Free boundary minimal surfaces in space forms and an eigenvalue problem

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1 Introduction

The study of minimal surfaces goes back to at least Lagrange in the 18th century. Perhaps the most classical problem in the field is Plateau's problem, which asks about the existence of a minimal surface with prescribed boundary in $\mathbb{R}^3$. Later, Courant and others asked about the existence of surfaces of minimal area whose boundary is free to move in some prescribed fixed surface in $\mathbb{R}^3$. Such surfaces are called free boundary minimal surfaces. Both of these questions still make sense when the ambient space is a Riemannian manifold more general than Euclidean space. The next simplest Riemannian manifolds are arguably the spheres $\mathbb{S}^n(r)$ and hyperbolic spaces $\mathbb{H}^n(r)$, the only complete, simply-connected spaces of constant (nonzero) curvature. In this work we will be looking at free boundary minimal submanifolds inside a geodesic ball in these spaces.

Few examples of such surfaces are known, but they include the totally geodesic ones and the hypersurfaces of revolution constructed by Mori [21] and do Carmo and Dajczer [2]. There are various works on the properties of these surfaces, such as Fraser and Schoen [10] on uniqueness, Freidin and McGrath [11, 12] on area bounds, Li and Xiong [19] on a gap theorem and Souam [23] on stability.

A different but related topic is that of extremal eigenvalue problems. The oldest such problem is likely to be the one posed by Rayleigh, who conjectured that the disk minimizes the first eigenvalue of the Dirichlet Laplacian among all planar domains of the same area. This was settled by Faber and Krahn leading to the celebrated Faber-Krahn inequality. Szegö and Weinberger showed that the disk maximizes the first positive eigenvalue of the Neumann Laplacian among planar domains of the same area.

In 1996, Nadirashvili [22] connected the problem of maximizing Laplace eigenvalues on a closed surface with closed minimal surfaces in the sphere, thus establishing a bridge between the study of minimal surfaces and that of extremal eigenvalue problems. This was later generalized by Fraser and Schoen [8], who related the maximization problem for Steklov eigenvalues on a compact surface with boundary to free boundary minimal surfaces in the Euclidean ball. The question that motivated some of our work in this report, particularly that in Section 5, is whether free boundary minimal surfaces in a geodesic ball in a space form have an associated extremal eigenvalue problem.

The report is organized as follows. In Section 2 we introduce several notions that will be used throughout, as well as describe the choices we have made in our notation. In Section 3 we discuss well known eigenvalue characterizations of minimal submanifolds of some simple ambient manifolds, and show that free boundary minimal submanifolds in space form balls admit analogous characterizations. We then apply these results in Section 4 to study the relation between volume and boundary volume in this class of submanifolds, as well as to some balancing properties. In Section 5 we describe the results of Nadirashvili and Fraser and Schoen mentioned above and begin preliminary work towards a possible generalization. Finally, in Section 6 we highlight questions and potential approaches that appear important or promising.

2 Definitions, Notation and Conventions

We follow the Einstein summation convention, so that an index that shows up once as a subscript and once as a superscript in the same term is assumed to be summed over from 1 to the dimension of the space in question. In particular, we write coordinates with a superscript for consistency.

All our manifolds and Riemannian metrics are assumed to be smooth (C^∞). Manifolds with boundary are assumed to have smooth boundary. By submanifold we mean immersed submanifold.

Let (M, g) be a Riemannian manifold of dimension n, and let Σ be a submanifold of dimension k, both possibly with boundary. We will always equip submanifolds with the induced metric. We denote the second fundamental form of Σ in M by $\mathrm{II} = \mathrm{II}^{\Sigma\subset M} : \Gamma(T\Sigma) \times \Gamma(T\Sigma) \to \Gamma(N\Sigma)$, where $T\Sigma$ and $N\Sigma$ denote the tangent and normal bundles of Σ, respectively. II can be seen as a section of $T^*\Sigma \otimes T^*\Sigma \otimes N\Sigma$, and then the mean curvature vector $H = H^{\Sigma\subset M} \in \Gamma(N\Sigma)$ is defined as the trace of II with respect to the metric. Σ is said to be **minimal in** M if its mean curvature vector $H^{\Sigma\subset M}$ is identically zero.

If M and Σ have nonempty boundary and $\partial\Sigma \subset \partial M$, then Σ is said to be a **free boundary minimal submanifold of** M if Σ is minimal in M and it meets ∂M orthogonally along $\partial\Sigma$.

We will use the outward unit conormal many times in what follows. Suppose Σ is compact with nonempty boundary. The **outward unit conormal** η **to** Σ **along** $\partial\Sigma$ is the unique vector field along $\partial\Sigma$ that is tangent to Σ, normal to $\partial\Sigma$, has unit length and points away from Σ.

There is no universal agreement as to the sign of the Laplace operator. Whenever we speak of the **Laplacian**, or the Laplace operator on (M, g), we will be talking about the operator defined by

$$-\Delta_g u = -\operatorname{div}(\nabla u)$$

for any $u \in C^\infty(M)$, where ∇u is the gradient of u with respect to the metric g.

In the next section we will encounter Minkowski space, a particularly simple example of a pseudo-Riemannian manifold. A **pseudo-Riemannian manifold** (M, g) is a smooth manifold M equipped with a pseudo-Riemannian metric g, that is, a symmetric covariant 2-tensor field that is nondegenerate at each point and has constant signature. Most of the definitions above can be adapted to the pseudo-Riemannian setting with only minor modifications. We note just a couple that one should keep in mind. A **Riemannian submanifold** of a pseudo-Riemannian manifold is a submanifold on which the induced metric is positive definite. A tangent vector v is a **unit vector** if $g(v, v) = \pm 1$, and a collection of tangent vectors $\{v_i\} \subset T_pM$ is **orthonormal** if $g(v_i, v_j) = \pm\delta_{ij}$ for each i and j.

Finally, we always integrate functions on an oriented Riemannian manifold with respect to the Riemannian volume form, and on a nonorientable Riemannian manifold with respect to the Riemannian density. So we will omit the form or density from our integrals, except where this might lead to confusion.

3 Minimal Surfaces and their Eigenvalue Characterizations

Our starting point is that of a submanifold of Euclidean or Minkowski space. Recall that **Minkowski space** $\mathbb{R}^{n,1}$ is just $\mathbb{R}^{n+1}$ with its standard smooth structure, equipped with the pseudo-Riemannian metric

$$g = (dx^1)^2 + \cdots + (dx^n)^2 - (dx^{n+1})^2.$$

The proof of the following standard result in the Euclidean case can be found in [4, Proposition 1.7]. The proof for Minkowski space needs only minor modifications.

