Small Cell Networks: Speed Based Power Allocation

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***Abstract*—Small cell networks promise to boost the capacity and provide good QoS (Quality of Service) to cell edge users. However they pose serious challenges, especially in the case of high speed unidirectional (e.g., users traveling on highways or in metros) users, in the form of frequent handovers. The extent of the handover losses depend greatly upon the speed of the user and it is shown previously that optimal cell size increases with speed. Thus in scenarios with diverse users (speeds spanning over large ranges), it would be inefficient to serve all of them using common cell radius and it is practically infeasible to design different cell sizes for different speeds. We alternatively propose to allocate power to an user based on its speed. Higher power virtually increases the cell size. We obtain an expression for optimal power division, optimal for busy probability and load factor (a QoS influenced by handover losses), for any given average power constraint and cell size. We show that the optimal power depends linearly upon the speed. We reinforce the theory via numerical simulations, illustrating good improvements in the performance (upto 60% in busy probability and 30% in load factor) with optimal power law. The improvement in performance increases as either the path loss increases or as the range of speeds to be supported by the system increases.**

***Index Terms*—Small cell networks, drop probability, power allocation, constraint optimization**

I. Introduction

Short radius cells appear as a promising solution to boost the capacity of mobile networks and to respond to the increasing demand for video oriented bandwidth demanding services (see [3], [4]), even for cell edge users. One can serve the users at higher data rates (resulting in smaller service times for communicating the same information) while operating at reasonable transmission power. However, operating with short radius cells (Small Cells (SC) or Pico Cells (PC)) raises major issues in terms of mobility management. Indeed, as the cell size decreases, the handover (HO) rate critically increases especially for high or medium speed users which results in un-acceptable rate of call drops: the frequency of handovers (HOs) increases and the risk for insufficient resources/unavailable servers gets higher. In these conditions, the calls drop before completion of the service, with higher probabilities.

Recently proposals are made to install small base stations on the lamp posts along the side of major streets like high ways (see [4] for more details) and we study such scenarios.

In [5], the above mentioned trade-offs are studied and a cell size that is optimal for appropriate metrics (like expected service time, load factor and drop or busy probability etc) is obtained. These optimal cell sizes are derived as a function of various parameters like, user velocity profile, wireless medium properties (in terms of propagation co-efficient), the transmission power etc. One of the important conclusions of the study is: higher the velocity, larger is the optimal cell size. Based on the results in [5], it is suggested in [2] to also boost the power for high or medium speed users in comparison with the low speed users. In this paper, we work further on this idea.

One can achieve good performance if cell sizes are varied based on speed. However, it would be a practically infeasible procedure to vary the physical cell size. One may divide the users to high and low speed classes and one may (for example) double the cell size for high speed users by switching the base station only when it crosses the boundary of the second base station from the serving one. But this makes possible only coarse control, which may not improve the performance significantly. We instead propose to vary the transmit power based on the speed, which can be controlled to any required level of fineness. The higher power virtually increases the cell size: a user at a far away point with higher transmit power can be served at the same transmission rate as that of a user standing at a nearby point with smaller power (capacity of the far away user increases with higher power).

In the first part of the paper, we classify the users into finite number of classes (each class specified by a range of user speeds), allocate same power to all the users in a class and this common power can vary across the classes. Our objective is to find a (finite) power allocation vector, subject to a constraint on the average power used, when all the users operate within the same cell. We obtained a closed form expression for the optimal power allocation (optimal for busy probability and load factor, criterion that trades-off between better service rates and frequent HOs) for any given cell size and average power constraint. By this allocation: higher the speed, higher the power allocated. Disparities in the powers allocated to various user classes increase with increase in the path-loss factor and or the cell size.

There would be situations in which one could estimate the user speed accurately and then it would be better to allocate power based on the precise speed of the user. We pose the (average power) constrained optimization problem with the optimizer being a power function (power as a function of speed), as an optimal control problem and obtain the optimal solution using Hamilton-Jacobi Bellman (HJB) equation and Lagrange multipliers. The optimal power function/law turns out to be (affine) linear in user speed and satisfies the average power constraint. Further, we notice that the discrete optimal power law (when finite number of user classes were used) converges to the continuous one, obtained utilizing HJB equations, as the number of user classes increases to infinity. We also show via simulations that the performance of the system improves as the number of velocity classes increases and best performance is obtained with the continuous optimal linear power law.

There is one major difference between our previous paper [5] and the current paper, with respect to the system model. We consider in both the cases uni-directional users traveling on a line while deriving service from series of base stations. In [5], it is assumed that the rate of communication can be changed continually that too with "maximal" transmission rates (i.e., capacity). This gives "maximal" performance. But in reality this is not possible and we consider a radically different situation in this paper: communication can happen at one of the N given choices of the transmission rates. At any given time, the rates chosen are less than the "maximal" transmission rate (capacity) at that time.

Related work: For an excellent survey on power control in wireless networks, the reader is referred to [10] and the references therein (eg. [11]–[15]). Most of the existing algorithms aim at either optimizing the total power spent keeping the QoS above a required level and or optimize the QoS while keeping the power utilized within a given budget. Better algorithms are designed based on the speed of the user (e.g., [16]–[18] are few of them). These algorithm either vary a control parameter or vary a step size of the adaptive algorithm so that the algorithm is better suited to the user mobility. They discuss about the convergence issues associated with mobility patterns. While here, we consider a simple system which usually operates in small cell networks with one dimensional users: rate is adapted based on the received SNR, which varies in a periodic fashion because the distance between the unidirectional user and the serving base stations changes periodically. Given such a system, we obtain the performance which is sensitive to the speed of the user because of frequent HOs. We come up with classification of users based on their speed and allocation of power to a class so as to improve the performance of the overall system which shares resources among all the users. Most of the existing algorithms attempt to overcome the hurdles created by difficult phenomenon one of them being high speed variations. While we are designing optimal systems accepting and expecting these variations and based on these variations. Here we have an optimal policy which can either be stored as a lookup table (in case of finite classes) or the formula can be remembered and system can allocate power using this stored information and the estimate of the user speed. One can use this in self organizing networks.

