rangle$  and if we introduce the dimensionless time  $\tau = t/J$ , it simplifies even more, becoming

$$\frac{i}{N} \partial_\tau |\Psi\rangle = \left( -\frac{2}{N} \hat{J}_x + \frac{2\Lambda}{N^2} \hat{J}_z^2 \right) |\Psi\rangle \quad (16)$$

We now want to find a differential equation for the coefficients  $c_m$  of eq. (13). In order to do that, we are going to project eq. (16) onto  $\langle m|$ . Moreover, we are going to use the fact that  $\hat{J}_x = \frac{1}{2} (\hat{J}_+ + \hat{J}_-)$ . If we are to project onto  $\langle m|$  we should remember how the ladder operators  $\hat{J}_\pm$  act. In particular, we can see we are only interested in those states  $|k\rangle$  such that  $\hat{J}_\pm |k\rangle = \beta_k |m\rangle$  with  $\beta_k$  some coefficient depending on the quantum number  $k$ . We are interested in such states as they are the ones which the projection on  $\langle m|$  is non zero

and we can see that for a fixed  $m$  only the states  $|m \pm 1\rangle$  meet the requirements. Take for example  $|m + 1\rangle$  then

$$\langle m | \hat{J}_- | m + 1 \rangle = \sqrt{\left(\frac{N}{2} + m + 1\right) \left(\frac{N}{2} - m\right)} \langle m | m \rangle = \beta_m \quad (17)$$

where we set  $\beta_m = \sqrt{\left(\frac{N}{2} + m + 1\right) \left(\frac{N}{2} - m\right)}$ .

By putting everything back together, we obtain

$$\langle m | \frac{i}{N} \partial_t | \Psi \rangle = \langle m | \tilde{H}_S | \Psi \rangle \quad (18)$$

$$\frac{i}{N} \frac{d}{dt} c_m(t) = -\frac{2}{N} (b_m c_{m+1}(t) + b_{m-1} c_{m-1}(t)) + \frac{2\Lambda}{N^2} m^2 c_m(t) \quad (19)$$

where in this case we set  $b_m = \beta_m/N$ . The result in eq. (19) gives us a Schrödinger equation for the coefficients  $c_m$  which is discrete. We now need to move from a discrete formulation to a continuous one and we are going to do that by performing a change of variable. If we look at the definition of  $b_m$ , we see that we can collect the  $N/2$  term as shown in the following

$$b_m = \frac{1}{N} \sqrt{\left(\frac{N}{2} + m + 1\right) \left(\frac{N}{2} - m\right)} = \frac{1}{N} \sqrt{\frac{N^2}{4} \left(1 + \frac{m}{N/2} + \frac{1}{N/2}\right) \left(1 - \frac{m}{N/2}\right)} \quad (20)$$

and if we define the continuous variable  $z = \frac{m}{N/2}$  and  $h = \frac{1}{N/2}$ , we obtain

$$\frac{1}{2} \sqrt{(1+z+h)(1-z)} := b_h(z) \quad (21)$$

where we can see that  $b_h(z-h) = \sqrt{(1+z)(1-z-h)}$  which can be mapped back to  $b_{m-1}$ . Additionally, if we define  $\sqrt{N/2} c_m = \psi(z)$  we see that  $\psi(z \pm h)$  can be mapped to  $c_{m \pm 1}$ . Finally, by recalling that for a function  $f(x)$  we have  $f(x \pm \epsilon) = e^{\pm \epsilon \partial_x} f(x)$  we can rewrite eq. (19) as

$$\frac{1}{2} i h \partial_t \psi(z) = -\frac{1}{2} [e^{-i\hat{p}} b_h(z) + b_h(z) e^{i\hat{p}}] \psi(z) + \frac{1}{2} \Lambda z^2 \psi(z) \quad (22)$$

where  $\hat{p} = -i h \partial_z$ . If we want to mimic the calculations made in [JDTM<sup>+</sup>12] we need to perform a Taylor expansion of both the  $e^{\pm i\hat{p}}$  part and the  $b_h(z)$  function up to the second order in  $h$  such as

$$e^{-i\hat{p}} \simeq 1 \pm h \partial_z - \frac{1}{2} h^2 \partial_z^2 \quad (23)$$

$$b_h(z) \simeq 1 + h \partial_h b_h(z)|_{h=0} + \frac{1}{2} h^2 \partial_h^2 b_h(z)|_{h=0}. \quad (24)$$

By carrying out the calculations, we obtain the following Schrödinger equation

$$i h \partial_t \psi(z) = -h^2 \partial_z (b_0(z) \partial_z \psi(z)) + [\Lambda z^2 - 2b_0(z)] \psi(z) \quad (25)$$

where  $b_0(z) = \sqrt{1-z^2}$ . We can retrieve equation (7) in [JDTM<sup>+</sup>12] by setting a new  $\tilde{h} \equiv h/2 = 1/N$ . In the following we will stick to definition of  $h = 1/N/2$  instead of using the other definition.

The last approximation we need to perform in order to obtain an oscillator-like Schrödinger equation for this system is given by neglecting the  $z$  dependence of the effective mass term and expanding the  $\sqrt{1-z^2}$  term into  $1 - z^2/2$  in the external potential term. We can finally write down the Schrödinger equation

$$i h \partial_t \psi(z) = H_{ho} \psi(z) \quad (26)$$

where the Hamiltonian of the system is given by

$$H_{ho} = -h^2 \partial_z^2 + (1 + \Lambda) z^2 = -h^2 \partial_z^2 + \frac{1}{4} \omega^2 z^2 \quad (27)$$

if we set  $\omega^2 \equiv 4(1 + \Lambda)$ .

## 2.2 Approximation to Harmonic Oscillator

### References

- [GO07] R Gati and M K Oberthaler. A bosonic josephson junction. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 40:R61–R89, 5 2007.
- [JDMP10] B. Juliá-Díaz, J. Martorell, and A. Polls. Bose-einstein condensates on slightly asymmetric double-well potentials. *Physical Review A*, 81, 6 2010.
- [JDTM<sup>+</sup>12] B. Julia-Diaz, E. Torrontegui, J. Martorell, J. G. Muga, and A. Polls. Fast generation of spin-squeezed states in bosonic josephson junctions. *Physical Review A*, 86, 12 2012.