Laboratory 4 - 07/11/16 - Solution

Exercise 1

The domain, described by the interval $(\alpha, \beta] \times (0, T_f]$, is discretized by choosing a spatial step $h > 0$ and a time step $\Delta t > 0$, defining the grid points (x_j, t^n) as follows

$$x_j = \alpha + (j-1)h \quad j = 1, \dots, N_h,$$

and

$$t^n = (n-1)\Delta t \quad n = 1, \dots, N_t\,,$$

where $N_h := \frac{\beta-\alpha}{h} + 1$, $N_t := \frac{T_f}{\Delta t} + 1$. We report the implementation of the Upwind (UW) method in `UWhyperbolic.m`. For this scheme, we approximate the spatial derivative with a one-side discretization, whereas for the time derivative we use the forward Euler method, thus obtaining the following scheme at time t^n and node x_j:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_j^n - U_{j-1}^n}{h} = 0, \quad n = 1, \dots, N_t - 1, \quad j = 2, \dots, N_h.$$

We observe that the transport direction determines the direction of the one-side discretization in the UW scheme, i.e. towards the left when $a > 0$ (as in this case), towards the right when $a < 0$. Rearranging the terms, the scheme reads:

$$U_j^{n+1} = U_j^n - a\,\lambda(U_j^n - U_{j-1}^n), \quad n = 1, \dots, N_t - 1, \quad j = 2, \dots, N_h,$$

where $\lambda := \frac{\Delta t}{h}$.

Exercise 2

We report the implementation of the UW and Lax-Wendroff (LW) methods in `GenericHyperbolic.m`. Both methods can be written directly from a centered scheme by adding a term proportional to the numerical diffusion:

$$U_j^{n+1} = U_j^n - a\frac{\lambda}{2}(U_{j+1}^n - U_{j-1}^n) + \mu^N \Delta t \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}. \quad (1)$$

In other terms, these schemes can be interpreted as explicit/centered schemes applied to a "perturbed/stabilized" problem, namely:

$$\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} - \mu^N \frac{\partial^2 y}{\partial x^2} = 0.$$

By properly choosing μ^N in (1), we can obtain both the UW and the LW schemes: in particular, we have $\mu^N = \frac{|a|h}{2}$ for the UW method and $\mu^N = \frac{a^2\Delta t}{2}$ for the LW method. Note that, even though we are now solving a slightly different problem, we recover the original transport equation when μ^N goes to zero, i.e. when h or Δt goes to zero.

As we saw in Laboratory 3 for the FE/C method, the centered schemes can not be applied to the rightmost node (x_{N_h}). For this reason, we will use again the following extrapolation:

$$U_{N_h}^{n+1} = (1 - \lambda a)U_{N_h}^n + \lambda a\, U_{N_h-1}^n.$$

We recall that the LW scheme can be formally obtained starting from the following Taylor's expansion:

$$\begin{aligned}
y(t^{n+1}, x_j) &= y(t^n, x_j) + \Delta t \frac{\partial}{\partial t} y(t^n, x_j) + \frac{\Delta t^2}{2}\frac{\partial^2}{\partial t^2} y(t^n, x_j) + \mathcal{O}(\Delta t^3) \\
&= y(t^n, x_j) - \Delta t a \frac{\partial}{\partial x} y(t^n, x_j) - a\frac{\Delta t^2}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial t} y(t^n, x_j) + \mathcal{O}(\Delta t^3) \\
&= y(t^n, x_j) - \Delta t a \frac{\partial}{\partial x} y(t^n, x_j) + a^2\frac{\Delta t^2}{2}\frac{\partial^2}{\partial x^2} y(t^n, x_j) + \mathcal{O}(\Delta t^3),
\end{aligned}$$

where we have used the following relation

$$\frac{\partial}{\partial t} y = -a\frac{\partial}{\partial x} y.$$

Exercise 3

We report in file `Lab4Es3.m` the Matlab instructions to solve the problem with the UW and the LW methods. The numerical results obtained with the different schemes are shown in Figure 1.

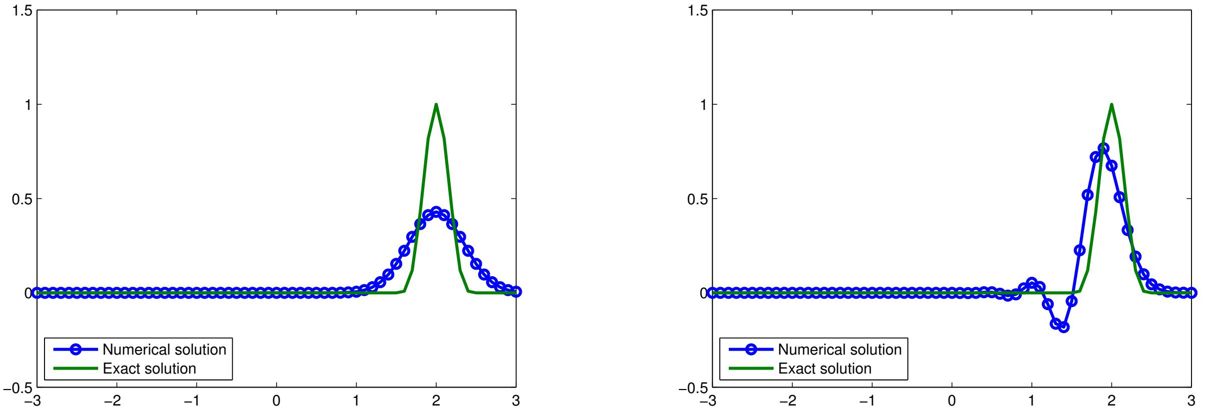

Figure 1: Numerical solution obtained with UW (left) and LW (right) at time $t = 2$ for $a = 1$, $\Delta t = 0.05$ and $h = 0.1$ (CFL $= \frac{a\Delta t}{h} = 0.5$).

