Laboratory 5 - 14/11/16 - Solution

Exercise 1

Backward Euler/centered scheme. The domain, described by the interval $(\alpha, \beta] \times (0, T_f]$, is discretized by choosing a spatial step $h > 0$ and a time step $\Delta t > 0$, defining the grid points (x_j, t^n) as follows

$$x_j = \alpha + (j-1)h, \quad j = 1, \ldots, N_h,$$

and

$$t^n = (n-1)\Delta t, \quad n = 1, \ldots, N_t\,,$$

where $N_h := \frac{\beta-\alpha}{h} + 1$, $N_t := \frac{T_f}{\Delta t} + 1$. We report the implementation of the BE/C method in `BEhyperbolic.m`. For this method, we approximate the spatial derivative with a centered scheme, in combination with a backward Euler scheme for the time discretization, obtaining the following scheme at time t^n and node x_j

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2h} = 0, \quad n = 1, \ldots, N_t - 1,\ j = 2, \ldots, N_h - 1.$$

Rearranging the terms, the scheme reads

$$a\frac{\lambda}{2}U_{j+1}^{n+1} + U_j^{n+1} - a\frac{\lambda}{2}U_{j-1}^{n+1}, = U_j^n, \quad n = 1, \ldots, N_t - 1,\ j = 2, \ldots, N_h - 1. \tag{1}$$

where $\lambda := \frac{dt}{h}$. Notice that, differently from the UW and LW schemes, in this case we can not write an explicit formula for U_j^{n+1}, but we have to solve (for each n) a linear system instead. For the last node ($j = N_h$), since we can not use a centered scheme, we use the implicit UW scheme

$$\frac{U_{N_h}^{n+1} - U_{N_h}^n}{\Delta t} + a\frac{U_{N_h}^{n+1} - U_{N_h-1}^{n+1}}{h} = 0,$$

which leads to the following equation:

$$U_{N_h}^{n+1}(1 + a\lambda) - a\lambda U_{N_h-1}^{n+1} = U_{N_h}^n. \tag{2}$$

Equations (1) and (2) lead to the solution of the following linear system at each time step

$$A\mathbf{U}^{n+1} = \mathbf{F},$$

where $A \in \mathbb{R}^{(N_h-1)\times(N_h-1)}$ is a tridiagonal matrix and $\mathbf{F} \in \mathbb{R}^{(N_h-1)}$ the known term:

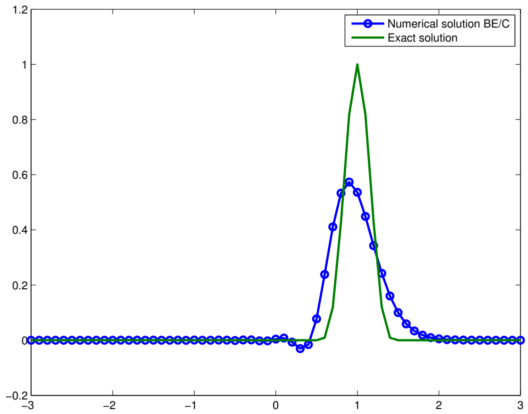
$$A = \begin{bmatrix} 1 & a\frac{\lambda}{2} & 0 & \dots & & & \dots & 0 \\ -a\frac{\lambda}{2} & 1 & a\frac{\lambda}{2} & 0 & \dots & & \dots & 0 \\ 0 & -a\frac{\lambda}{2} & 1 & a\frac{\lambda}{2} & 0 & \dots & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & & \dots & -a\frac{\lambda}{2} & 1 & a\frac{\lambda}{2} & 0 \\ 0 & \dots & & & \dots & -a\frac{\lambda}{2} & 1 & a\frac{\lambda}{2} \\ 0 & \dots & & & & 0 & -a\lambda & 1+a\lambda \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} U_2^n + a\frac{\lambda}{2}U_{inflow}^{n+1} \\ U_3^n \\ U_4^n \\ \vdots \\ U_{N_h-1}^n \\ U_{N_h}^n \end{bmatrix}.$$

