Aproximación por mínimos cuadrados

Motivación. Ajuste discreto lineal. Ajuste polinómico. Ajustes potencial y exponencial. Ejemplos con Python. Ajuste continuo polinómico, Funciones ortogonales. Polinomios de Legendre. Polinomios de Chebishev.

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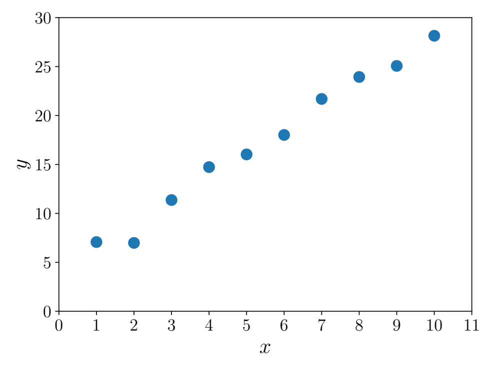
Ajuste discreto

Datos:

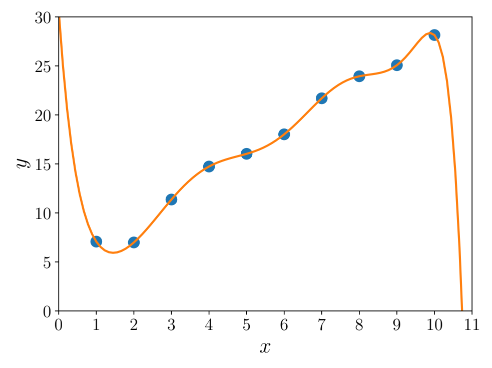
x_i	y_i	x_i	y_i
1	7.07	6	18.02
2	6.99	7	21.69
3	11.37	8	23.94
4	14.73	9	25.07
5	16.03	10	28.15

Figura:



Ajuste exacto: polinomio de grado 9

$$
\begin{aligned}
P_9(x) = 30.63 - 55.47x + 53.23x^2 \\
- 30.82x^3 12.24x^4 - 3.21x^5 \\
+ 0.53x^6 - 0.053x^7 + 0.0029x^8 \\
- 6.4 \times 10^{-5} x^9
\end{aligned}
$$



Ajuste exacto: polinomio de grado 9

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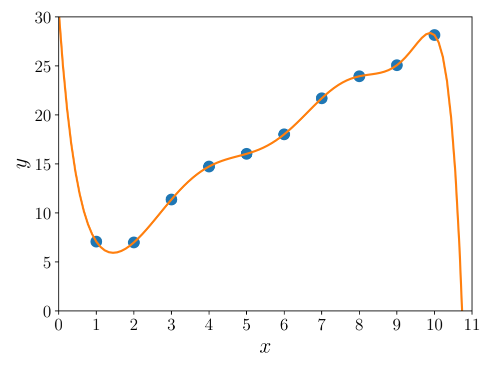

Aproximación lineal:

$$y = a_1\,x + a_0$$

- Problema **minimax**:

$$E_\infty(a_0, a_1) = \max_{1 \le i \le 10} \{|y_i - (a_1 x_i + a_0)|\}$$

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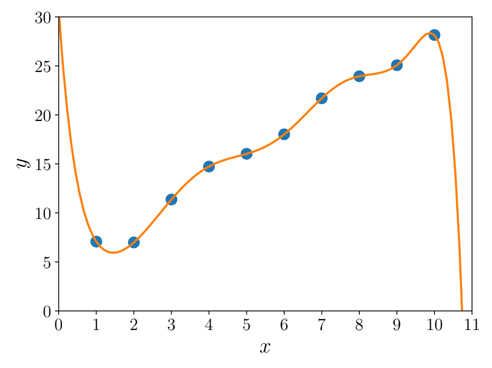

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$$E_1(a_0, a_1) = \sum_{i=1}^{10} |y_i - (a_1 x_i + a_0)|$$

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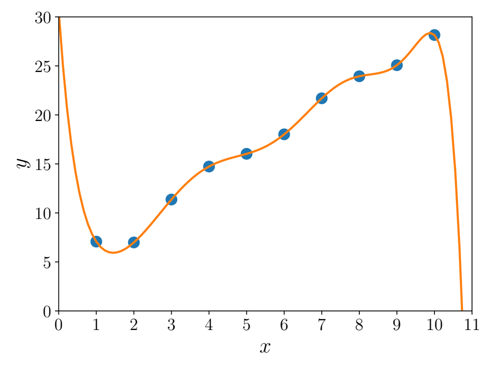

Aproximación lineal:

$$y = a_1\, x + a_0$$

- Problema **minimax**:

$$E_\infty(a_0, a_1) = \max_{1 \le i \le 10} \{|y_i - (a_1 x_i + a_0)|\}$$

- Desviación absoluta:

$$E_1(a_0, a_1) = \sum_{i=1}^{10} |y_i - (a_1 x_i + a_0)|$$

- **Mínimos cuadrados:**

$$E_2(a_0, a_1) = \sum_{i=1}^{10} [y_i - (a_1 x_i + a_0)]^2$$

Para el conjunto $\{(x_i, y_i)\}_{i=1}^{m}$, **minimizar** respecto de a_0, a_1:

$$E \equiv E_2(a_0, a_1) = \sum_{i=1}^{m} [y_i - (a_1 x_i + a_0)]^2$$

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$$\frac{\partial E}{\partial a_0} = 0 \quad \text{y} \quad \frac{\partial E}{\partial a_1} = 0$$