Proposition 1. *Let Σ be a k-dimensional Riemannian submanifold of Euclidean space $\mathbb{R}^{n+1}$ or Minkowski space $\mathbb{R}^{n,1}$. Then the mean curvature vector field H of Σ in $\mathbb{R}^{n+1}$ or $\mathbb{R}^{n,1}$ is given by*

$$H = \left(\Delta_\Sigma x^i\right) \frac{\partial}{\partial x^i}.$$

In words, the i-th component of the mean curvature vector in standard coordinates is $\Delta_\Sigma x^i$. The following is now immediate.

Corollary 2. *A Riemannian submanifold* Σ^k *of* $\mathbb{R}^{n+1}$ *or* $\mathbb{R}^{n,1}$ *is minimal if and only if*

$$\Delta_\Sigma x^i = 0$$

for $i = 1, \ldots, n+1$.

We will use $\mathbb{S}^n$ to denote the unit sphere centered at the origin in $\mathbb{R}^{n+1}$. By $\mathbb{H}^n$ we mean the n-dimensional hyperbolic space of radius 1. In the hyperboloid model, $\mathbb{H}^n$ is the Riemannian submanifold of $\mathbb{R}^{n,1}$ defined by

$$(x^1)^2 + \cdots + (x^n)^2 - (x^{n+1})^2 = -1, \qquad x^{n+1} > 0.$$

We can use Proposition 1 and a knowledge of $\mathrm{II}^{\mathbb{S}^n \subset \mathbb{R}^{n+1}}$, $\mathrm{II}^{\mathbb{H}^n \subset \mathbb{R}^{n,1}}$, to obtain a characterization of minimal submanifolds of the sphere or hyperbolic space.

Proposition 3. *A submanifold* Σ^k *of* $\mathbb{S}^n$ *is minimal if and only if*

$$\Delta_\Sigma x^i + kx^i = 0$$

for $i = 1, \ldots, n+1$.

A submanifold Σ^k *of* $\mathbb{H}^n$ *is minimal if and only if*

$$\Delta_\Sigma x^i - kx^i = 0$$

for $i = 1, \ldots, n+1$.

Proof. For the case of the sphere we refer to [4, Lemma 2.37], and give only the proof for hyperbolic space here.

We have $\Sigma \subset \mathbb{H}^n \subset \mathbb{R}^{n,1}$, so from the definition,

$$\mathrm{II}^{\Sigma \subset \mathbb{R}^{n,1}} = \mathrm{II}^{\Sigma \subset \mathbb{H}^n} + \mathrm{II}^{\mathbb{H}^n \subset \mathbb{R}^{n,1}}.$$

Fix $p \in \Sigma$ and take an orthonormal basis $(E_1, \ldots, E_k)$ of $T_p\Sigma$. Taking the trace of the previous equation with respect to directions tangent to Σ, we have

$$\begin{aligned} H^{\Sigma \subset \mathbb{R}^{n,1}} &= H^{\Sigma \subset \mathbb{H}^n} + \sum_{i=1}^{k} \mathrm{II}^{\mathbb{H}^n \subset \mathbb{R}^{n,1}}(E_i, E_i) \\ &= H^{\Sigma \subset \mathbb{H}^n} + kN, \end{aligned}$$

where $N(x^1, \ldots, x^{n+1}) = x^i \partial_i$ is a unit normal vector field on $\mathbb{H}^n$.

Hence $\Sigma \subset \mathbb{H}^n$ is minimal if and only if $H^{\Sigma \subset \mathbb{R}^{n,1}} - kN = 0$, that is, if and only if $\Delta_\Sigma x^i - kx^i = 0$ for $i = 1, \ldots, n+1$. □

Now we turn to analogous results for free boundary minimal submanifolds. Fraser and Schoen [9] observed that free boundary minimal submanifolds of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ could be characterized in terms of eigenfunctions of a Dirichlet-to-Neumann operator with eigenvalue 1. We will meet this operator in Section 5, so for now we can state their result as follows.

Proposition 4. *Let Σ^k be a submanifold of $\mathbb{B}^n$ with nonempty boundary such that $\partial\Sigma \subset \partial\mathbb{B}^n$. Then Σ is a free boundary minimal submanifold in $\mathbb{B}^n$ if and only if*

$$\begin{cases} \Delta_\Sigma x^i = 0 & \text{in } \Sigma, \\ \partial_\eta x^i = x^i & \text{on } \partial\Sigma, \end{cases}$$

for $i = 1, \dots, n$.

We will show there is an analogous characterization of free boundary minimal submanifolds in a space form ball. Starting with the sphere, let $p_N = (0, \dots, 0, 1) \in \mathbb{S}^n$ be the "north pole", take $r \in (0, \pi)$ and let $B = B_r(p_N)$ be the closed geodesic ball in $\mathbb{S}^n$ centered at p_N with radius r.

Proposition 5. *Let Σ^k be a submanifold of B with nonempty boundary such that $\partial\Sigma \subset \partial B$. Then Σ is a free boundary minimal submanifold in B if and only if*

$$\begin{cases} \Delta_\Sigma x^i + kx^i = 0 & \text{in } \Sigma, \\ \partial_\eta x^i = \cot(r)\, x^i & \text{on } \partial\Sigma, \end{cases} \tag{1}$$

for $i = 1, \dots, n$, and

$$\begin{cases} \Delta_\Sigma x^{n+1} + kx^{n+1} = 0 & \text{in } \Sigma, \\ \partial_\eta x^{n+1} = -\sin(r) & \text{on } \partial\Sigma. \end{cases} \tag{2}$$

Proof. Being minimal in B is the same as being minimal in $\mathbb{S}^n$, so by Proposition 3, Σ is minimal in B if and only if $\Delta_\Sigma x^i + kx^i = 0$ for $i = 1, \dots, n+1$.

Now we compute ν, the outward unit conormal to B along ∂B. Let $p = (x^1, \dots, x^{n+1}) \in \partial B$. Project p onto the hyperplane $\{x^{n+1} = 0\}$ and normalize to get

$$\hat{p} = \frac{(x^1, \dots, x^n, 0)}{\sin r},$$

since $\sin r$ is the Euclidean distance between p and the x^{n+1}-axis. Note that $\{p_N, \hat{p}\}$ form an orthonormal basis for the 2-dimensional subspace of $\mathbb{R}^{n+1}$ containing p_N and p. Let $\gamma(t) = (\cos t)p_N + (\sin t)\hat{p}$ for $t \in [0, r]$. Then γ is a unit-speed radial geodesic in B, and $\gamma(r) = p$, so

$$\begin{aligned} \nu(p) &= \gamma'(r) \\ &= \cot(r)\, x^1 \frac{\partial}{\partial x^1} + \cdots + \cot(r)\, x^n \frac{\partial}{\partial x^n} - \sin(r) \frac{\partial}{\partial x^{n+1}}. \end{aligned}$$

This allows us to express the free boundary condition. Recall that η is the outward unit conormal to Σ along $\partial\Sigma$. We write $\eta\, x^i$ for the directional derivative of the function x^i in the direction η. Then Σ meets ∂B orthogonally along $\partial\Sigma$

exactly when $\eta = \nu$, which is equivalent to

$$\begin{aligned}\partial_\eta x^i &= \eta\, x^i \\ &= \nu\, x^i \\ &= \nu^i \\ &= \cot(r)\, x^i,\end{aligned}$$

for $i = 1, \ldots, n$, and

$$\partial_\eta x^{n+1} = -\sin(r).$$

$\square$

A very similar result holds in hyperbolic space. As before, let $p_N = (0, \ldots, 0, 1) \in \mathbb{H}^n$, take any $r > 0$ and let $B = B_r(p_N)$ be the closed geodesic ball in $\mathbb{H}^n$ centered at p_N with radius r.