Notice that equation 2 modifies the last row of A, whereas the first element of $\mathbf{F}$ is modified because of the inflow boundary condition. Indeed, if we write equation 1 for the first node where the equation is solved ($j = 2$), we obtain

$$a\frac{\lambda}{2}U_3^{n+1} + U_2^{n+1} = U_2^n + a\frac{\lambda}{2}U_{inflow}^{n+1},$$

since $U_1^{n+1} = U_{inflow}^{n+1}$ is the known inflow boundary condition, so that the corresponding term goes to the right-hand side of the equation.

We report in Figure 1 the analytical and numerical solution obtained with CFL number equal to 0.5 and 2.0 at $t = 1$.



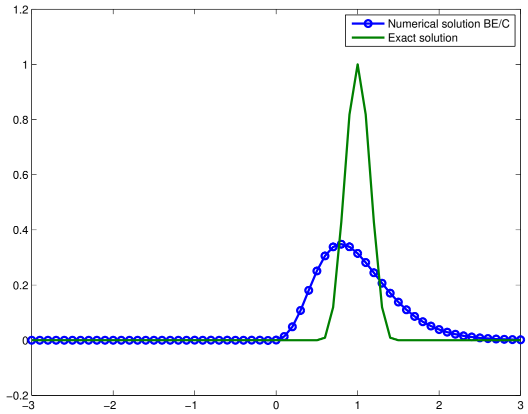


Figure 1: Numerical solution obtained with the BE/C with CFL=0.5 (left) and CFL=2.0 (right) at $t = 1$.

We remark that the numerical solution is always absolutely stable, even in the case of CFL greater than 1. This is a typical feature of the implicit method, which is however characterized by an increase in the computational cost (with respect to the explicit methods) at each time step, due to the solution of a tridiagonal linear system instead of a set of decoupled equations. Furthermore, the choice of the time step obviously influences the accuracy of the solution (see Fig. 1, right).

Exercise 2

To numerically prove the order of accuracy of the UW and LW schemes, we assume that the methods are of order r in space and s in time:

$$||\boldsymbol{e}(h,\Delta t)\|_\infty = \mathcal{O}(h^r) + \mathcal{O}(\Delta t^s),$$

where $\boldsymbol{e}(h,\Delta t)$ is the error obtained with spatial step h and time step Δt. Thus, in order to estimate the error reduction, we can use the following formula

$$\frac{\|\boldsymbol{e}(h,\Delta t)\|_\infty}{\|\boldsymbol{e}(h/2,\Delta t/2)\|_\infty} \simeq \frac{h^r + \Delta t^s}{(h/2)^r + (\Delta t/2)^s} \simeq 2^p,$$

where $p = \min(r,s)$. Hence, the order of accuracy can be estimated by

$$p \simeq log_2\left(\frac{\|\boldsymbol{e}(h,\Delta t)\|_\infty}{\|\boldsymbol{e}(h/2,\Delta t/2)\|_\infty}\right).$$

We report in file `Lab5Es2.m` the computation of the order of accuracy of both methods by using a for loop in which, at each iteration, we divide in half the value of h and Δt (with a constant CFL number equal to 0.5). Note that inside the loop we use the function `GenericHyperbolicErr.m`, which was obtained by modifying the function `GenericHyperbolic.m` implemented in the previous laboratory in order to compute the error.

For the UW and LW schemes, we have $r = s$, so the formula for the estimation of p gives both the space and time order of accuracy. In particular, for the UW scheme we obtain

```
p =

    0.3820    0.5326    0.6886    0.8153    0.8988    0.9470,
```

while for the LW scheme we have

```
p =

    1.0705    1.8621    1.9841    1.9990    1.9998    2.0000.
```

These results prove that the order of accuracy (both in space and time) of the UW scheme is 1, while the order of accuracy (both in space and time) of the LW scheme is 2.