Paper Notes

- SIR : Susceptible - Infected - Recovered
- Unique model params :
  - Initial # of susceptible individuals
  - Proportionality factor b/w Detected # of positives with the actual # of infected
- 2D grid search + weighted least squares optimization
- SIR is an eg. of Collective model
- Collective models :
  - Smaller # of parameters
  - Differential / Difference time equations
  - Parameters Need to be identified from the data
  - Usually requires soln. of non-convex optimization problems
- Methodology :
  - Assume model structure + underlying parameters
  - Construct a grid of parameters on which the model has non-linear dependence.
  - At each pt., identify all other params via convex optimization.
  - Optimal param estimate is chosen by minimizing some objective function over the grid.
  - Works with collective models due to a low numbers of parameters

- Not extremely reliable for long-term predictions.
- Long term prediction algorithm: weighted average of multi-step predictions performed by starting the simulation at all available initial conditions.

- $S(t)$ = # of individuals susceptible of contracting infection at t.
  $I(t)$ = # of infected individuals at t
  $R(t)$ = # of individuals that recovered upto t.
  $D(t)$ = # of deceased individuals upto t.

- Kermack - McKendrick equations:

$$S(t+1) = S(t) - \beta \frac{S(t)\,I(t)}{S(t)+I(t)}$$

$$I(t+1) = I(t) + \beta \frac{S(t)\,I(t)}{S(t)+I(t)} - \gamma I(t) - \nu I(t)$$

$$R(t+1) = R(t) + \gamma I(t)$$

$$D(t+1) = D(t) + \nu I(t)$$

β = Infection Rate
γ = Recovery Rate
ν = Mortality Rate
(time expressed in days)

- Underlying assumptions:
    - Recovered individuals are no longer susceptible to infection
    - Any other causes of death are disregarded

- $\tilde{I}(t)$ = Portion of symptomatic individuals

$I(t) = \alpha \tilde{I}(t), \quad \alpha \geq 1$

Model on Detection basis:

$$\left.\begin{aligned}
\tilde{S}(t+1) &= \tilde{S}(t) - \beta \frac{\tilde{S}(t)\tilde{I}(t)}{\tilde{S}(t)+\tilde{I}(t)} \\
\tilde{I}(t+1) &= \tilde{I}(t) + \beta \frac{\tilde{S}(t)\tilde{I}(t)}{\tilde{S}(t)+\tilde{I}(t)} - \gamma \tilde{I}(t) - \nu \tilde{I}(t) \\
\tilde{R}(t+1) &= \tilde{R}(t) + \gamma \tilde{I}(t) \\
D(t+1) &= D(t) + \alpha \nu \tilde{I}(t)
\end{aligned}\right] \text{- SIRD model}$$

$$\Rightarrow \tilde{S}(t) = \frac{1}{\alpha} S(t)$$

$$\tilde{R}(t) = \frac{1}{\alpha} R(t)$$

- Want to estimate $S(t_0), \alpha, \beta, \gamma, \nu$
- Let P: Population under observation

$$\Rightarrow S(t_0) = \omega P, \quad \omega \in [0,1]$$

$$\tilde{S}(t) = \frac{\omega}{\alpha} P - \tilde{I}(t) - \tilde{R}(t) - \tilde{D}(t)$$

Potential Model: Optimistic Reinfection Model.

- Ideas:
  - Include a reinfection parameter
  - Introduce an "optimistic" testing model:
    - Testing rate $\alpha(t)$ begins at some initial condition $\alpha(t_0)$
    - Eventually plateaus at 1,

$$\frac{dS}{dt} = -\beta S I + \gamma I$$

$$\frac{dI}{dt} = \beta S I - \gamma I - \nu I + \mu R$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

$$\frac{dD}{dt} = \nu I$$

$$\frac{dRe}{dt} = \mu R$$

$$\alpha(t) = \tan^{-1}\left(\alpha\left(t + \frac{t_0}{\alpha}\right)\right)$$

, μ = reinfection rate,

$\tilde{I}(t) = \alpha(t) I(t)$

Let (S^*, I^*) be the equilibrium pt of the set of DEs.