Proposition 6. *Let Σ^k be a submanifold of B with nonempty boundary such that $\partial\Sigma \subset \partial B$. Then Σ is a free boundary minimal submanifold in B if and only if*

$$\begin{cases} \Delta_\Sigma x^i - kx^i = 0 & \text{in } \Sigma, \\ \partial_\eta x^i = \coth(r)\, x^i & \text{on } \partial\Sigma, \end{cases} \tag{3}$$

for $i = 1, \ldots, n$, and

$$\begin{cases} \Delta_\Sigma x^{n+1} - kx^{n+1} = 0 & \text{in } \Sigma, \\ \partial_\eta x^{n+1} = \sinh(r) & \text{on } \partial\Sigma. \end{cases} \tag{4}$$

Proof. The proof parallels the previous one, except that we replace the Euclidean metric with the Minkowski metric.

Being minimal in B is the same as being minimal in $\mathbb{H}^n$, so by Proposition 3, Σ is minimal in B if and only if $\Delta_\Sigma x^i - kx^i = 0$ for $i = 1, \ldots, n+1$.

Now we compute ν, the outward unit conormal to B along ∂B. Let $p = (x^1, \ldots, x^{n+1}) \in \partial B$. Project p onto the hyperplane $\{x^{n+1} = 0\}$ and normalize to get

$$\hat{p} = \frac{(x^1, \ldots, x^n, 0)}{\sinh r},$$

since $\sinh r$ is the Minkowski distance between p and the x^{n+1}-axis. Note that $\{p_N, \hat{p}\}$ form an orthonormal basis for the 2-dimensional subspace of $\mathbb{R}^{n,1}$ containing p_N and p. Let $\gamma(t) = (\cosh t)p_N + (\sinh t)\hat{p}$ for $t \in [0, r]$. Then γ is a unit-speed radial geodesic in B, and $\gamma(r) = p$, so

$$\begin{aligned}\nu(p) &= \gamma'(r) \\ &= \coth(r)\, x^1 \frac{\partial}{\partial x^1} + \cdots + \coth(r)\, x^n \frac{\partial}{\partial x^n} + \sinh(r) \frac{\partial}{\partial x^{n+1}}.\end{aligned}$$

With the same notation as before, Σ meets ∂B orthogonally along $\partial\Sigma$ exactly when $\eta = \nu$, which is equivalent to

$$\partial_\eta x^i = \coth(r)\, x^i,$$

for $i = 1, \dots, n$, and

$$\partial_\eta x^{n+1} = \sinh(r).$$

□

4 Applications

These results have many consequences for the study of minimal submanifolds. For example, the characterization of minimal submanifolds in Euclidean space in terms of harmonic functions implies the convex hull property, which states that compact minimal submanifolds of Euclidean space must lie in the convex hull of their boundary. This characterization is also important in the proof of the monotonicity formula for minimal submanifolds of Euclidean space. The first part of Proposition 3 ties the study of closed minimal submanifolds in $\mathbb{S}^n$ with that of Laplace eigenfunctions and eigenvalues. A well known open problem related to this connection is Yau's conjecture [26, Problem 100], which asks whether the first positive eigenvalue of the Laplacian on an embedded minimal hypersurface in $\mathbb{S}^n$ is $n-1$.

We will focus on two known applications of Propositions 3 and 4 that turn out to generalize to free boundary minimal submanifolds in space form balls.

4.1 Volume and Boundary Volume

The first application is an interesting relation between the volume $|\Sigma|$ and the boundary volume $|\partial\Sigma|$ of a compact free boundary minimal submanifold Σ^k of $\mathbb{B}^n$. We give the proof as we will have opportunity to use some of it later on.

Proposition 7 ([9])**.** *If Σ^k is a compact free boundary minimal submanifold in $\mathbb{B}^n$, then $|\partial\Sigma| = k|\Sigma|$.*

Proof. Let $|x|^2 = \sum_{i=1}^n (x^i)^2$ denote the square distance function from the origin of $\mathbb{R}^n$. Then

$$\begin{aligned}\Delta_\Sigma |x|^2 &= \sum_{i=1}^n \left(2x^i \Delta_\Sigma x^i + 2|\nabla^\Sigma x^i|^2\right)\\ &= 2\sum_{i=1}^n |\nabla^\Sigma x^i|^2\\ &= 2k,\end{aligned}$$

where we have used the fact that for any k-dimensional submanifold of $\mathbb{R}^n$, we have $\sum_{i=1}^n |\nabla^\Sigma x^i|^2 = k$. Now

$$\begin{aligned} 2k|\Sigma| &= \int_\Sigma \Delta_\Sigma |x|^2 \\ &= \int_{\partial\Sigma} \partial_\eta |x|^2 \\ &= \int_{\partial\Sigma} \sum_{i=1}^n 2x^i \partial_\eta x^i \\ &= 2\int_{\partial\Sigma} \sum_{i=1}^n (x^i)^2 \\ &= 2|\partial\Sigma|. \end{aligned}$$

□

We would like to relate the volume and boundary volume of compact free boundary minimal submanifolds in a space form ball. Since we are working on geodesic balls of radius r, we should try to compare our results with those for the Euclidean ball $\mathbb{B}_r^n$ of radius r. In this case, it is easy to show that the free boundary condition expressed in Proposition 4 becomes $\partial_\eta x^i = \frac{x^i}{r}$, and hence the relation between volume and boundary volume of compact free boundary minimal submanifolds in $\mathbb{B}_r^n$ becomes $\frac{|\partial\Sigma|}{|\Sigma|} = \frac{k}{r}$.