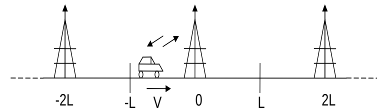


Fig. 1. User moving in a car while deriving service

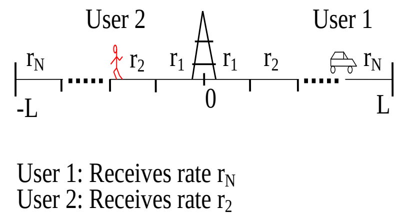


Fig. 2. Rate partitioning and user's movement

II. System Model

We consider users moving in a fixed direction. The users are moving in one direction (in a one dimensional line $[-D, D]$) and at high speeds, which vary negligibly during the call. One such example (see Figure 1) is when a user driving in a car derives its service from portable base stations (BSs) which are installed on street infrastructure (like lamp posts). The distance between successive base stations is $2L$. The users can move in one of the two directions with equal probability, i.e., with half probability. We assume symmetry in both the directions and hence any performance (e.g., busy probability, drop probability etc.), conditioned on the direction of the user, will be equal for both the directions. Thus, unconditional performance would be the same as the performance given a direction, say left to right. *So, without loss of generality we consider only users moving from left to right.* The user moves with speed V, which is uniformly distributed. We assume uniform arrivals. In small cell networks, the (adjacent) BSs are reasonably close and hence one has to design carefully with sufficient reuse factor to avail the advantages of small BS separations. In this work, we indeed assume this is the case, i.e., we assume no interference from the other cells.

Rate Regions: The cell is divided into $2N$ disjoint segments (depending upon distance from BS[1]) and users in a segment are served with the same transmission rate. Let $\{\mathbb{A}_n\}_{n\in\mathbb{N}}$ represent these segments (see Figure 2):

$$\begin{aligned}\mathbb{A}_n &:= \left[\frac{(n-1)L}{N}, \frac{nL}{N}\right] 1_{\{n>0\}} + \left[\frac{nL}{N}, \frac{(n+1)L}{N}\right] 1_{\{n<0\}};\\ \mathbb{N} &:= \{-N, \cdots, -1, 1, \cdots, N\}. \end{aligned} \quad (1)$$

Segments $\mathbb{A}_n, \mathbb{A}_{-n}$ placed at the same distance on either side of BS (Figure 2) are served with common rate r_n and these common rates decrease as the distance from BS increases. Let $\mathbb{R} := \{r_1, \ldots, r_N\}$ (decreasing set) represent

[1]In small cell networks (transmission at small distances), distance based propagation losses would be sufficient for deciding the theoretical rate limits as well as the practical transmission rates.

the ensemble of all possible transmission rates. The service rate changes once the user moves from one region to another.

Arrivals: There are two types of arrivals: 1) arrivals from external world (represented by subscript e and this subscript is used only when there is ambiguity) modeled as Poisson arrivals with parameter λ; 2) handover (HO) arrivals (subscript h) modeled again as Poisson[2] arrivals is the sub-stream of external arrivals whose service is not completed in the previous cell. Rate of arrivals into the cell of interest $[-L, L]$ (referred as cell 0) depends upon the cell dimension L and this is shown by either λ_L (for external arrivals) or $\lambda_{h;L}$ (for HO arrivals). For external arrivals, we assume [3] $\lambda_L = \lambda L$ while $\lambda_{h;L}$ will be calculated in later sections. Every arrival, brings along with it the marks (Φ, S), where $\Phi \in \mathbb{N}$ is the position of arrival with distribution $\Pi := \{\pi_n\}_n$ and S, the number of bytes to be transmitted, is exponentially distributed, i.e., $S \sim \mu exp^{-\mu t} dt$ for some $\mu > 0$.

Resources: A cell can attend K parallel calls. The power per transmission P can depend upon the speed of the user and we obtain the optimal power function given an average power constraint.

An example of $\mathbb{R}$: One can *choose the set of possible transmission rates, $\mathbb{R}$ and N based on the practical channel coding schemes* that would be used in the network design. The analysis presented can be extended to study a system with any given $\mathbb{R}$ and N. However, in this paper, we consider a specific example, which is important and which gives good closed form expressions. This specific $\mathbb{R}$ is obtained using low SNR approximation of the following theoretical (capacity) rate[4]:

$$r(d) := P\left(1_{\{d \le d_0\}} + r_0 \, |d|^{-\beta} \, 1_{\{d > d_0\}}\right) \text{ with } r_0 = d_0^{\beta}.$$

Here $r(d)$ is the rate at distance d, d_0 is a small lossless distance[5] while β is the propagation co-efficient. We consider a specific system which supports transmission at the maximum possible rate for the entire region. For example in $\mathbb{A}_n$, the farthest user will be at distance $|n|L/N$ and hence maximal transmission rate, that can be allocated, equals

$$r_n = r\left(\frac{|n|L}{N}\right) = r_0 P N^{\beta} L^{-\beta} |n|^{-\beta}. \tag{2}$$

Remark: *Alternatively, if the system under consideration can design modulation and or channel coding schemes so as to achieve (almost) ν percent of the theoretical rates where $\nu < 1$ is a fixed coefficient, then again the above rate structure is applicable (after absorbing ν into r_0 of (2)).*

[2] This is a commonly made assumption, see [8], [9].

[3] If the arrivals in the entire line segment $[-D, D]$ occur at rate λ' those in segment $[-L, L]$ occur at a smaller rate $\lambda_L = \lambda' Prob$(arrival in $[-L, L]$). For the special case of uniform arrivals (i.e., arrivals landing uniformly in $[-D, D]$) $\lambda_L = \lambda L$.

[4] For unit noise variance, capacity equals $log(1 + SNR)$, where signal to noise ratio $SNR = PA$ and attenuation $A = 1_{\{d \le d_0\}} + (d/d_0)^{-\beta} 1_{\{d > d_0\}}$. For low SNRs, $log(1 + SNR) \approx SNR$ and hence capacity equals PA.

[5] Typically d_0 is small and in this paper **we consider optimizing over cell sizes** $L > d_0 N$. That is, we always consider distances $d > d_0$ and hence use the simpler formula $r(d) = r_0 P d^{-\beta}$.

Handovers: When the user reaches the boundary $\{|\mathbf{x}| = L\}$ the call is handed over to the neighboring cell (if not completed and if free servers are available in the new cell).

Information to initiate HO: Every new connection requires s_h extra bytes to be exchanged to initiate it. The effect of these bytes (on the system performance) for a new call will be negligible (as it would be once), however one needs to consider their effect on HO calls. These bytes are usually very small in proportion to the actual information to be transmitted, i.e., $s_h \ll S$ with probability close to one. In particular we assume that $\mu s_h \ll 1$. We also assume that *s_h bytes are exchanged with probability very close to one (actually w.p.1), while user is in the last region r_{-N}.*

Notations: We denote the transpose by ^{t}. Calligraphic letters represent matrices. Mathbb letters represent sets (e.g., $\mathbb{N}$ - set of segment numbers, $\mathbb{R}$ - set of all possible transmission rates, $\mathbb{A}_n$ - rate region n). The contents inside two flower brackets represent either a set or an ordered tuple (as according to convenience): for example $\{r_n\}_n$ represents the set $\mathbb{R}$ while $\{\pi_n\}_n$ represents the ordered tuple Π. Lower case letters represent time index (k) or the segment index (n). Lower case bold letters represent the vectors.