$$\frac{dS}{dt} = f(S^*, I^*) + (S - S^*)\frac{\partial f}{\partial S} + (I - I^*)\frac{\partial f}{\partial I} + \cdots$$

$$\frac{dI}{dt} = g(S^*, I^*) + (S - S^*)\frac{\partial g}{\partial S} + (I - I^*)\frac{\partial g}{\partial I} + \cdots$$

$$\frac{d\alpha}{dt} = \frac{1}{1 + \left(\alpha\left(t + \frac{t_0}{\alpha}\right)\right)^2}$$

$$\frac{d\tilde{S}}{dt} = -\beta \tilde{S}(t)\tilde{I}(t) + \gamma \tilde{I}(t)$$

$$\frac{dI}{dt} = \beta \tilde{S}(t)\tilde{I}(t) - \gamma \tilde{I}(t) - \nu \tilde{I}(t) + \mu \tilde{R}(t)$$

$$\frac{dR}{dt} = \gamma \tilde{I}(t) - \mu \tilde{R}(t)$$

$$\frac{dD}{dt} = \nu \frac{1}{\alpha(t)} \tilde{I}(t)$$

$$\frac{d\tilde{R}e}{dt} = \mu \tilde{R}(t)$$

$$\begin{bmatrix} \tilde{S}(t) \\ \tilde{I}(t) \\ \tilde{R}(t) \\ D(t) \end{bmatrix}$$

$$\begin{bmatrix} \tilde{I}(t+1) - \tilde{I}(t) \\ \tilde{R}(t+1) - \tilde{R}(t) \\ D(t+1) - D(t) \\ \tilde{R}e(t+1) - \tilde{R}e(t) \end{bmatrix} = \phi_\omega(t) \begin{bmatrix} \beta \\ \gamma \\ \nu \\ \mu \end{bmatrix}$$

$$\phi_\omega(t) = \begin{bmatrix} \frac{\tilde{S}(t)\tilde{I}(t)}{S(t)+I(t)} & -\tilde{I}(t) & -\tilde{I}(t) & \tilde{R}(t) \\ 0 & \tilde{I}(t) & 0 & -\tilde{R}(t) \\ 0 & 0 & \frac{1}{\alpha(t)}\tilde{I}(t) & 0 \\ 0 & 0 & 0 & \tilde{R}(t) \end{bmatrix}$$

$$\frac{dS}{dt} = -\beta S I + \gamma I$$

$$\frac{dI}{dt} = \beta S I - \gamma I - \nu I + \mu R$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

$$\frac{dD}{dt} = \nu I$$

$$\frac{dRe}{dt} = \mu R$$

$\alpha(t) = \tan^{-1}\left(\alpha\left(t + \frac{t_0}{\alpha}\right)\right)$, μ = reinfection rate,
$\tilde{I}(t) = \alpha(t) I(t)$

Equilibrium pts.:

$$-\beta S I + \gamma I = 0$$
$$\beta S I - \gamma I - \nu \quad + \mu \quad = 0$$