Proposition 8. *If Σ^k is a compact free boundary minimal submanifold in $B_r(p_N) \subset \mathbb{S}^n$, then*

$$k \cot r \leq \frac{|\partial\Sigma|}{|\Sigma|} \leq k \csc r.$$

If Σ^k is a compact free boundary minimal submanifold in $B_r(p_N) \subset \mathbb{H}^n$, then

$$k \operatorname{csch} r \leq \frac{|\partial\Sigma|}{|\Sigma|} \leq k \coth r.$$

Proof. Let Σ^k be a compact free boundary minimal submanifold in $B_r(p_N) \subset \mathbb{S}^n$. By Proposition 5, we have $\partial_\eta x^{n+1} = -\sin(r)$ on $\partial\Sigma$. Integrating over $\partial\Sigma$,

$$\begin{aligned} -\sin(r)|\partial\Sigma| &= \int_{\partial\Sigma} \partial_\eta x^{n+1} \\ &= \int_\Sigma \Delta_\Sigma x^{n+1} \\ &= -k\int_\Sigma x^{n+1}, \end{aligned}$$

where we have used Proposition 5 again. Since $\cos r \leq x^{n+1} \leq 1$ on Σ, we find that

$$\cos(r)|\Sigma| \leq \int_\Sigma x^{n+1} \leq |\Sigma|,$$

and thus, using what we found above,

$$\cos(r)|\Sigma| \leq \frac{1}{k}\sin(r)|\partial\Sigma| \leq |\Sigma|,$$

$$k\cot(r) \leq \frac{|\partial\Sigma|}{|\Sigma|} \leq k\csc(r),$$

as required.

The proof of the hyperbolic case is completely analogous: integrate $\partial_\eta x^{n+1} = \sinh(r)$ over $\partial\Sigma$, use Proposition 6 and the fact that $1 \leq x^{n+1} \leq \cosh r$ on Σ to conclude. $\square$

Note that $\cot r \sim \csc r \sim 1/r$ as $r \to 0^+$, and so in the spherical case

$$\frac{|\partial\Sigma|}{|\Sigma|} \sim \frac{k}{r}$$

as $r \to 0^+$. Similarly, $\operatorname{csch} r \sim \coth r \sim 1/r$ as $r \to 0^+$, so in the hyperbolic case we also have

$$\frac{|\partial\Sigma|}{|\Sigma|} \sim \frac{k}{r}$$

as $r \to 0^+$. Thus both results degenerate into the Euclidean one for small $r > 0$.

Note also that, in the hyperbolic case, the proposition says that the volume and boundary volume of compact free boundary minimal submanifolds in B are comparable. In the spherical case, the same is true when $r \in (0, \pi/2)$, but for $r \geq \pi/2$ the lower bound says nothing and we are left with $|\partial\Sigma| \leq k\csc(r)|\Sigma|$.

4.2 Balancing

Our second application will be to the "balancing" of certain classes of minimal submanifolds. For a compact submanifold Σ^k (with or without boundary) of Euclidean or Minkowski space, we will say that Σ is balanced, or that its center of mass is at the origin, if

$$\int_\Sigma x^i = 0$$

for all $i = 1, \ldots, n+1$. The following well known result is an immediate consequence of Proposition 3.

Corollary 9. *A closed minimal submanifold Σ^k of $\mathbb{S}^n$ is balanced.*

A similar result for compact free boundary minimal submanifolds of $\mathbb{B}^n$ follows from Proposition 4.

Proposition 10. *If Σ^k is a compact free boundary minimal submanifold in $\mathbb{B}^n$, then both Σ and $\partial\Sigma$ are balanced.*

Proof. Given $i = 1, \ldots, n$,

$$\begin{aligned}
\int_{\partial\Sigma} x^i &= \int_{\partial\Sigma} \partial_\eta x^i \\
&= \int_\Sigma \Delta_\Sigma x^i \\
&= 0,
\end{aligned}$$

so $\partial\Sigma$ is balanced. To see that Σ is balanced, we use Green's formulas for the Laplacian to find that

$$\begin{aligned}
2k \int_\Sigma x^i &= \int_\Sigma x^i \Delta_\Sigma |x|^2 \\
&= \int_\Sigma |x|^2 \Delta_\Sigma x^i + \int_{\partial\Sigma} \left(x^i \partial_\eta |x|^2 - |x|^2 \partial_\eta x^i \right) \\
&= 0.
\end{aligned}$$

□

Now we generalize to free boundary minimal submanifolds in space form balls. We cannot expect these to be balanced since there is now a distinguished direction as seen from Euclidean or Minkowski space, namely that of the center of the geodesic ball. We have chosen $p_N = (0, \ldots, 0, 1)$ as the center of our geodesic balls, so for us the special direction is along the x^{n+1}-axis. The next proposition says that for a free boundary minimal submanifold Σ^k in $B_r(p_N) \subset \mathbb{S}^n$, the center of mass of both Σ and $\partial\Sigma$ lies on the x^{n+1}-axis.

Proposition 11. *Let Σ^k be a compact free boundary minimal submanifold in $B_r(p_N) \subset \mathbb{S}^n$. Then*

$$\int_{\partial\Sigma} x^i = \int_\Sigma x^i = 0$$

for $i = 1, \ldots, n$.

Proof. Start with the boundary. For any $i = 1, \ldots, n$,

$$\begin{aligned}
\cos(r)\cot(r) \int_{\partial\Sigma} x^i &= \int_{\partial\Sigma} x^{n+1} \partial_\eta x^i \\
&= \int_{\partial\Sigma} x^i \partial_\eta x^{n+1} + \int_\Sigma \left(x^{n+1} \Delta_\Sigma x^i - x^i \Delta_\Sigma x^{n+1} \right) \\
&= -\sin(r) \int_{\partial\Sigma} x^i + \int_\Sigma \left(x^{n+1}(-k x^i) - x^i(-k x^{n+1}) \right) \\
&= -\sin(r) \int_{\partial\Sigma} x^i,
\end{aligned}$$

where we have used Green's formulas for the Laplacian and Proposition 5. Since $\cos(r)\cot(r) + \sin(r) = \csc(r)$ is never zero for $r \in (0, \pi)$, we find that

$$\int_{\partial\Sigma} x^i = 0.$$

The result for Σ itself follows from this, as

$$\begin{aligned} -k \int_{\Sigma} x^i &= \int_{\Sigma} \Delta_{\Sigma} x^i \\ &= \int_{\partial\Sigma} \partial_\eta x^i \\ &= \cot(r) \int_{\partial\Sigma} x^i \\ &= 0. \end{aligned}$$

□

The same is true in hyperbolic space.

Proposition 12. *Let Σ^k be a compact free boundary minimal submanifold in $B_r(p_N) \subset \mathbb{H}^n$. Then*

$$\int_{\partial\Sigma} x^i = \int_{\Sigma} x^i = 0$$

for $i = 1, \ldots, n$.