Upper case letters represent system parameters. For example, D - dimension of Macrocell, L - dimension of small cell, P - Power per transmission, K - Number of servers, N - Total number of possible transmission rates (number of elements in $\mathbb{R}$) , $\Pi = \{\pi_n\}_n = \{Prob(\text{Arrival in segment } n)\}_n$ - Vector of arrival probabilities etc. Upper case letter can also represent random variables (V - Velocity, S - number of bytes to be transferred etc.).

III. User speed dependent Performance measures

In this section we derive the performance metrics, that capture the trade off between HO losses and the better service rates (mentioned in Introduction), for a given user speed profile. We consider non elastic traffic. Non elastic traffic comprises of users demanding immediate service. These users (e.g., voice calls) drop the call if it is not picked up immediately, i.e., if all the K servers are busy (see [1], [5]). The probability that a call is not picked up immediately is called the Busy probability and the probability that a call that was picked up is ever dropped before completing its service is called the Drop probability. Both these performance metrics depend upon HOs as well as the average service time and hence capture the required trade-off.

In this paper we work with metrics proportional to P_{Busy}, to be specific with load factor ρ (equation (8) derived in later sections). One can obtain P_{Drop} as in [5] and show that it can be approximately optimized by minimizing the load factor, but we omit its discussion here. Similarly in [5], we showed under certain conditions that the expected waiting time of an elastic user (an user who can wait for his turn but demands high quality of transmission) can be optimized by minimizing the load factor. We expect similar results to hold even in this case, but again we do not discuss these in this paper. *In all, we optimize the load factor ρ.*

Without loss of generality we consider cell 0, $[-L, L]$. Let ψ_n represent the probability that a call originated in rate segment n, $\mathbb{A}_n$, is completed before reaching boundary $\{L\}$ (one element set). The user is served at rate r_n and hence a total of $(L/N)(1/V)r_n$ bytes are transfered before he crosses $\mathbb{A}_n$ and then at rate r_{n+1} and so on till he reaches the boundary point L. Total number of bytes transfered during this transit equal $\sum_{m=n}^{N} r_m L/(NV)$ and so the user will complete its call (i.e., transfer of S (exponential) bytes of information) before leaving the cell with probability (conditioning on V):

$$\begin{aligned} \psi_n &= Prob\left(S < \frac{L}{NV}\sum_{m=n}^{N} r_m\right) \\ &= 1 - E\left[e^{-\frac{\mu L \sum_{m=n}^{N} r_m}{NV}}\right]. \end{aligned} \tag{3}$$

It is difficult to obtain the above Laplace transform and hence ψ_n, for uniformly distributed velocities. However one can obtain a good approximation if we assume, $V \sim \mathcal{U}([V_{min}, V_{max}])$ with V_{max} close to V_{min} and both away from 0. In the above $\mathcal{U}$ represents the uniform distribution. When the high speed users are partitioned to a large number of classes, each class will satisfy the above assumption. Then ($e^{-x} \approx 1 - x$ when x is small) one can approximate

$$\psi_n \approx \frac{\mu L \sum_{m=n}^{N} r_m}{N} E[1/V]. \tag{4}$$

On the other hand, one can approximate $1 - \psi_n$ directly with 1 as ψ_n in (4) is very small in comparison with 1. Let $P_{e,ho}$ ($P_{h,ho}$) represent the probability that a new (HO) call is handed over (again) to the neighboring cell. Recall here that we are modeling the HOs also as Poisson arrivals. A new call can arrive in any $\mathbb{A}_n$ with probability π_n while a handed over call always occur at $-L$, i.e., in rate region $\mathbb{A}_{-N}$ (calls are from left to right). Then by unconditioning (w.r.t. the event of arrival being in $\mathbb{A}_n$, i.e., conditioned on $\Phi = n$),

$$\begin{aligned} P_{e,ho} &= 1 - \sum_{n=-N}^{N} \pi_n \psi_n = 1 - PL^{1-\beta} E[1/V] C_{e,ho} \text{ with} \\ C_{e,ho} &:= \frac{\mu}{N} r_0 N^{\beta} \sum_{n=-N}^{N} \pi_n \sum_{m=n}^{N} |m|^{-\beta}. \end{aligned} \tag{5}$$

In the above $C_{e,ho}$ is an appropriate constant and is obtained by substituting r_m from (2). Every HO call needs exchange of extra s_h control bytes and an HO call arrives only in $\mathbb{A}_{-N}$. Thus using similar logic ($C_{h,ho}$ is another constant like $C_{e,ho}$),

$$\begin{aligned} P_{h,ho} = 1 - \psi_{-N,h} &:= 1 - Prob\left(S + s_h < \frac{L}{NV}\sum_{m=-N}^{N} r_m\right) \\ &\approx 1 - \left(C_{h,ho} PL^{1-\beta} E[1/V] - \mu s_h\right) \text{ with} \\ C_{h,ho} &:= \frac{\mu}{N} r_0 N^{\beta} \sum_{n=-N}^{N} |n|^{-\beta}. \end{aligned} \tag{6}$$

1) Expected Service time: the average amount of time for which a user is served in a cell. Let b_n represent the average time for which the service is received in cell 0, given call originated in region n. When *high speed users travel in small cells*, with high probability, a call is not completed in one cell and hence one can approximate b_n with the time taken to reach the boundary:

$$b_n = \frac{L}{N}(N - n) E\left[\frac{1}{V}\right].$$

The service time of a new call (b_e) and that of a handed over call (b_h) on average equals (by unconditioning)[6]:

$$\begin{aligned} b_e &= \sum_{i=-N}^{N} b_n \pi_n = C_{b,e} LE[1/V], \quad C_{b,e} := \sum_{n=-N}^{N} \pi_n \frac{N-n}{N} \\ b_h &= b_{-N} = C_{b,h} LE[1/V], \qquad C_{b,h} := 2. \end{aligned} \tag{7}$$

The expectation in $E[1/V]$ is with respect to the distribution of the user speeds, which is uniform for new arrivals. However, for the one in b_h (average service time for HO users) the corresponding distribution may not be uniform. The higher speed users are handed over more frequently than the lower speed ones. So, the distribution of the HO user velocities need not be uniform even if the new users arrive with uniform speeds. However as in [5, Appendix B], one can show that the HO user speed distribution converges to uniform distribution as the cell size reduces to zero and hence we approximate the HO user speed distribution with uniform distribution again. Further, clearly, this approximation is more accurate as the support of uniform distribution ($V_{max} - V_{min}$) decreases. Or equivalently when the number of velocity based user classes, considered in section V, increases.