Regarding the Upwind method, the numerical solution is characterized by an additional diffusion, with values that are damped down for increasing values of time. This is due to the artificial viscosity introduced by the method. However, no error in phase (*dispersion error*) is obtained.

The numerical solution obtained with the LW scheme, instead, does not exhibit damping, but it shows a light delay with respect to the exact solution (*dispersion error*). Moreover, we notice that the numerical solution obtained with the LW method exhibits oscillations where a jump in the first derivative of the initial datum occurs (*Gibbs phenomenon*). Such oscillations are "good" oscillations, since they do not blow up for increasing time.

We remark that the addition of numerical viscosity is particularly important when no physical viscosity is present in the equation we are discretizing, as is the case of the transport equation. Indeed, we proved in Laboratory 3 that the FE/C method, which does not introduce any numerical viscosity, is unconditionally absolutely unstable.

We finally recall that both the UW and LW schemes produce stable solutions provided that the CFL condition is satisfied, i.e. $\frac{a\Delta t}{h} \leq 1$. Indeed, by decreasing h and/or increasing Δt such that the CFL condition is not satisfied, the solution is not absolutely stable. As examples, we report in Figures 2 and 3 the solutions obtained with the two methods in scenarios in which the CFL is not satisfied.

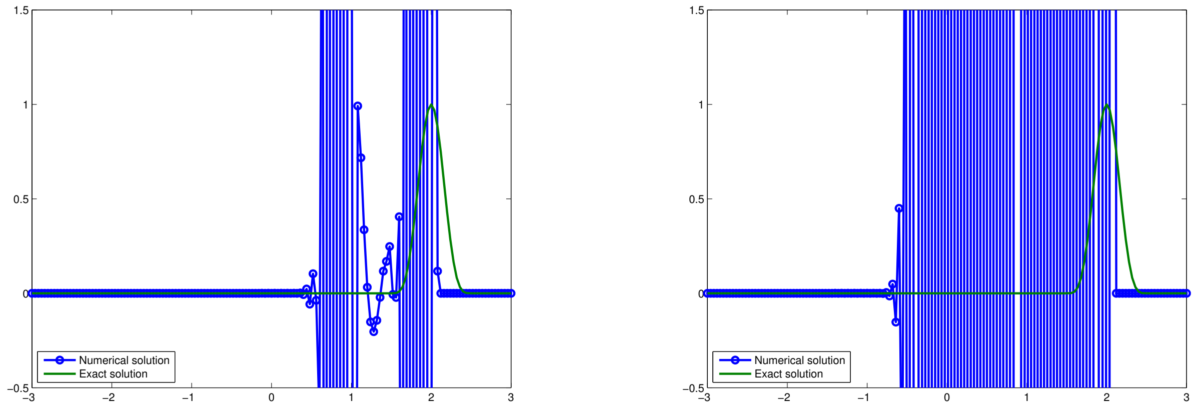

Figure 2: Numerical solution obtained with UW (left) and LW (right) at time $t = 2$ for $a = 1$, $h = 0.04$ and $\Delta t = 0.05$ (CFL $= \frac{a\Delta t}{h} = 1.25$).

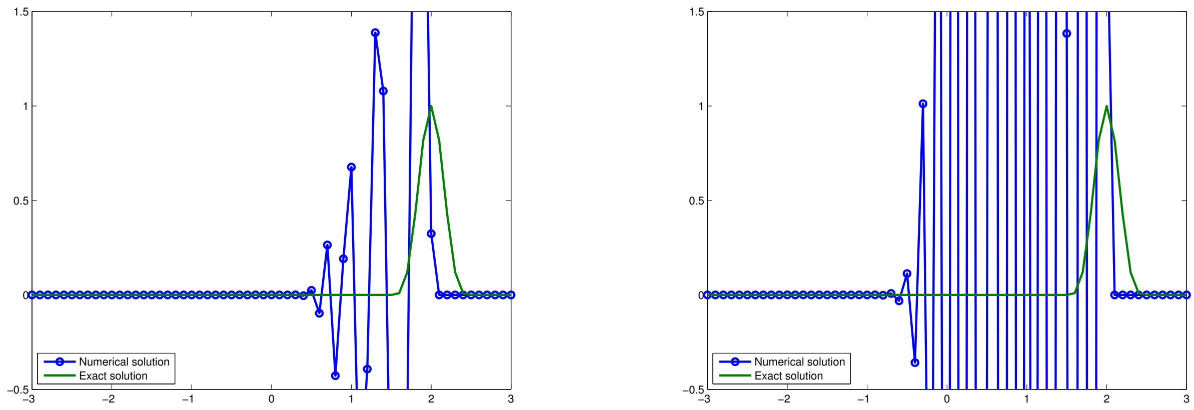

Figure 3: Numerical solution obtained with UW (left) and LW (right) at time $t = 2$ for $a = 1$, $h = 0.1$ and $\Delta t = 0.125$ (CFL $= \frac{a\Delta t}{h} = 1.25$).