Proof. Arguing as in the previous proof and using Proposition 6, we find that

$$\cosh(r)\coth(r) \int_{\partial\Sigma} x^i = \sinh(r) \int_{\partial\Sigma} x^i.$$

Since $\cosh(r)\coth(r) - \sinh(r) = \operatorname{csch}(r)$ is never zero for $r > 0$, we conclude that

$$\int_{\partial\Sigma} x^i = 0$$

for all $i = 1, \ldots, n$. The result for Σ follows essentially as before. □

5 Minimal Surfaces and Extremal Eigenvalue Problems

From now on we will only consider 2-dimensional surfaces, for some of the results we discuss do not generalize to higher dimensions [3].

5.1 Closed Surfaces and Laplace Eigenvalues

Let M be a closed surface (compact without boundary). Each Riemannian metric g on M gives rise to a Laplace operator Δ_g whose spectrum consists of a sequence

$$0 = \lambda_1(g) < \lambda_2(g) \leq \lambda_3(g) \leq \cdots \to \infty,$$

where each eigenvalue is repeated according to its multiplicity. Note carefully that we denote the eigenvalue 0 by $\lambda_1(g)$ and not by $\lambda_0(g)$, which goes against standard practice. We are forced to do this because we will encounter operators whose first eigenvalue is not zero, and it is important that we name the eigenvalues consistently. So for us the first eigenvalue always has a subscript 1.

The notation $\lambda_k(g)$ is meant to emphasize that we are thinking of each eigenvalue as a functional on the space $\mathcal{R}(M)$ of Riemannian metrics on M.

Let us consider $\lambda_2(g)$, the first positive eigenvalue of the Laplacian. This turns out to be unbounded above on $\mathcal{R}(M)$, because it scales like

$$\lambda_2(cg) = \frac{1}{c}\lambda_2(g)$$

for any $c > 0$. We can, however, consider the scale invariant quantity $\lambda_2(g)\,|M|_g$, where $|M|_g$ is the area of M equipped with the metric g. This "normalized eigenvalue" was shown to be bounded above by Yang and Yau [25], and Li and Yau [20]. A natural question then is: are there maximizing metrics? This is a hard question in general, but the answer is yes for at least some surfaces, such as the sphere $\mathbb{S}^2$ [17], the real projective plane $\mathbb{RP}^2$ [20], the torus $\mathbb{T}^2$ and the Klein bottle [22],

Another basic question one can ask is: if a maximizing metric does exist, then what can we say about its geometry?

Theorem 13 ([22])**.** *Let M be a closed surface and suppose $g_0 \in \mathcal{R}(M)$ is such that*

$$\lambda_2(g_0)\,|M|_{g_0} = \sup_{g \in \mathcal{R}(M)} \lambda_2(g)|M|_g.$$

Then there exist independent λ_2-eigenfunctions $u_1, \ldots, u_{n+1}$ $(n \geq 2)$ which, after rescaling the metric, give an isometric minimal immersion $u = (u_1, \ldots, u_{n+1})$ of M into $\mathbb{S}^n$.

In particular, $u(M)$ is a minimal surface in $\mathbb{S}^n$ and the maximizing metric g_0 is a constant multiple of the induced metric on $u(M)$ from $\mathbb{S}^n$.

Let $S^2(M)$ be the space of symmetric covariant 2-tensor fields on M, equipped with the $L^2(M)$ inner product

$$(\omega, \eta)_{L^2(M)} = \int_M \langle \omega, \eta \rangle \, da_{g_0}.$$

We give an outline of the proof following the exposition in [5] and [7].

Proof outline. **Part 1:** We are assuming the existence of a maximizer g_0 for our functional, so g_0 should satisfy the associated Euler-Lagrange equations. It turns out that the possibility of a changing multiplicity for $\lambda_2(g)$ implies that our functional is not differentiable in general. Nevertheless, for any smooth path of metrics $g(t)$, it can be shown that $\lambda_2(t)$ is Lipschitz and hence differentiable almost everywhere.

Now compute this derivative where it exists. Let $g(t)$ be a smooth family of metrics with unit area and such that $g(0) = g_0$. Fix a time t_* and assume $\lambda_2(t)$ is differentiable at $t = t_*$. Pick a λ_2-eigenfunction u_* at time t_*, normalized so that $\int_M u_*^2 \, da_* = 1$. Take any smooth family of functions u_t on M such that $u_{t_*} = u_*$ and $\int_M u_t \, da_t = 0$ for all t. For example, we may take

$$u_t = u_* - \frac{1}{|M|_{g(t)}} \int_M u_* \, da_t.$$

Then each u_t is an admissible function in the variational characterization of $\lambda_2(t)$, so

$$F(t) = \int_M |\nabla u_t|^2 \, da_t - \lambda_2(t) \int_M u_t^2 \, da_t$$

is nonnegative. Since $F(t_*) = 0$ by construction, we have $\dot{F}(t_*) = 0$. Computing and rearranging gives

$$\dot{\lambda}_2(t_*) = -\left(q_{t_*}(u_*), \dot{g}(t_*)\right)_{L^2(M)},$$

where

$$q_t(u) = du \otimes du - \frac{1}{2}\left(|\nabla u|^2 - \lambda_2(t) u^2\right) g(t)$$

for any λ_2-eigenfunction u at time t.

We can use this to approximate the one-sided derivatives of $\lambda_2(t)$ at $t = 0$, and then the fact that g_0 is a maximizing metric gives the existence of some λ_2-eigenfunction u at time $t = 0$ such that

$$\left(q_0(u), \dot{g}(0)\right)_{L^2(M)} = 0.$$

Now construct as many admissible variations of the metric g_0 as possible. It turns out that for any $\omega \in S^2(M)$ with $(g_0, \omega)_{L^2(M)} = 0$, there exists a smooth family of metrics $g(t)$ with unit area, $g(0) = g_0$ and $\dot{g}(0) = \omega$. So the conclusion of this part of the argument is that for any $\omega \in S^2(M)$ with $(g_0, \omega)_{L^2(M)} = 0$, there exists a λ_2-eigenfunction u such that

$$(q_0(u), \omega)_{L^2(M)} = 0.$$

Part 2: Interpreting the conclusion of the previous part geometrically suggests that g_0 belongs to the convex hull of the image of q_0, and also indicates how to prove it. That allows us to write

$$g_0 = \sum_{i=1}^{n+1} q_0(u^i)$$

for some collection of λ_2-eigenfunctions $(u^1, \dots, u^{n+1})$. Simplifying gives the result. $\square$

5.2 Compact Surfaces with Boundary and Steklov Eigenvalues

Consider now a compact Riemannian surface with boundary (M, g). The metric determines a Dirichlet-to-Neumann operator $T_g : C^\infty(\partial M) \to C^\infty(\partial M)$ by

$$T_g u = \partial_\eta \hat{u},$$

where $\hat{u} \in C^\infty(M)$ is the harmonic extension of u to M. That is, $\hat{u}$ is the unique solution to

$$\begin{cases} \Delta_g \hat{u} = 0 & \text{in } M \\ \hat{u} = u & \text{on } \partial M. \end{cases}$$

Uniqueness follows from the maximum principle, and then existence follows from the Fredholm alternative.