Performance metrics obtained in the remaining part of this section, section III, are derived in a way similar to that in [5] and [1]. These derivations are obtained using the stochastic equivalence of calls going out of and coming into cell 0 (details in [1], [5]). We brief the derivations and more details can be found in the two papers. These derivations are required for obtaining the main results of the paper (section V).

2) HO Arrival rate: We obtain the rate, $\lambda_{h;L}$, at which HOs occur. Let λ_L represent the fraction of the new calls that arrive in the cell of interest $[-L, L]$ which equals λL for uniform arrivals (for appropriate $\lambda > 0$). A fraction of the new arrivals as well as HO calls get converted (again) to HO calls. The calls that have not finished their service before reaching the boundary are exactly this fraction, whose value is given by $P_{e,ho}$ and $P_{h,ho}$ respectively for new arrivals and the HO calls. By memory less nature of S (the bytes to be transferred) we have a stochastic equivalence between the calls entering and leaving the cell (see [5], [1]). Using this, $\lambda_{h;L}$ satisfies the fixed point equation, $\lambda_{h;L} = \lambda_L P_{e,ho} + \lambda_{h;L} P_{h,ho}$. Hence, from equations (5), (6)

$$\lambda_{h;L} = \frac{\lambda_L P_{e,ho}}{1 - P_{h,ho}} = \lambda L \frac{1 - PL^{1-\beta} E[1/V] C_{e,ho}}{PL^{1-\beta} E[1/V] C_{h,ho} - \mu s_h}.$$

3) Overall expected service time: (considering HO and new arrivals) equals (see (7)),

$$\bar{b} = \left(\frac{\lambda_L}{\lambda_L + \lambda_{h,L}} b_e + \frac{\lambda_{h;L}}{\lambda_L + \lambda_{h,L}} b_h\right).$$

By ergodicity, the probability that a given call is new, equals the ratio of the arrival rate of the new/external calls and the arrival rate of all the calls and hence the above result.

4) Load factor: The product of the average service time and the arrival rate gives the rate at which the overall load is arriving into the system and this product divided by the number of servers is called load factor, which is given by:

$$\rho = \frac{1}{K}(\lambda_L + \lambda_{h;L})\bar{b}.$$

[6]The time taken for exchange of s_h HO bytes has to be included in the time of the cell utilized by the user and hence the (HO) service time b_h. But since the rate of HOs is high and we are approximating b_h by the average time taken to traverse the entire cell, b_h does not change with s_h.

On Simplifying, we have

$$\rho = \frac{\lambda L^2 E[1/V]}{K}\left(C_{b,e} + C_{b,h}\frac{1 - PL^{1-\beta}E[1/V]C_{e,ho}}{PL^{1-\beta}E[1/V]C_{h,ho} - \mu s_h} \right). \quad (8)$$

5) Busy Probability: A small cell catering to non elastic traffic can be modeled by an M/G/K/K queue (as in [5]). Then using Erlang's loss formula, the busy probability can be calculated as,

$$P_{Busy} = \frac{\rho^K/K!}{\sum_{k=0}^{K}\rho^k/k!}. \quad (9)$$

Busy probability, depends upon L only via ρ and both are differentiable in L (see [5, Theorem 5] for similar details) and by differentiating twice one can immediately obtain the following:

Lemma 1: Optimizers of ρ and P_{Busy} are same, i.e.,

$$L_\rho^* := \arg\inf_L \rho = \arg\inf_L P_{Busy}(L) =: L^*_{P_{Busy}}. \ \square$$

By Lemma 1, the cell size optimizing the P_{Busy} is same as that optimizing load factor. In a similar way the power allocation affects busy probability only via the load factor ρ. Hence in the coming sections, we obtain the quantities that optimize the load factor, ρ.

IV. Maximum velocity supported by system

Before we proceed further, we would like to note down a limit on the speeds that can be supported, by a system with maximum N possible transmission rates and with transmission power P. This is equivalent of [5, Theorem 2] obtained for systems with 'maximal' transmission rates. A user entering at $-L$ when moving with speed V, can transfer in a cell, at maximum (see the discussions while deriving ψ_n given by (3) and using (2))

$$g^N(L) := \frac{L}{NV}\sum_{m=-N}^{N} r_m = C_{h,ho}\frac{PL^{1-\beta}}{\mu V}, \quad (10)$$

bytes of information, out of which s_h are used for HO purposes. So, useful communication is possible only when $g^N(L) > s_h$ with probability one. With $\beta > 1$ (the practical range of path loss factors), $g^N(L)$ reduces with L and so useful communication is not possible for any cell size if $g^N(Nd_0)$ itself is less than s_h and hence we have:

Theorem 1: When $\beta > 1$, there exists a limit on the maximum velocity that can be supported by the system for a given power P

$$V_{lim}(P) := \frac{1}{s_h} r_0 \sum_{n=-N}^{N} |n|^{-\beta} P d_0^{1-\beta}. \ \square$$

From the above theorem, given a set of system parameters, useful communication can be achieved by increasing the power P. In section V, we assume the transmission power P to be sufficient enough to ensure useful communication.

V. Optimal power: for cell size L, constraint $\bar{P}$

A. Finite number of User classes

The users are divided into different classes based on their speeds. Divide the (speed) interval $[V_{min}, V_{max}]$ into I disjoint intervals $\{\mathcal{I}_i\}_{i \le I}$ and classify users into one of the I classes based on their speed. From (8), load factor ρ depends upon the user speed profile only via $E[1/V]$ and so we are interested in the conditional expectation

$$\Upsilon_i := E\left[\frac{1}{V} \middle| V \in \mathcal{I}_i \right].$$

Let $p_i := Prob(V \in \mathcal{I}_i)$ represent the probability of class i and P_i the transmit power allocated to class i.

