Laboratory 3 - 24/10/16 - Solution

Exercise 1

We report the implementation of the forward Euler/centered method in the file `FEhyperbolic.m`. The domain, described by the interval $(\alpha,\beta]\times(0,T_f]$, is discretized by choosing a space step $h>0$ and a time step $\Delta t>0$ and by defining the grid points (x_j,t^n) as follows

$$x_j=\alpha+(j-1)h \quad j=1,\cdots,N_h,$$

$$t^n=(n-1)\Delta t \quad n=1,\cdots,N_t\,,$$

where $N_h:=\dfrac{\beta-\alpha}{h}+1$ and $N_t:=\dfrac{T_f}{\Delta t}+1$.
In the forward Euler/centered scheme (FE/C), we approximate the spatial derivative with a centered scheme, in combination with a forward Euler scheme for the time discretization, thus obtaining the following scheme at time t^n in the node x_j:

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}+a\frac{U_{j+1}^n-U_{j-1}^n}{2h}=0, \quad n=1,\dots,N_t-1, \quad j=2,\dots,N_h-1.$$

Rearranging the terms, the FE/C method can be written as follows:

$$U_j^{n+1}=U_j^n-\frac{a\lambda}{2}\left(U_{j+1}^n-U_{j-1}^n\right), \quad n=1,\dots,N_t-1, \quad j=2,\dots,N_h-1,$$

where $\lambda:=\frac{\Delta t}{h}$.
For the rightmost node x_{N_h}, we can not apply this scheme, since we would need a node that is outside the domain. For this reason, we use the following extrapolation:

$$U_{N_h}^{n+1}=(1-\lambda a)U_{N_h}^n+\lambda a\,U_{N_h-1}^n.$$

Note that this extrapolation comes from an upwind discretization of the equation at the node x_{N_h}.

We remark that in the implementation of the method proposed in the file `FEhyperbolic.m` the solution at each time step is not stored but just plotted. This is a choice, justified by the fact that in this case we do not need to compute derived quantities from the numerical solution.

Exercise 2

We report in the file `Lab3Es2.m` the Matlab instructions to solve the problem with the forward Euler/centered scheme by choosing different values of h and Δt. Figure 1 shows the numerical results obtained at time $t = 4.5$.

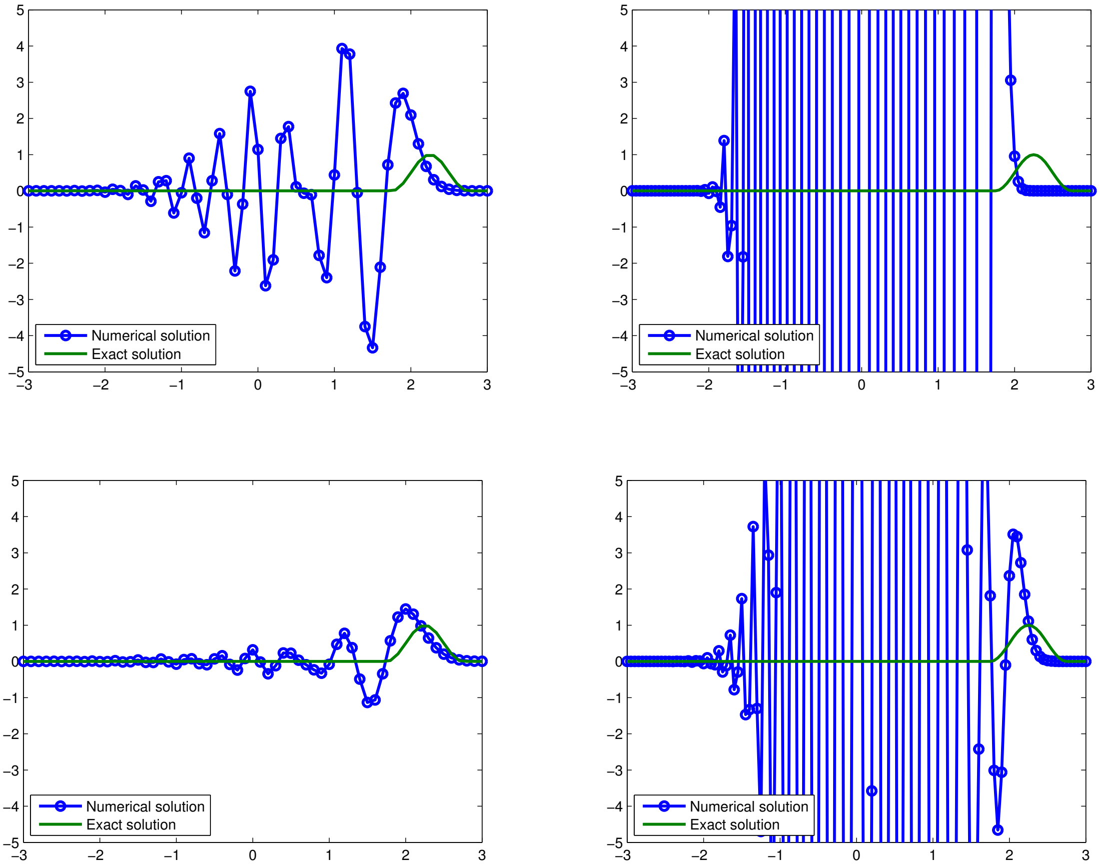
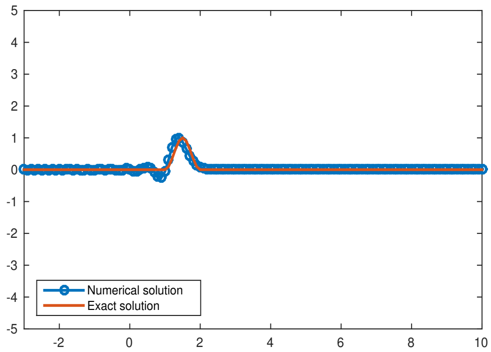


Figure 1: Numerical solutions obtained with the FE/C method, with $a = 0.5$ at time $t = 4.5$. Top-left: $h = 0.1$ and $\Delta t = 0.1$. Top-right: $h = 0.05$ and $\Delta t = 0.1$. Bottom-left: $h = 0.1$ and $\Delta t = 0.05$. Bottom-right: $h = 0.05$ and $\Delta t = 0.05$.

We can see that the solution shows numerical oscillations for any chosen h and Δt. Indeed, this scheme is unconditionally absolutely unstable. This property makes this scheme, in practice, unusable.

We point out that even though the numerical solution seems to be absolutely stable by choosing Δt very small, e.g. $\Delta t = 0.01$ with $h = 0.1$, for large values of t the oscillations become evident and the numerical solution blows up, see Figure 2.



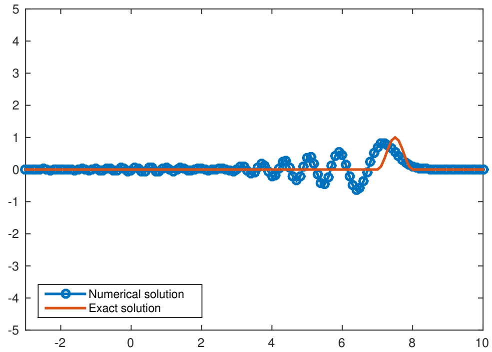


Figure 2: Numerical solutions obtained with the FE/C method, with $a = 0.5$, $h = 0.1$ and $\Delta t = 0.01$ at time $t = 4.5$ (left) and $t = 15$ (right).