The Dirichlet-to-Neumann operator T_g is self-adjoint, positive semidefinite and has a discrete spectrum consisting of a sequence

$$0 = \sigma_1(g) < \sigma_2(g) \leq \sigma_3(g) \leq \cdots \to \infty,$$

where each eigenvalue is repeated according to its multiplicity. Once again, note the nonstandard indexing of the eigenvalues. Eigenvalues of the Dirichlet-to-Neumann map T_g are called Steklov eigenvalues.

We will also think of each Steklov eigenvalue $\sigma_k(g)$ as a functional on $\mathcal{R}(M)$. The first positive eigenvalue $\sigma_2(g)$ is unbounded above on $\mathcal{R}(M)$, because it scales like

$$\sigma_2(cg) = \frac{1}{\sqrt{c}}\sigma_2(g)$$

for any $c > 0$. We can, as before, consider the scale invariant quantity $\sigma_2(g)\,|\partial M|_g$, which was shown to be bounded above by Fraser and Schoen [9]. The same two questions that we discussed above apply here: are there maximizing metrics? And if yes, then what can we say about them?

We will not get into the first question here, except to note that Fraser and Schoen [8] have proved the existence of a maximizing metric for $\sigma_2(g)\,|\partial M|_g$ on any compact surface with boundary, genus 0 and an arbitrary number of boundary components. As to the second question, we have the following counterpart to Theorem 13.

Theorem 14 ([8]). *Let M be a compact surface with boundary and suppose $g_0 \in \mathcal{R}(M)$ is such that*

$$\sigma_2(g_0)\,|\partial M|_{g_0} = \sup_{g \in \mathcal{R}(M)} \sigma_2(g)\,|\partial M|_g.$$

Then the multiplicity of $\sigma_2(g_0)$ is at least 2 and, after rescaling g_0 so that $\sigma_2(g_0) = 1$, there exist independent σ_2-eigenfunctions $u^1, \dots, u^n$, $(n \geq 2)$, such that

$$u = (u^1, \dots, u^n) : M \to \mathbb{B}^n$$

is a proper branched conformal minimal immersion that is an isometry on ∂M.

5.3 The Dirichlet-to-Neumann map for the Helmholtz equation

Let (M, g) be a compact Riemannian surface with nonempty boundary. Given $\lambda \in \mathbb{R}$, we call

$$\Delta_g u + \lambda u = 0$$

the λ-Helmholtz equation, or just the Helmholtz equation if the specific value of λ is not important. We will denote the spectrum of the Dirichlet Laplacian on (M, g) by $\mathrm{Spec}(-\Delta_g^D)$.

If $\lambda \notin \mathrm{Spec}(-\Delta_g^D)$, then define the Dirichlet-to-Neumann operator for the Helmholtz equation $T : C^\infty(\partial M) \to C^\infty(\partial M)$ by

$$Tu = \partial_\eta(Eu),$$

where $Eu \in C^\infty(M)$ is the unique solution of

$$\begin{cases} \Delta_g(Eu) + \lambda(Eu) = 0 & \text{in } M \\ Eu = u & \text{on } \partial M. \end{cases} \tag{5}$$

The fact that (5) has a unique solution follows from the Fredholm alternative: there are no nontrivial solutions to the homogeneous problem by hypothesis, so the inhomogeneous problem (5) is uniquely solvable for any boundary data in $C^\infty(\partial M)$. Note that when $\lambda = 0$ this is just the Dirichlet-to-Neumann operator of the previous subsection.

If $\lambda \in \mathrm{Spec}(-\Delta_g^D)$, then we can still define a Dirichlet-to-Neumann operator for the λ-Helmholtz equation but on a restricted domain. Let W be the eigenspace of λ as an eigenvalue of $-\Delta_g^D$. Define

$$(D_\eta W)^\perp = \left\{ u \in C^\infty(\partial M) : \int_{\partial M} u\, \partial_\eta w = 0 \ \text{ for all } w \in W \right\}.$$

In words, $(D_\eta W)^\perp$ is the space of functions on the boundary of M that are $L^2(\partial M)$-orthogonal to the normal derivatives of the Dirichlet Laplace eigenfunctions in W. Then, again by the Fredholm alternative, the boundary value problem (5) has a solution if and only if $u \in (D_\eta W)^\perp$. The solution is unique up to addition of a function in W, and it is then possible to pick a unique solution Eu such that $\partial_\eta(Eu) \in (D_\eta W)^\perp$. So, in this case, we take the Dirichlet-to-Neumann map for the λ-Helmholtz equation $T : (D_\eta W)^\perp \to (D_\eta W)^\perp$ to be given by

$$Tu = \partial_\eta(Eu),$$

where $Eu \in C^\infty(M)$ is the unique solution of (5) such that $\partial_\eta(Eu) \in (D_\eta W)^\perp$.

Note that both the extension operator E and the Dirichlet-to-Neumann map for the Helmholtz equation T depend on the metric and the parameter λ, so we should technically be writing E_g^λ and T_g^λ. However, this would overburden the notation and make our expressions harder to read, so we will drop g and λ from the operators' names whenever we are working with a fixed metric and a fixed parameter.

Just as Proposition 4 connects free boundary minimal surfaces in $\mathbb{B}^n$ with eigenfunctions of the Dirichlet-to-Neumann map, so Propositions 5 and 6 connect free boundary minimal surfaces in space form balls with eigenfunctions of the Dirichlet-to-Neumann map for the Helmholtz equation. More precisely, Proposition 5 says that Σ is a free boundary minimal surface in $B \subset \mathbb{S}^n$ if and only if the Euclidean coordinates restrict to eigenfunctions of the Dirichlet-to-Neumann map with parameter $\lambda = 2$, the first n coordinates with eigenvalue $\cot r$, and the last one with eigenvalue $-\tan r$ (since $x^{n+1} = \cos r$ on $\partial\Sigma$). Proposition 6 has a similar interpretation, but the relevant map is the one with parameter $\lambda = -2$, and the eigenvalues are $\coth r$ and $\tanh r$.

