# Laboratory 8 - 12/12/16 - Solution

## Exercise 1

a) The domain, described by the interval $(\alpha, \beta)$, is discretized by choosing a spatial step $h > 0$ and defining the grid points $x_j$ as follows

$$x_j = jh, \quad j = 0, \dots, M,$$

where $M := \frac{\beta-\alpha}{h}$ is the number of space intervals (thus, the number of nodes is $M+1$). We report in the file `AdvectionDiffusionUpwind.m` the implementation of the Upwind scheme for the resolution of the proposed problem. This is obtained by discretizing the advection term with a decentered scheme, obtaining

$$ay'(x_j) \simeq a\frac{y(x_j) - y(x_{j-1})}{h}, \quad j = 1, \dots, M-1.$$

and by applying a second order and centered scheme for the diffusion term, yielding

$$-\mu y''(x_j) \simeq -\mu\frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{h^2}, \quad j = 1, \dots, M-1.$$

These lead to the following numerical equations:

$$-\mu\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + a\frac{U_j - U_{j-1}}{h} = f(x_j), \quad j = 1, \dots, M-1,$$

where $U_j \simeq y(x_j)$. In particular, in the first and last node of the internal nodes, i.e., $x_1$ and $x_{M-1}$, we obtain

$$-\mu\frac{U_2 - 2U_1 + U_0}{h^2} + a\frac{U_1 - U_0}{h} = f(x_1)$$
$$-\mu\frac{U_2 - 2U_1}{h^2} + a\frac{U_1}{h} = f(x_1) + \mu\frac{y_\alpha}{h^2} + a\frac{y_\alpha}{h}$$

and

$$-\mu\frac{U_M - 2U_{M-1} + U_{M-2}}{h^2} + a\frac{U_{M-1} - U_{M-2}}{h} = f(x_{M-1})$$
$$-\mu\frac{-2U_{M-1} + U_{M-2}}{h^2} - a\frac{U_{M-1} - U_{M-2}}{2h} = f(x_{M-1}) + \mu\frac{y_\beta}{h^2}.$$

We notice that the equations are not written for the boundary nodes since we have Dirichlet boundary conditions.

In algebraic form, we have the following linear system of size $M-1$

$$A\boldsymbol{U} = \boldsymbol{F},$$

where $A = \mu A^\mu + a A^a$, with

$$A^\mu = \frac{1}{h^2}\begin{bmatrix} 2 & \text{-1} & 0 & \dots & & & \dots & 0 \\ \text{-1} & 2 & \text{-1} & 0 & \dots & & \dots & 0 \\ 0 & \text{-1} & 2 & \text{-1} & 0 & \dots & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & & \dots & \text{-1} & 2 & \text{-1} & 0 \\ 0 & \dots & & & \dots & \text{-1} & 2 & \text{-1} \\ 0 & \dots & & & & \dots & \text{-1} & 2 \end{bmatrix},$$

$$A^a = \frac{1}{h}\begin{bmatrix} 1 & 0 & 0 & \dots & & & \dots & 0 \\ \text{-1} & 1 & 0 & 0 & \dots & & \dots & 0 \\ 0 & \text{-1} & 1 & 0 & 0 & \dots & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & & \dots & \text{-1} & 1 & 0 & 0 \\ 0 & \dots & & & \dots & \text{-1} & 1 & 0 \\ 0 & \dots & & & & \dots & \text{-1} & 1 \end{bmatrix},$$

$$\boldsymbol{F} = \begin{bmatrix} f(x_1) + \mu\dfrac{y_\alpha}{h^2} + a\dfrac{y_\alpha}{h} \\ f(x_2) \\ \vdots \\ \vdots \\ \vdots \\ f(x_{M-2}) \\ f(x_{M-1}) + \mu\dfrac{y_\beta}{h^2} \end{bmatrix}.$$

With $h = 0.1$, we obtain the solution depicted in Figure 1, left (see file `Lab8Es1.m`). Note that the Upwind scheme does not show numerical oscillations, proving its stability even when $Pe > 1$ . The price to pay is that the numerical solution is diffusive and that the order of accuracy is only 1 (see Exercise 2).

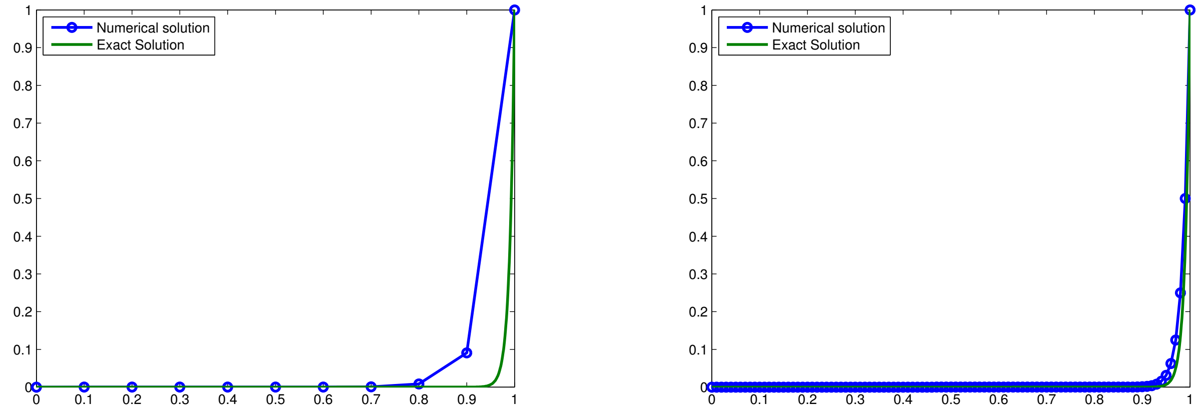


Figure 1: Numerical solution obtained with the Upwind scheme for $h = 0.1$ (left) and for $h = 0.01$ (right).

b) We report in the file `AdvectionDiffusionCenteredStabilized.m` the implementation of the centered finite difference scheme for the resolution of the proposed problem with a generic stabilization. With this implementation, the stabilization is added just by changing the definition of $\mu$, namely substituting $\mu$ with $\mu_h = \mu(1+\phi(Pe))$, where $\phi(Pe)$ is a generic function depending on the Peclet number. This corresponds to the addition of numerical viscosity to the physical one. In particular, it is easy to verify that we can obtain the Upwind scheme by choosing $\phi(Pe) = Pe$, which corresponds to adding numerical viscosity equal to $\mu Pe$. By choosing $\phi(Pe) = 0$, we can recover the second-order centered scheme.

## Exercise 2

The order of accuracy $p$ of the two methods can be estimated by the formula

$$p \simeq log_2 \left( \frac{\|\boldsymbol{e}(h)\|_\infty}{\|\boldsymbol{e}(h/2)\|_\infty} \right).$$

For the centered scheme, we obtain

```
p = 2.0047    2.0012    2.0003,
```

while for the Upwind scheme, we obtain

```
p = 0.9447    0.9712    0.9853,
```

as expected from theory (see file `Lab8Es2.m`).