Laboratory 7 - 05/12/16 - Solution

1) The domain, described by the interval (α, β), is discretized by choosing a spatial step $h > 0$ and defining the grid points x_j as follows

$$x_j = jh, \quad j = 0, \ldots, M,$$

where $M := \frac{\beta-\alpha}{h}$ is the number of space intervals (thus, the number of nodes is $M+1$). We report in the file `AdvectionDiffusionCentered.m` the implementation of the second order and centered finite difference scheme for the resolution of the proposed problem. By discretizing the advection term with a centered scheme of the second order, we obtain

$$ay'(x_j) \simeq a\frac{y(x_{j+1}) - y(x_{j-1})}{2h}, \quad j = 1, \ldots, M-1.$$

By applying a second order and centered scheme for the diffusion term, we get

$$-\mu y''(x_j) \simeq -\mu\frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{h^2}, \quad j = 1, \ldots, M-1.$$

These lead to the following numerical equations:

$$-\mu\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} + a\frac{U_{j+1} - U_{j-1}}{2h} = f(x_j), \quad j = 1, \ldots, M-1,$$

where $U_j \simeq y(x_j)$. In particular, in the first and last node of the internal nodes, i.e., x_1 and x_{M-1}, we obtain

$$-\mu\frac{U_2 - 2U_1 + U_0}{h^2} + a\frac{U_2 - U_0}{2h} = f(x_1)$$
$$-\mu\frac{U_2 - 2U_1}{h^2} + a\frac{U_2}{2h} = f(x_1) + \mu\frac{y_\alpha}{h^2} + a\frac{y_\alpha}{2h}$$

and

$$-\mu\frac{U_M - 2U_{M-1} + U_{M-2}}{h^2} + a\frac{U_M - U_{M-2}}{2h} = f(x_{M-1})$$
$$-\mu\frac{-2U_{M-1} + U_{M-2}}{h^2} - a\frac{U_{M-2}}{2h} = f(x_{M-1}) + \mu\frac{y_\beta}{h^2} - a\frac{y_\beta}{2h}.$$

We notice that the equations are not written for the boundary nodes since we have Dirichlet boundary conditions.

In algebraic form, we have the following linear system of size $M-1$

$$A\boldsymbol{U} = \boldsymbol{F},$$

where $A = \mu A^{\mu} + a A^{a}$, with

$$A^{\mu} = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \dots & & & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & & \dots & -1 & 2 & -1 & 0 \\ 0 & \dots & & & \dots & -1 & 2 & -1 \\ 0 & \dots & & & & \dots & -1 & 2 \end{bmatrix},$$

$$A^{a} = \frac{1}{2h}\begin{bmatrix} 0 & 1 & 0 & \dots & & & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & & \dots & 0 \\ 0 & -1 & 0 & 1 & 0 & \dots & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & & \dots & -1 & 0 & 1 & 0 \\ 0 & \dots & & & \dots & -1 & 0 & 1 \\ 0 & \dots & & & & \dots & -1 & 0 \end{bmatrix},$$

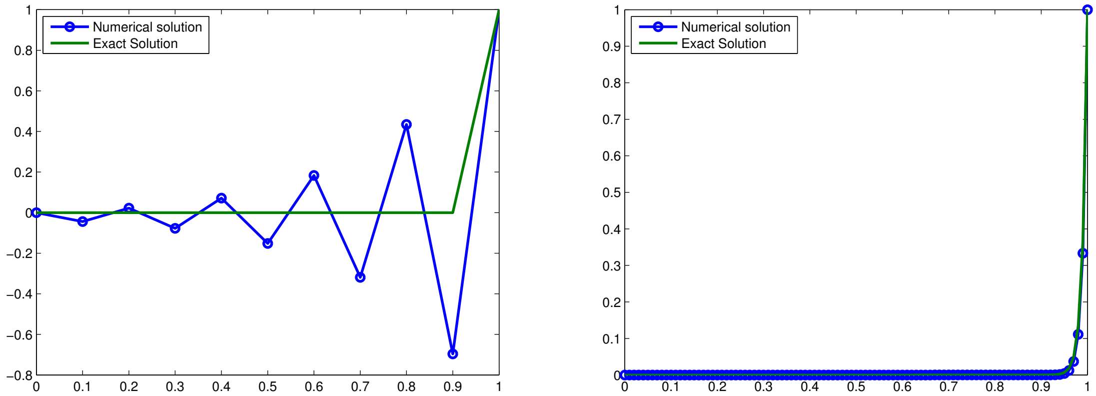
$$\boldsymbol{F} = \begin{bmatrix} f(x_1) + \mu\frac{y_\alpha}{h^2} + a\frac{y_\alpha}{2h} \\ f(x_2) \\ \vdots \\ \vdots \\ \vdots \\ f(x_{M-2}) \\ f(x_{M-1}) + \mu\frac{y_\beta}{h^2} - a\frac{y_\beta}{2h} \end{bmatrix}.$$

With $h = 0.1$ we obtain the solution depicted in Figure 1a (see file `Lab7.m`).

2) To avoid the oscillations in the numerical solution computed with the centered scheme, it is necessary to have

$$Pe_h = \frac{ah}{2\mu} < 1,$$

so $h < \frac{2\mu}{a}$. For example, with $h = 0.01$ we obtain the stable solution plotted in Figure 1b.



(a) $h = 0.1$ (b) $h = 0.01$

Figure 1: Numerical solution with the second order and centered scheme.