5.3.1 Basic properties

From now on we assume $\lambda \notin \operatorname{Spec}(-\Delta_g^D)$. For any such λ it is known that the Dirichlet-to-Neumann operator T^λ is self-adjoint, its spectrum is discrete and bounded below. We will discuss these properties below. Before that, we state Green's formulas for the operator $L = \Delta_g + \lambda$ as they show up several times in the following. For any $u, v \in C^\infty(M)$, we have

$$\int_M uLv = \int_{\partial M} u\,\partial_\eta v - \int_M \langle \nabla u, \nabla v\rangle + \lambda \int_M uv,$$

which is an immediate consequence of Green's formula for the Laplacian. This implies that

$$\int_M (uLv - vLu) = \int_{\partial M} (u\,\partial_\eta v - v\,\partial_\eta u) \tag{6}$$

and

$$\int_M uLu = \int_{\partial M} u\,\partial_\eta u - \int_M |\nabla u|^2 + \lambda \int_M u^2. \tag{7}$$

Now we show that self-adjointness is an immediate consequence of (6).

Lemma 15. *T is self-adjoint.*

Proof. Given $u, v \in C^\infty(\partial M)$,

$$\begin{aligned}(Tu, v)_{L^2(\partial M)} - (u, Tv)_{L^2(\partial M)} &= \int_{\partial M} (v\,\partial_\eta(Eu) - u\,\partial_\eta(Ev)) \\ &= \int_M (Ev\,L(Eu) - Eu\,L(Ev)) \qquad \text{by (6)}, \\ &= 0\end{aligned}$$

since $L(Eu) = L(Ev) = 0$. $\square$

Since T is self-adjoint, we have an associated quadratic form

$$\begin{aligned}Q(u) &= (Tu, u)_{L^2(\partial M)} \\ &= \int_M |\nabla(Eu)|^2 - \lambda \int_M (Eu)^2 \qquad \text{by (7)},\end{aligned}$$

and a Rayleigh quotient

$$\begin{aligned}R(u) &= \frac{Q(u)}{||u||^2_{L^2(\partial M)}} \\ &= \frac{\int_M |\nabla(Eu)|^2 - \lambda \int_M (Eu)^2}{\int_{\partial M} u^2}.\end{aligned}$$

As usual, if $u \in C^\infty(\partial M)$ is an eigenfunction of T with eigenvalue σ, then $R(u) = \sigma$. So showing the spectrum of T is bounded below is the same as showing the Rayleigh quotient is bounded below. We discuss the cases of negative, zero, or positive λ separately. This trichotomy seems to be unavoidable, as each case gives rise to qualitatively different behavior for solutions of $\Delta_g u + \lambda u = 0$ and hence for the Dirichlet-to-Neumann operator T^λ.

If $\lambda < 0$, then the Rayleigh quotient is positive for all $u \in C^\infty(\partial M) \setminus \{0\}$, so all the eigenvalues of T^λ are positive. If $\lambda = 0$, then the Rayleigh quotient is nonnegative, so all the eigenvalues of T^λ are nonnegative.

For $\lambda > 0$, the Rayleigh quotient can take negative values. We have seen one example of such a function, namely the coordinate function x^{n+1} restricted to the boundary of a free boundary minimal surface $\Sigma \subset B_r(p_N) \subset \mathbb{S}^n$ where $r \in (0, \pi/2)$. It does not seem to be obvious that the Rayleigh quotient is bounded below when $\lambda > 0$, but it is true and follows for example from [1, Theorem 4.15].

It is also well known [13, 15] that the spectrum of T^λ is discrete and consists of a sequence

$$\sigma_1^\lambda(g) < \sigma_2^\lambda(g) \leq \sigma_3^\lambda(g) \leq \cdots \to \infty,$$

where each eigenvalue is repeated according to its multiplicity. While both [13] and [15] refer to general properties of pseudodifferential operators, it is not hard to see that for $\lambda \leq 0$, a more direct approach like that in [14, Section 8.12] yields the same result. It is not clear whether the same argument works for $\lambda > 0$.

We should note that even though our interest in the Dirichlet-to-Neumann operator for the Helmholtz equation stems from its connection with free boundary minimal surfaces in space form balls, this operator has been used for other purposes in geometry. Notably, it is the key ingredient in Friedlander's proof [13] that

$$\lambda_{k+1}^N \leq \lambda_k^D$$

for all $k \geq 1$, where λ_k^N and λ_k^D are respectively the eigenvalues of the Neumann Laplacian and Dirichlet Laplacian on any bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. This is one reason we chose to always index the eigenvalues starting from 1.

5.3.2 The interval: an explicit example

An interval on the real line is the simplest setting in which one can consider a Dirichlet-to-Neumann map. An interval is of course one-dimensional and hence the theory we have discussed for surfaces does not apply directly. Nonetheless, several interesting properties of the Dirichlet-to-Neumann map for the Helmholtz equation are already present on an interval, and here we can obtain explicit expressions for everything we care about, so we consider this example in some detail now.

We work on the interval $[0, L]$ for some $L > 0$, equipped with the Euclidean metric. Its boundary consists of two points $x = 0$ and $x = L$, and so a function u on the boundary is described by an ordered pair $(u(0), u(L))$ of real numbers. Given $\lambda \notin \operatorname{Spec}(-\Delta^D) = \{\left(\frac{k\pi}{L}\right)^2 : k \geq 1\}$, the λ-Helmholtz extension of (a, b) is the function u on $[0, L]$ solving

$$\begin{cases} u'' + \lambda u = 0 & \text{in } (0, L), \\ u(0) = a, \quad u(L) = b. \end{cases}$$

Case 1: $\lambda < 0$.

The λ-Helmholtz extension of (a, b) is given by

$$u(x) = \frac{b - a\cosh\left(\sqrt{-\lambda}L\right)}{\sinh\left(\sqrt{-\lambda}L\right)} \sinh\left(\sqrt{-\lambda}x\right) + a\cosh\left(\sqrt{-\lambda}x\right)$$

for $x \in [0, L]$. The Dirichlet-to-Neumann map for the Helmholtz equation $T^\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ is, by definition, given by $T^\lambda(a, b) = (-u'(0), u'(L))$. Using the expression for u above, we find that

$$T^\lambda(a, b) = \frac{\sqrt{-\lambda}}{\sinh\left(\sqrt{-\lambda}L\right)} \left(a\cosh\left(\sqrt{-\lambda}L\right) - b, b\cosh\left(\sqrt{-\lambda}L\right) - a\right).$$

This map has two simple eigenvalues,

$$\sigma_1^\lambda = \frac{\sqrt{-\lambda}}{\sinh\left(\sqrt{-\lambda}L\right)}\left(\cosh\left(\sqrt{-\lambda}L\right) - 1\right)$$

and

$$\sigma_2^\lambda = \frac{\sqrt{-\lambda}}{\sinh\left(\sqrt{-\lambda}L\right)}\left(\cosh\left(\sqrt{-\lambda}L\right) + 1\right).$$

The σ_1^λ-eigenfunctions are the constant functions on the boundary, that is, the functions of the form (a, a) for any $a \in \mathbb{R} \setminus \{0\}$. The σ_2^λ-eigenfunctions are the functions of the form $(a, -a)$ for any $a \in \mathbb{R} \setminus \{0\}$.