HO rates of the different user classes can be calculated as before:

$$\lambda_{h;L,i} = \lambda_{L,i}\frac{P_{e,ho,i}}{1 - P_{h,ho,i}} = \lambda L p_i \frac{1 - P_i L^{1-\beta}\Upsilon_i C_{e,ho}}{P_i L^{1-\beta}\Upsilon_i C_{h,ho} - \mu s_h}.$$

Similarly the expected service time for different user classes can be calculated and then the overall load factor ρ simplifies[7]:

$$\begin{aligned}\rho(L, \mathbf{P}) &= \frac{1}{K}\left(\lambda_L b_e + \sum_i \lambda_{h;L,i} b_{h,i} \right) \quad (11)\\ &= \frac{\lambda L^2}{K}\sum_i p_i \Upsilon_i \left(C_{b,e} + C_{b,h}\frac{1 - \delta_i C_{e,ho}}{\delta_i C_{h,ho} - \mu s_h} \right),\end{aligned}$$

where $\delta_i := P_i \Upsilon_i L^{1-\beta}$. We are interested in power allocation $\mathbf{P} := \{P_1, \cdots, P_I\}$ which minimizes ρ while the average power satisfies following constraint:

$$\sum_i p_i P_i \le \bar{P}.$$

By assuming that power constraint $\bar{P}$ is sufficiently large enough to ensure useful communication as explained in the previous section, one can solve this optimization problem and we obtain:

Theorem 2: Assume $C_{h,ho} - \mu s_h C_{e,ho} > 0$ and assume that the following matrix is positive definite,

$$\mathcal{P}_V := \begin{bmatrix} p_1 + \frac{p_1^2}{p_I} & \frac{p_1 p_2}{p_I} & \cdots & \frac{p_1 p_{I-1}}{p_I} \\ \frac{p_1 p_2}{p_I} & p_2 + \frac{p_2^2}{p_I} & \cdots & \frac{p_2 p_{I-1}}{p_I} \\ & & \vdots & \\ \frac{p_1 p_{I-1}}{p_I} & \frac{p_{I-1} p_2}{p_I} & \cdots & p_{I-1} + \frac{p_{I-1}^2}{p_I} \end{bmatrix} > 0.$$

Then the power allocation that minimizes ρ given by (11) while satisfying the average power constraint equals:

$$P_i^*(L; \bar{P}) = \bar{P} + \frac{\mu s_h L^{\beta-1}}{C_{h,ho}}\left(\frac{1}{\Upsilon_i} - \sum_j p_j \frac{1}{\Upsilon_j} \right). \quad (12)$$

Proof: Proof is available in Appendix A $\square$

From (5), (6) $C_{h,ho} \ge C_{e,ho}$ and from our earlier assumptions, $\mu s_h \ll 1$. So, the first assumption would be satisfied easily. The second assumption is often satisfied, for example when the users are classified into I uniform classes, i.e., when

$$\mathcal{I}_i = V_{min} + \left[\frac{i-1}{I}, \frac{i}{I} \right] (V_{max} - V_{min}) \text{ and so } p_i = \frac{1}{I} \text{ for all } i.$$

From (12) we obtain the following properties of the optimal power division: 1) allocated power increases with the speed range of users in a class (users with lower Υ_i get higher power), which in turn implies an increased virtual cell; 2) the disparity in the allocated powers between different classes increases with cell size, the disparity in speeds and the path-loss factor. *Thus from (12) in a medium with large (path) losses, one needs to allocate larger power to high speed users in order to improve the overall system performance. Intuitively otherwise, large speed users hold on to the system resources for longer time periods which can deteriorate the performance of the low speed users also. And the same applies to the case when the system has to support wide range of user speeds ($V_{max} - V_{min}$).*

B. Continuous optimal power law using HJB equations

When one has an accurate estimate of the velocity, it might be optimal to do finer power allocation, i.e., transmit power is varied continually based on the precise value of velocity. Further, as mentioned in the previous sections, the modeling in-accuracies (while obtaining handover rates, handover velocity distribution etc) reduce when the number of classes increase. With the continuous case of this subsection, we expect to model more accurately.

[7]Load factor ρ equals the product of total average arrival rate ($\bar{\lambda} := \lambda_L + \sum_i \lambda_{h;L,i}$) and average service time divided by K. An arrival turns out to be a new arrival with probability $\lambda_L/\bar{\lambda}$, it turns out to be an HO arrival of class i with probability $\lambda_{h;L,i}/\bar{\lambda}$ and so average service time equals $\left(\lambda_L b_e + \sum_i \lambda_{h;L,i} b_{h,i}\right)/\bar{\lambda}$.

We assume that user velocity is a continuous random variable with density $p_V(v)$ and with support in the range $[V_{min}, V_{max}]$ for some $0 \le V_{min} < V_{max} < \infty$. For uniform random variables, $p_V(v) \equiv 1/(V_{max} - V_{min})$.

We would like to obtain an optimal power law, which is basically a function of velocity v and which optimizes the load factor ρ. By power law we meant a function $P(.)$ which maps velocity v to a positive real number, that represents the power allocated to user traveling with speed v. We are only interested in those functions, whose average is constrained by $\bar{P}$, i.e., we are interested in policies, $\{P(.)\}$, that satisfy:

$$\int_{V_{min}}^{V_{max}} P(v) p_V(v) dv \le \bar{P}.$$

Let B represent the random service time (the time for which the cell resources are utilized by the user). The conditional average service times $b_e(v)$, $b_h(v)$, the probabilities of service not getting completed $P_{e,ho}(v)$, $P_{h,ho}(v)$ and the HO rates $\lambda_{h;L}(v)$ can be computed as in the previous section, after conditioning on the user velocity V. They depend upon the power policy $P(.)$. It is easy to see that these conditional expressions will be same as in the previous section except that $\Upsilon_v = E[1/V | V \in (v - dv, v + dv)]$ (conditioned that V is in an infinitesimal interval around v) has to be replaced by $1/v$, i.e., for example:

$$\begin{aligned} b_e(v) &= \frac{C_{b,e} L}{v}, \quad P_{e,ho}(v) = 1 - \frac{P(v) L^{1-\beta} C_{e,ho}}{v} \\ b_h(v) &= \frac{C_{b,h} L}{v}, \quad P_{h,ho}(v) = 1 - \frac{P(v) L^{1-\beta} C_{h,ho}}{v} + \mu s_h. \end{aligned}$$

Overall arrival rate: By conditioning and un-conditioning on velocity V, the overall arrival rate including the HOs, equals (E is the expectation with respect to random velocity V):

$$\bar{\lambda} = E[\lambda_L(V) + \lambda_{h;L}(V)] = \lambda L E\left[1 + \frac{P_{e,ho}(V)}{1 - P_{h,ho}(V)}\right] \quad (13)$$

