Laboratory 2 - 17/10/16

Ordinary differential equations

An ordinary differential equation (ODE) is an equation that contains derivatives with respect to a unique independent variable. In the following, we will consider only first order differential equations and, in particular, we will consider the so-called *Cauchy problem*:

Find $y : \mathcal{I} \to \mathbb{R}$ such that

$$\begin{cases} \dot{y}(t) = f\left(t, y(t)\right) \quad \forall t \in \mathcal{I}, \\ y(t_0) = y_0, \end{cases} \tag{1}$$

where $\mathcal{I} \subset \mathbb{R}$, f is a given function and $\dot{y} = dy/dt$. Moreover, t_0 is the time wherein we impose the initial condition y_0 in order to fix a unique solution between the infinite admissible ones of problem (1).

The forward (explicit) Euler method allows to compute an approximate solution of the Cauchy problem through the following sequence:

$$u^{n+1} = u^n + \Delta t\, f(t^n, u^n), \quad n = 1, \dots, N_t - 1,$$

with N_t being the number of temporal nodes chosen to discretize interval $\mathcal{I}$. This scheme is obtained by using the forward Euler formula to approximate $\dot{y}(t)$ at each node t^n, $n = 1, \dots, N_t - 1$.

In the backward (implicit) Euler method we approximate $\dot{y}(t)$ with the backward Euler formula, obtaining the following sequence:

$$u^{n+1} = u^n + \Delta t\, f(t^{n+1}, u^{n+1}), \quad n = 1, \dots, N_t - 1,$$

thus leading, in general, to the solution of a non-linear equation.

Exercise 1

Consider the following reference problem:

$$\begin{cases} \dot{y}(t) = \lambda y(t), \quad t \in (t_0, T^f), \\ y(t_0) = y_0. \end{cases} \tag{2}$$

a) Implement the forward Euler method for the solution of (2).

b) Consider the following differential equation:

$$\begin{cases} \dot{y}(t) = -2y(t) \quad t \in (0, 12], \\ y(t_0) = 1. \end{cases} \tag{3}$$

Solve (3) with the forward Euler method implemented at the previous point using $\Delta t = 0.05$ and evaluate the accuracy of the numerical solution through a comparison with the exact one. Repeat the same evaluation with $\Delta t = 1.2$ and interpret the result.

c) Estimate the order of accuracy of the forward Euler method.

Exercise 2

a) Implement the backward Euler method for the solution of the reference problem (2).

b) Solve (3) with the backward Euler method using $\Delta t = 0.05$ and $\Delta t = 1.2$. Compare the computed solutions with the ones obtained at point 1b).

Exercise 3

Consider the following differential equation:

$$\begin{cases} \dot{y}(t) = \dfrac{1}{1+t^2} - 2y(t)^2 \quad t \in (0, 10], \\ y(t_0) = 0, \end{cases} \tag{4}$$

with solution

$$y(t) = \frac{t}{1+t^2}.$$

a) Modify the code related to the forward Euler method previously implemented to solve a generic Cauchy problem. In this case, the Matlab function will need $f(t, y)$ as additional input parameter (e.g. defined as an *inline* function).

b) Solve problem (4) with the forward Euler method choosing $\Delta t = 0.95$, $\Delta t = 0.9$ and $\Delta t = 0.2$ and compare the numerical solutions with the exact one.