Case 2: $\lambda > 0$, $\lambda \notin \operatorname{Spec}(-\Delta^D)$.

The λ-Helmholtz extension of (a, b) is given by

$$u(x) = \frac{b - a\cos(\sqrt{\lambda}L)}{\sin(\sqrt{\lambda}L)}\sin(\sqrt{\lambda}x) + a\cos(\sqrt{\lambda}x)$$

for $x \in [0, L]$, and then

$$T^\lambda(a, b) = \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}L)}\left(a\cos(\sqrt{\lambda}L) - b, b\cos(\sqrt{\lambda}L) - a\right).$$

We have two simple eigenvalues,

$$\frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}L)}\left(\cos(\sqrt{\lambda}L) - 1\right) \quad \text{and} \quad \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}L)}\left(\cos(\sqrt{\lambda}L) + 1\right).$$

Note that their order changes whenever λ crosses an eigenvalue of the Dirichlet Laplacian on $[0, L]$. The eigenfunctions of the eigenvalue listed on the left are the constants on the boundary, and the eigenfunctions of the eigenvalue listed on the right are of the form $(a, -a)$ for any $a \in \mathbb{R} \setminus \{0\}$.

5.3.3 Scaling of the eigenvalues

Now we study the eigenvalues $\sigma_k^\lambda(g)$ as functionals on $\mathcal{R}(M)$. Recall that the scaling behavior of both Laplace eigenvalues on closed surfaces and Steklov eigenvalues on compact surfaces with boundary implied these were unbounded above. The eigenvalues $\sigma_k^\lambda(g)$ do not seem to scale in any comparable way, essentially due to the fact that $\Delta_g + \lambda$ does not scale with the metric.

We can still say something however. For any $c > 0$ and $v \in C^\infty(M)$,

$$\begin{aligned} &\Delta_{cg} v + \lambda v = 0 \\ \implies\ &c^{-1}\Delta_g v + \lambda v = 0 \\ \implies\ &\Delta_g v + (\lambda c)v = 0. \end{aligned}$$

So given $u \in C^\infty(\partial M)$, we find that for any $c > 0$,

$$E_{cg}^\lambda u = E_g^{c\lambda} u$$

Since our Dirichlet-to-Neumann map T_g^λ is the composition of the extension operator E_g^λ and the normal derivative operator, it is easy to see that

$$T_{cg}^\lambda u = \frac{1}{\sqrt{c}}\, T_g^{c\lambda} u,$$

and thus

$$\sigma_k^\lambda(cg) = \frac{1}{\sqrt{c}}\, \sigma_k^{c\lambda}(g). \tag{8}$$

It turns out this is enough to conclude that most eigenvalues are unbounded above as functionals on a space of metrics.

Lemma 16. *Given $\lambda \in \mathbb{R}$ and $k \geq 2$, the functional $\sigma_k^\lambda(g)$ is unbounded above on $\{g \in \mathcal{R}(M) : \lambda \notin \operatorname{Spec}(-\Delta_g^D)\}$.*

The result is not true for $k = 1$ as $\sigma_1^0(g) = 0$ for all $g \in \mathcal{R}(M)$.

Proof. It suffices to prove the Lemma for $k = 2$. Let $\lambda \in \mathbb{R}$ and pick a metric $g \in \mathcal{R}(M)$ such that $\lambda_1^D(g) > \lambda$. Then $\lambda \notin \operatorname{Spec}(-\Delta_{cg}^D)$ for any $c \in (0,1)$, and from (8) we know that

$$\sigma_2^\lambda(cg) = \frac{1}{\sqrt{c}}\, \sigma_2^{c\lambda}(g). \tag{9}$$

Friedlander [13, Lemma 2.3] showed the eigenvalues $\sigma_k^\lambda(g)$ are continuous functions of λ on any interval that does not intersect $\operatorname{Spec}(-\Delta_g^D)$. Therefore, as $c \to 0^+$, we have $\sigma_2^{c\lambda}(g) \to \sigma_2^0(g) > 0$, and hence the right hand side of (9) tends to $+\infty$. $\square$

This means that, as before, some normalization is needed if we are to try to maximize $\sigma_k^\lambda(g)$ on a space of metrics. In this case there is no obvious scale invariant quantity analogous to the ones we discussed previously, but we can still restrict the space of metrics g under consideration to those with $|\partial M|_g = 1$. This at least excludes any scaling of such a metric. Is it the case that the eigenvalues $\sigma_k^\lambda(g)$ are bounded above on the space of metrics g with $\lambda \notin \operatorname{Spec}(-\Delta_g^D)$ and $|\partial M|_g = 1$?

6 Future Work

There are many questions related to the contents of this report that we have not yet been able to answer. Perhaps one of the most important is: on a compact Riemannian surface with boundary (M, g), what are the σ_1^λ-eigenfunctions?

For the Laplacian on a closed Riemannian surface or the Dirichlet-to-Neumann map T^0 on a compact Riemannian surface with boundary, identifying the first eigenfunctions is trivial. The first eigenvalue is zero and the first eigenfunctions are the nonzero constants. For a nonzero λ, we have seen that the first eigenvalue of T^λ will in general not be zero. Moreover, a nonzero constant function in the interior cannot solve the λ-Helmholtz equation. However, I think it is possible that the first eigenfunctions are the nonzero constants *on the boundary*, at least for $\lambda < \lambda_1^D$, and that would have several implications. Note that, on the interval, this is true for $\lambda < \lambda_1^D$, but false for $\lambda \in (\lambda_1^D, \lambda_2^D)$.

Another main problem is the lack of explicit examples. I have not yet been able to compute solutions to the Helmholtz equation on simple surfaces with given boundary data. It would be important to have examples on which to test ideas such as the one in the previous paragraph.

We have worked with Euclidean coordinates from the beginning mainly because the interior equations in Propositions 5 and 6 were known. In some ways it would be more natural to work with intrinsic coordinates on $\mathbb{S}^n$ or $\mathbb{H}^n$ when looking at free boundary minimal submanifolds of geodesic balls in these spaces. For example, working with normal coordinates centered at the center of our geodesic ball would make all coordinates indistinguishable, in contrast with the situation in Euclidean coordinates.

Finally, as mentioned in the introduction, we would like to see whether maximizing metrics for the eigenvalues $\sigma_k^\lambda(g)$ are related to free boundary minimal surfaces in a space form ball. Although not included here, we have been able to compute the first variation of these eigenvalues (where it exists) along a path of smooth metrics, as in the proof of Theorem 13. The next steps involve new difficulties, in part due to the presence of the boundary, and in part due to the Helmholtz equation.

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