Average of the overall service time B: Let $Z = (H_O, V)$ represent a joint random variable in which the first component is a indicator that it is due to HO, i.e., $H_O = 1$ implies it is an HO call. The following are the events given that a call has arrived. We first consider the HO case. Let $\ddot{V}$ and $\ddot{H}_O$ represent the velocity and the call type (HO or external) in the previous cell before the Handover to the current cell. That is, $\ddot{H}_O = 1$ implies it was a HO call in the previous cell and is again handed over to the current cell. For any set $\mathcal{A}$, since the velocity of a user remains constant during the call:

$$\begin{aligned} &P(H_0 = 1, V \in \mathcal{A}) = P(H_0 = 1, \ddot{V} \in \mathcal{A}) \\ &\quad = P(H_0 = 1, \ddot{V} \in \mathcal{A}, \ddot{H}_O = 0) + P(H_0 = 1, \ddot{V} \in A, \ddot{H}_O = 1) \end{aligned}$$

By stationarity $P(\ddot{H}_O = 0) = P(HO = 0)$ and this equals the probability that a call arrived is a new call. By ergodicity this equals $\lambda_L / \bar{\lambda}$. Thus, by conditioning first on $\ddot{H}_O$ and then on $\ddot{V}$ we obtain:

$$\begin{aligned} &P(H_0 = 1, \ddot{V} \in \mathcal{A}, \ddot{H}_O = 0) \\ &\qquad = P(H_0 = 1 \ddot{V} \in \mathcal{A} | \ddot{H}_O = 0) P(\ddot{H}_O = 0) \\ &\qquad = \int_A P_{e,ho}(v) p_V(v) dv \, \frac{\lambda_L}{\bar{\lambda}}. \end{aligned}$$

Considering infinitesimal interval $vdv := [v - dv, v + dv]$ and conditioning as in the previous equation we obtain:

$$\begin{aligned} &P(H_0 = 1, \ddot{V} \in vdv, \ddot{H}_O = 1) \\ &\quad = P(H_0 = 1 | \ddot{V} \in vdv, \ddot{H}_O = 1) P(\ddot{H}_O = 1, \ddot{V} \in vdv) \\ &\quad \approx P_{h,ho}(v) P(\ddot{H}_O = 1, \ddot{V} \in vdv). \end{aligned}$$

By stationarity and memoryless property of S, $P(\ddot{H}_O = 1, \ddot{V} \in vdv) = P(H_O = 1, V \in vdv)$ and thus,

$$P(H_0 = 1, V \in vdv) \approx \frac{P_{e,ho}(v) p_V(v) dv}{1 - P_{h,ho}} \frac{\lambda_L}{\bar{\lambda}}. \quad (14)$$

In a similar way,

$$\begin{aligned} P(H_0 = 0, V \in vdv) &= P(V \in vdv | H_0 = 0) \frac{\lambda_L}{\bar{\lambda}} \\ &\approx p_V(v) dv \frac{\lambda_L}{\bar{\lambda}}. \quad (15) \end{aligned}$$

Note that service requirements S is independent of $Z = (H_0, V)$. Hence, using equations(13)-(15) and conditioning on Z, the average of the overall service time equals:

$$\bar{b} = \frac{E\left[b_e(V) + b_h(V) \frac{P_{e,ho}(V)}{1 - P_{h,ho}(V)}\right]}{E\left[1 + \frac{P_{e,ho}(V)}{1 - P_{h,ho}(V)}\right]} = \frac{L}{E\left[1 + \frac{P_{e,ho}(V)}{1 - P_{h,ho}(V)}\right]} \quad (16)$$

$$\int_{V_{min}}^{V_{max}} \left(C_{b,e} + C_{b,h} \frac{1 - \frac{P(v) L^{1-\beta} C_{e,ho}}{v}}{\frac{P(v) L^{1-\beta} C_{h,ho}}{v} - \mu s_h}\right) \frac{p_V(v)}{v} dv. \quad (17)$$

And so the load factor equals:

$$\rho = \frac{\lambda L^2}{K} \int_{V_{min}}^{V_{max}} \left(C_{b,e} + C_{b,h} \frac{1 - \frac{P(v) C_{e,ho}}{L^{\beta-1} v}}{\frac{P(v) C_{h,ho}}{L^{\beta-1} v} - \mu s_h}\right) \frac{p_V(v)}{v} dv. \quad (18)$$

We are interested in power law $\{P^*(.)\}$ which minimizes the load factor ρ under the average power constraint. One can rewrite:

$$\frac{1 - \frac{P(v) C_{e,ho}}{L^{\beta-1} v}}{\frac{P(v) C_{h,ho}}{L^{\beta-1} v} - \mu s_h} = -\frac{C_{e,ho}}{C_{h,ho}} + \frac{1 - \mu s_h \frac{C_{e,ho}}{C_{h,ho}}}{\frac{P(v) C_{h,ho}}{L^{\beta-1} v} - \mu s_h}.$$

We again assume, $C_{e,ho} \mu s_h < C_{h,ho}$ and so the numerator of the second factor is positive and after leaving out the constants the optimal power law is given by:

$$P^*(.) = \arg \inf_{P(.)} \int_{V_{min}}^{V_{max}} \frac{1}{P(v) C_{h,ho} - v L^{\beta-1} \mu s_h} p_V(v) dv$$

$$\text{subject to} \int_{V_{min}}^{V_{max}} p_V(v) P(v) \le \bar{P}. \quad (19)$$

With ϱ representing the Lagrange multiplier we minimize:

$$\int_{V_{min}}^{V_{max}} \left(\frac{1}{P(v) C_{h,ho} - v L^{\beta-1} \mu s_h} - \varrho (P(v) - \bar{P})\right) p_V(v) dv.$$

This is a state independent optimal control problem, for any given ϱ and can be solved using Hamilton-Jacobi-Bellman (HJB) equation (see for e.g., [6], [7]):

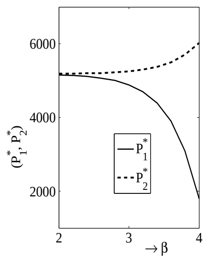
$$\frac{dU}{dv} = p_V(v) \inf_P \left\{\frac{1}{C_{h,ho} P - \mu s_h L^{\beta-1} v} - \varrho P\right\} \quad (20)$$

where U, the value function, is given by:

$$U(v) := \inf_{P(.)} \int_v^{V_{max}} \left[\frac{1}{P(v) C_{h,ho} - v L^{\beta-1} \mu s_h} - \varrho P(v)\right] p_V(v) dv.$$