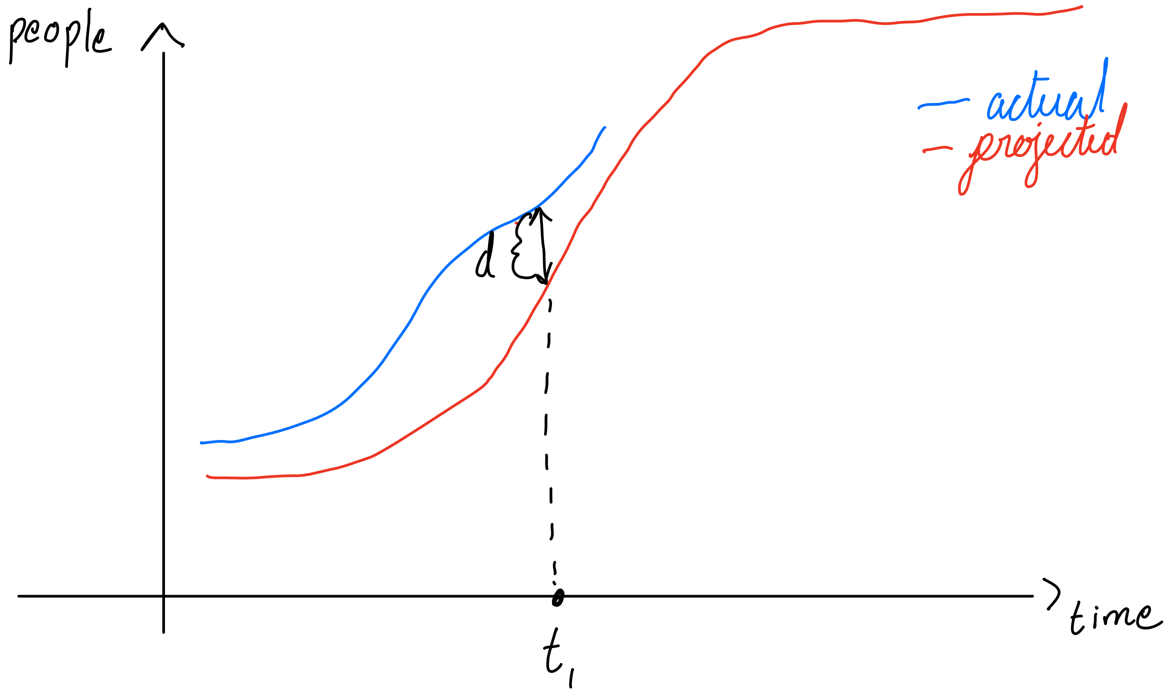
$I = 0 \Rightarrow S =$ whatever
$R = 0$

$$J = \begin{bmatrix} -\beta I & -\beta S + \gamma & 0 & 0 & 0 \\ \beta I & \beta S - \gamma - \nu & \mu & 0 & 0 \\ 0 & \gamma & -\mu & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \end{bmatrix}$$

$$J_{eq} = \begin{bmatrix} 0 & \gamma & 0 & 0 & 0 \\ 0 & -\gamma-\nu & \mu & 0 & 0 \\ 0 & \gamma & -\mu & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \end{bmatrix}$$

$$\lambda = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{\gamma}{2} - \frac{\mu}{2} - \frac{\beta S}{2} - \frac{1}{2} (\\ \frac{1}{2} (\beta S \quad \mu \quad \gamma \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \left[\right. \right.$$



$$re(t_1) = \underbrace{r}_{\text{random parameter } \in [0,1]} \cdot d = r\,(\text{actual}(t_1) - \text{projected}(t_1)).$$

Some stability analysis

$$J_{eq} = \begin{bmatrix} 0 & -\beta S + \gamma & 0 & 0 & 0 \\ 0 & \beta S - \gamma - \nu & \mu & 0 & 0 \\ 0 & \gamma & -\mu & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \end{bmatrix}$$

at $S = S_0$

the non-zero eigenvalues of this system (MATLAB) are

$$\lambda = -\frac{1}{2}(\beta S_0 + \gamma + \mu + \nu) \pm \frac{1}{2}\sqrt{D}, \text{ where}$$

$$D = (\beta S_0 + \gamma + \mu + \nu)^2 - 4\mu(\beta S_0 + \nu)$$

In real world situations, $S_0 \gg \beta, \gamma, \mu, \nu$

$$\Rightarrow \lambda \approx -\frac{1}{2}\beta S_0 \pm \sqrt{(\beta S_0)^2 - 4\mu\beta S_0}$$

i) $D < 0$

$$\Rightarrow (\beta S_0)^2 < 4\mu \beta S_0$$

$$\Rightarrow \beta S_0 < 4\mu \Rightarrow \beta S_0 - 4\mu < 0$$

$$-\frac{1}{2}\beta S_0 \pm \sqrt{(\beta S_0)^2 - 4\mu\beta S_0}$$

$$\beta S_0^2 - 4\mu\beta S_0 \geqslant 0$$