Laboratory 6 - 28/11/16 - Solution

1) The domain, described by the interval $(0, L)$, is discretized by choosing a spatial step $h > 0$ and defining the grid points x_j as follows

$$x_j = jh, \quad j = 0, \dots, M,$$

where $M := \frac{L}{h}$ is the number of space intervals (thus, the number of nodes is $M + 1$). We report in the file `Elliptic.m` the implementation of the second order and centered finite difference scheme for the resolution of the proposed problem. By writing at each internal node x_j the scheme

$$-\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = F_j, \qquad j = 1, \dots, M-1,$$

where U_j is the numerical solution at node x_j, we obtain the following linear system of size $M - 1$

$$A\boldsymbol{U} = \boldsymbol{F},$$

where

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \dots & & & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & & \dots & -1 & 2 & -1 & 0 \\ 0 & \dots & & & \dots & -1 & 2 & -1 \\ 0 & \dots & & & & \dots & -1 & 2 \end{bmatrix}, \ \boldsymbol{F} = \begin{bmatrix} f(x_1) + \dfrac{y_0}{h^2} \\ f(x_2) \\ \vdots \\ \vdots \\ \vdots \\ f(x_{M-2}) \\ f(x_{M-1}) + \dfrac{y_L}{h^2} \end{bmatrix} \tag{1}$$

Note that the scheme is written only in the internal nodes, since the solution in the boundary points (x_0 and x_M) is known (Dirichlet condition). Notice also that the values of F_1 and F_{M-1} are modified by the Dirichlet conditions.

2) Figure 1 represents the solution for the case $h = 0.025$ (see file `Lab6b.m`).

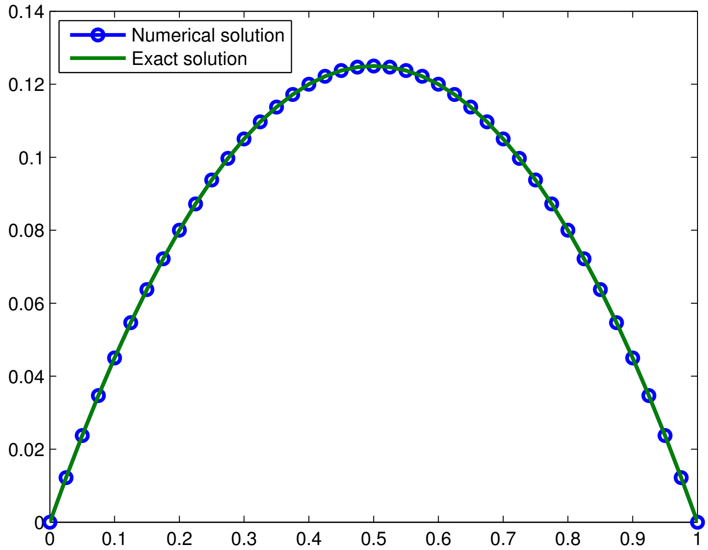

Figure 1: Numerical solution with $h = 0.025$.

3) We report in file `Lab6c.m` the computation of the order of accuracy of the method. Recall that the order of accuracy p can be estimated with the following formula

$$p \simeq log_2 \left(\frac{\|\boldsymbol{e}(h)\|_\infty}{\|\boldsymbol{e}(h/2)\|_\infty} \right).$$

Another way to estimate (qualitatively) the order of accuracy of the method is to plot (in logarithmic scale) the error as function of h. In Figure 2, we report such plot for both cases a) and b).

a) In the first case, we notice from Figure 2 that the error has order of magnitude close to the machine epsilon. Indeed, from the theory we have that the error between the numerical solution and the exact one satisfies

$$\max_j |y(x_j) - U_j| \leq Ch^2 \max_{x\in[0,L]} |f''(x)|,$$

where C is a suitable constant. Since in this case $f = 1$, the error is in fact zero. Still, we notice from the figure that the error slightly increases for decreasing h. This happens because the condition number of A increases for increasing dimension of the problem, so that the rounding errors due to the machine computations become more evident.

b) In the second case, we see from the figure that the order of accuracy of the method is 2, in accordance with the theory. Indeed, by considering 6 successive halvings of h starting from $h = 0.025$, we obtain

```
p =

    2.0013    2.0003    2.0001    2.0000    2.0000.
```

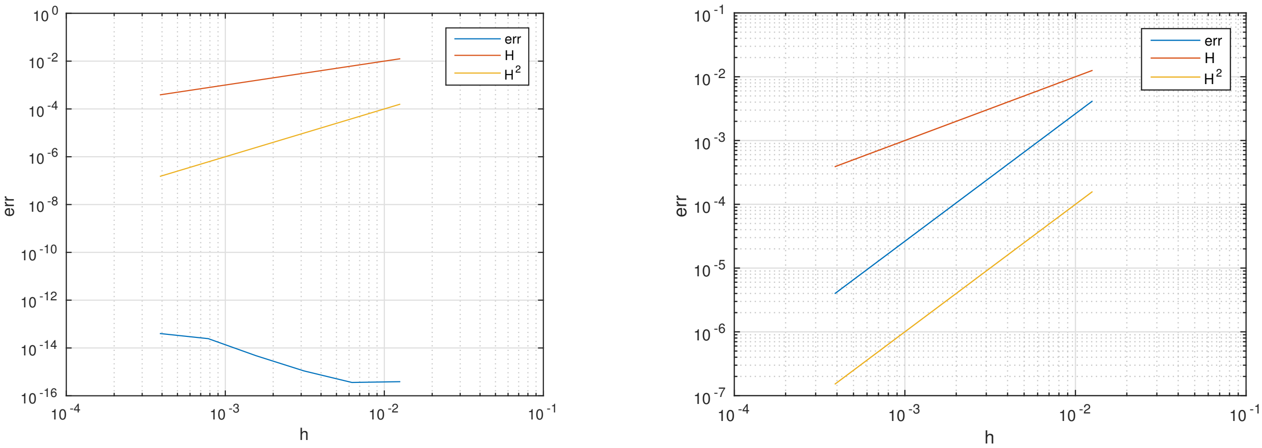

Figure 2: Error as function of h for case a) (left) and case b) (right).