The value function U satisfies the HJB equation (20) when it is continuously differentiable (e.g., [6], [7]) and we will see that this indeed is the case. The optimal control $P^*()$ equals the minimizer of the optimization in the right hand side of the HJB equation (20) (see for example [6], [7]). This can be computed easily for any v and

$$P^*(v) = \frac{1}{C_{h,ho}} \mu s_h L^{\beta-1} v + \sqrt{\frac{1}{C_{h,ho} \varrho}}.$$



Settings
$N = 3 \; d_0 = 5$
$r_0 = .4d_0^\beta \; \gamma = 0.5$
$K = 500$
$\lambda = 10^{-7} \; s_h = 0.2$
$\mu = 0.000001$
$(\Upsilon_1, \Upsilon_2) = (.067, .011)$
$(p_1, p_2) = (0.2, 0.8)$
$\bar{P} = 0.0001, \, L = 160$

Fig. 3. Power allocated optimally to two user classes

The average power equals:

$$\int_{V_{min}}^{V_{max}} P^*(v) p_V(v) dv = \frac{1}{C_{h,ho}} \mu s_h L^{\beta-1} E[V] + \sqrt{\frac{1}{C_{h,ho}\varrho}}.$$

Equating the above to average power constraint $\bar{P}$, we obtain the optimal Lagrange multiplier ϱ and then the optimal power law:

Theorem 3: The power function $P^*(.)$ that optimizes ρ given by (18), while satisfying the average power constraint equals:

$$P^*(v) = \bar{P} + \frac{1}{C_{h,ho}} \mu s_h L^{\beta-1} \left(v - E[V]\right). \qquad \square \tag{21}$$

One can easily verify that the continuous power law (21) equals[8] the limit of the discrete optimal power policy (12). Thus we obtained optimal power law via two different methods for continuous control and the two methods are resulting in the same solution. Also, the optimal power allocation increases linearly with the speed v.

The load factor at optimal power law parametrized by threshold $\bar{P}$ equals:
$\rho^*(L; \bar{P}) =$

$$\begin{aligned} &\frac{\lambda L^2}{K} \int_{V_{min}}^{V_{max}} \left(\frac{C_{\rho,1}}{v} + \frac{C_{\rho,2}}{\bar{P} L^{1-\beta} C_{h,ho} - E[V] \mu s_h} \right) p_V(v) dv \\ &= \frac{\lambda L^2}{K} \left(C_{\rho,1} E\left[\frac{1}{V}\right] + \frac{C_{\rho,2}}{\bar{P} L^{1-\beta} C_{h,ho} - E[V] \mu s_h} \right) \end{aligned} \tag{22}$$

with $C_{\rho,1} := C_{b,e} - \dfrac{C_{b,h} C_{e,ho}}{C_{h,ho}}, C_{\rho,2} := C_{b,h} \left(1 - \mu s_h \dfrac{C_{e,ho}}{C_{h,ho}}\right)$.

VI. Numerical Examples

We reinforce the theory developed so far using numerical examples. We first consider a two class example in Figure 3 (with settings as mentioned in the figure) and study the differences in the optimal powers allocated to the two sets of users. We plot the two optimal powers (given by (12)) as a function of β, the path loss factor. We notice: more the losses (path-loss factor) are, more diverse the two allocated powers are. Thus the requirement of power allocation increases as the (path) losses increase.

Hence after we consider examples with continuous optimal power law. In all these examples, we refer the minimum speed V_{min} by V_1 and the maximum speed V_{max} by V_2. We first study the improvement in busy probability (see Figure 4), obtained using optimal power law. By and large parameter setting remains same as in Figure 3 except for $\mu = 10^{-3}, \beta = 3, L = 885, \bar{P} = 10000$. K = 9 and 17 is taken for first and second part of Figure 4 respectively. Quantities ρ^* and P^*_{busy} are the performance (ρ given by (18) and P_{Busy} given by (9)) using optimal power law (given by (21)) while ρ and P_{busy} are the performance obtained by allocating equal power to all users (i.e., when $P(v) \equiv \bar{P}$). In Figure 4, we plot the two sets of performance for different velocity profiles. More specifically,

[8]As one increases the number of user classes I to infinity, the conditional expectation Υ converges to $1/V$ while the sum for any I equals, $\sum_i p_i \Upsilon_i = E[1/V]$, the unconditional expectation.

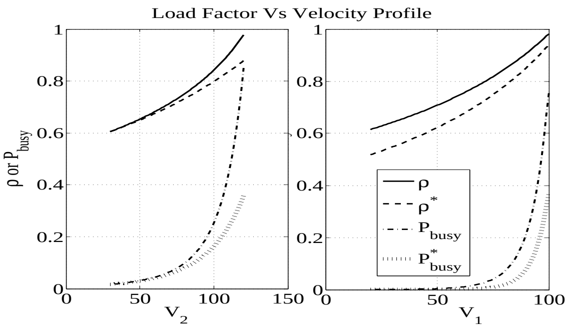


Fig. 4. Comparison of performance with and without optimal power allocation

we kept V_1 fixed at 15 and varied V_2 from 30 to 120 in first part of Figure 4 while V_2 is kept fixed at 135 and V_1 varies from 20 to 100 in second part. From both the figures, it is clear that we obtain significant improvement (upto 60%) in busy probability with optimal power law.

While conducting the experiments we studied the improvement obtained with finite classes. We notice that the performance with finite classes converges to that of the continuous one only with $I \geq 10$. This shows that one need very fine control (more number of user classes) to obtain the best performance (for example, in Figure 4 we have upto 60% improvement in P_{Busy} with continuous optimal power law).

Performance comparison

When the users are classified based on their speed into finite number of classes, we estimate the handover rates $(\{\lambda_{h;L,i}\}_{i \leq I})$ corresponding to each class. That is, we obtain the rate at which handovers occur, which is treated to be common for all the users in a class. However, it is easy to verify that, in reality the handover rates vary monotonically with the speed of the user. Thus with finer number of classes one may have modeling issues. We expect the expression (18) for ρ, obtained considering continuous variations in speed V, to model the actual load factor more accurately. In future, we wish to establish this either via numerical simulations or via theoretical analysis. We are considering (18) for comparing the finite power polices (12) with the continuous one (21).

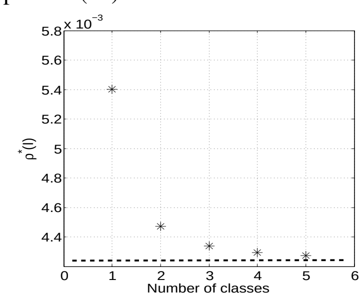


Fig. 5. ρ^* decreases with I and approaches to one with continuous law

When one can estimate the speed accurately, one can use finer power allocation and we expect this to improve the performance significantly. This is established in Figure 5. Experimental setting for this figure is same as that of Figure 3 except $\mu = 10^{-3}, \beta = 2, \bar{P} = 185, V_{min} = 20$ and $V_{max} = 1000$. We obtain the curve in the figure by calculating equation (18) with power policies given in (12) for different values of I (the stars in Figure 5). We also plot (18) with continuous optimal policy (21) as a straight line (a dash line in Figure 5). We notice that the load factor with optimal power law using finite classes decreases monotonically towards that of the continuous optimal power law. Thus one obtains best performance when one uses the optimal linear power law.

Conclusions

High speed users pose serious design problems for small cell networks in the form of frequent hand-overs. Further, the optimal cell (dimension) design is sensitive to the speed of the user. For efficient design one needs to vary inter base station distances based on the speed of the user (see [5]). However it is practically infeasible to do so. Alternately, we propose to reflect the speed based (optimal) design variations in the allocated powers. We obtain a closed form expression for optimal power allocation (optimal for load factor and or busy probability), which allocates different transmit powers to different user (speed) classes, for any given average power constraint and the cell size. The optimal allocation ensures larger power to higher speeds and the differences in the powers allocated increase with path loss factor and the disparities in the speeds. If it is possible to estimate user speed accurately, one can obtain a very fine optimal power control with speed. We show that the optimal power varies linearly with the user speed and this result is obtained using Hamilton-Jacobi-Bellman equations. We observe via numerical simulations that, there is large improvement in busy probability and load factor when optimal power allocation is used instead of equal powers for all users. The improvement in performance increases as the number of user classes increases and one obtains the best improvement with continuous optimal power law. Further systems with large (path) losses and or the ones which support wide variations of user speeds, improve significantly with optimal power allocation.

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APPENDIX A

Proof of Theorem 2: The average power constraint can be satisfied with equality by substituting

$$P_I = \frac{1}{p_I}\left(\bar{P} - \sum_{i<I} p_i P_i\right),$$

and then (with $\mathbf{P}_{-I} := (P_1, \cdots, P_{I-1})$)

$$\begin{aligned}
\rho(L;\bar{P}) &:= \rho\left(L, \left(\mathbf{P}_{-I}, \frac{\bar{P} - \sum_{i<I} p_i P_i}{p_I}\right)\right) \\
&= \frac{\lambda L^2}{K}\sum_{i<I} p_i \Upsilon_i \left(C_{b,e} + C_{b,h}\frac{1-\delta_i C_{e,ho}}{\delta_i C_{h,ho} - \mu s_h}\right) \\
&\quad + \frac{\lambda L^2}{K} p_I \Upsilon_I \left(C_{b,e} + C_{b,h}\frac{1-\bar{\delta}(\mathbf{P}_{-I}) C_{e,ho}}{\bar{\delta}(\mathbf{P}_{-I}) C_{h,ho} - \mu s_h}\right), \\
\bar{\delta}(\mathbf{P}_{-I}) &:= \frac{1}{p_I}\left(\bar{P} - \sum_{i<I} p_i P_i\right)\Upsilon_I L^{1-\beta}.
\end{aligned}$$

We obtain the optimizer of the load factor ρ via the zeros of the derivatives (if they exist). The partial derivatives (for all $i < I$) are given by:

$$\begin{aligned}
\frac{d\rho}{dP_i} &= \frac{\lambda}{K} C_{b,h} p_i \Upsilon_i^2 L^{3-\beta} \frac{C_{e,ho}\mu s_h - C_{h,ho}}{\left(\delta_i C_{h,ho} - \mu s_h\right)^2} \qquad (23) \\
&\quad - \frac{\lambda}{K} C_{b,h} p_i \Upsilon_I^2 L^{3-\beta} \frac{C_{e,ho}\mu s_h - C_{h,ho}}{\left(\bar{\delta}(\mathbf{P}_{-I}) C_{h,ho} - \mu s_h\right)^2} \\
&= \frac{\lambda}{K} C_{b,h} p_i L^{3-\beta}\left(C_{e,ho}\mu s_h - C_{h,ho}\right) \qquad (24) \\
&\quad \left(\frac{\Upsilon_i^2}{\left(\delta_i C_{h,ho} - \mu s_h\right)^2} - \frac{\Upsilon_I^2}{\left(\bar{\delta}(\mathbf{P}_{-I}) C_{h,ho} - \mu s_h\right)^2}\right) (25)
\end{aligned}$$

One can easily obtain the zero of equation (23), i.e., the equilibrium point. Equating the partial derivatives to zero, $\partial\rho/\partial P_i = 0$ for all i, the optimal power allocation $\mathbf{P}(L, \bar{P})$, for a given cell size and average power constraint $\bar{P}$ equals:

$$\begin{aligned}
\mathbf{P}^*(L) &= \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 1 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ p_1 & p_2 & p_3 & \cdots & p_{I-1} & p_I \end{bmatrix}^{-1} \frac{\mu s_h L^{\beta-1}}{\Upsilon_I C_{h,ho}} \mathbf{v}_\mu \\
\mathbf{v}_\mu &:= \begin{bmatrix} \frac{\Upsilon_1 - \Upsilon_I}{\Upsilon_1}, & \frac{\Upsilon_2 - \Upsilon_I}{\Upsilon_2}, & \cdots, & \frac{\Upsilon_{I-1} - \Upsilon_I}{\Upsilon_{I-1}}, & \bar{P} \end{bmatrix}. \qquad (26)
\end{aligned}$$

This can be solved easily and

$$\begin{aligned}
P_i^*(L;\bar{P}) &= \bar{P} + \frac{\mu s_h L^{\beta-1}}{C_{h,ho}} \sum_{j<I} p_j \left(\frac{1}{\Upsilon_I} - \frac{1}{\Upsilon_j}\right) \\
&\quad - \frac{\mu s_h L^{\beta-1}}{C_{h,ho}}\left(\frac{1}{\Upsilon_I} - \frac{1}{\Upsilon_i}\right).
\end{aligned}$$

This simplifies to (12). Differentiating (23) again the Hessian matrix equals:

$$\frac{2C_{h,ho}\lambda C_{b,h} L^{4-2\beta}\left(C_{h,ho} - C_{e,ho}\mu s_h\right)\Upsilon_I^3}{K\left(\bar{\delta}(\mathbf{P}_{-I}) C_{h,ho} - \mu s_h\right)^3}\mathcal{P}_V,$$

where $\mathcal{P}_V$ is defined in the hypothesis of the theorem. This is a positive definite matrix under the given assumptions and hence $\mathbf{P}^*$ given by (12) is indeed a minimizer. $\square$