Esto es:

$$0 = \frac{\partial}{\partial a_0} \sum_{i=1}^{m} [y_i - (a_1 x_i + a_0)]^2 = 2 \sum_{i=1}^{m} (y_i - a_1 x_i - a_0)(-1)$$

$$0 = \frac{\partial}{\partial a_1} \sum_{i=1}^{m} [y_i - (a_1 x_i + a_0)]^2 = 2 \sum_{i=1}^{m} (y_i - a_1 x_i - a_0)(-x_i)$$

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Ecuaciones normales:

$$a_0 \cdot m + a_1 \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i$$

$$a_0 \sum_{i=1}^{m} x_i + a_1 \sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i \, y_i$$

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Solución:

$$a_0 = \frac{\sum_{i=1}^{m} x_i^2 \sum_{i=1}^{m} y_i - \sum_{i=1}^{m} x_i y_i \sum_{i=1}^{m} x_i}{m \left(\sum_{i=1}^{m} x_i^2 \right) - \left(\sum_{i=1}^{m} x_i \right)^2}$$

$$a_1 = \frac{m \sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i}{m \left(\sum_{i=1}^{m} x_i^2 \right) - \left(\sum_{i=1}^{m} x_i \right)^2}$$

Ejemplo: Encontrar la recta de mínimos cuadrados:

x_i	y_i	x_i^2	$x_i y_i$	$P(x_i) = 2.437x_i + 3.903$
1	7.07	1	7.07	6.34
2	6.99	4	13.97	8.78
3	11.37	9	34.10	11.21
4	14.73	16	58.92	13.65
5	16.03	25	80.14	16.09
6	18.02	36	108.10	18.52
7	21.69	49	151.85	20.96
8	23.94	64	191.52	23.40
9	25.07	81	225.62	25.83
10	28.15	100	281.49	28.27
55	173.04	385	1152.77	$E \approx 6.62$

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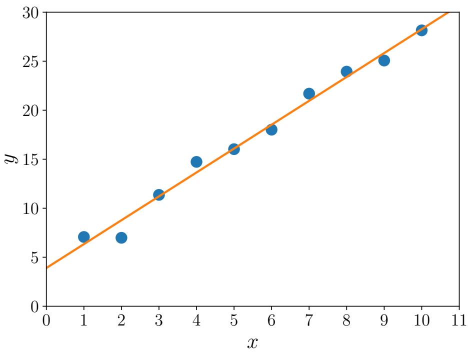
10	28.15	100	281.49	28.27
55	173.04	385	1152.77	$E \approx 6.62$

$$a_0 = \frac{385(173.04) - 55(1152.77)}{10(385) - 55^2} = 3.903$$

$$a_1 = \frac{10(1152.77) - 55(173.04)}{10(385) - 55^2} = 2.437$$



Mínimos cuadrados polinomiales:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$\{(x_i, y_i)\}, i = 1, \ldots m$, $n < m - 1$. Minimizar:

$$\begin{aligned}
E &= \sum_{i=1}^{m} [y_i - P_n(x_i)]^2 \\
&= \sum_{i=1}^{m} y_i^2 - 2\sum_{i=1}^{m} P_n(x_i) y_i + \sum_{i=1}^{m} [P_n(x_i)]^2 \\
&= \sum_{i=1}^{m} y_i^2 - 2\sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j\right) y_i + \sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j\right)^2 \\
&= \sum_{i=1}^{m} y_i^2 - 2\sum_{j=0}^{n} a_j \left(\sum_{i=1}^{m} y_i x_i^j\right) + \sum_{j=0}^{n}\sum_{k=0}^{n} a_j a_k \left(\sum_{i=1}^{m} x_i^{j+k}\right)
\end{aligned}$$

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$$E = \sum_{i=1}^{m} [y_i - P_n(x_i)]^2$$

$$= \sum_{i=1}^{m} y_i^2 - 2\sum_{i=1}^{m} P_n(x_i) y_i + \sum_{i=1}^{m} [P_n(x_i)]^2$$

$$= \sum_{i=1}^{m} y_i^2 - 2\sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j\right) y_i + \sum_{i=1}^{m} \left(\sum_{j=0}^{n} a_j x_i^j\right)^2$$

$$= \sum_{i=1}^{m} y_i^2 - 2\sum_{j=0}^{n} a_j \left(\sum_{i=1}^{m} y_i x_i^j\right) + \sum_{j=0}^{n}\sum_{k=0}^{n} a_j a_k \left(\sum_{i=1}^{m} x_i^{j+k}\right)$$

Minimización: $\partial E / \partial a_j = 0, j = 0, 1, \ldots n$.

$$0 = \frac{\partial E}{\partial a_j} = -2\sum_{i=1}^{m} y_i x_i^j + 2\sum_{k=0}^{n} a_k \sum_{i=1}^{m} x_i^{j+k}$$

$(n+1)$ **ecuaciones normales**:

$$\sum_{k=0}^{n} a_k \sum_{i=1}^{m} x_i^{j+k} = \sum_{i=1}^{m} y_i x_i^j, \; j = 0, 1, \ldots n$$

$$
\begin{aligned}
a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 + \cdots + a_n \sum_{i=1}^{m} x_i^n &= \sum_{i=1}^{m} y_i x_i^0 \\
a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 + a_2 \sum_{i=1}^{m} x_i^3 + \cdots + a_n \sum_{i=1}^{m} x_i^{n+1} &= \sum_{i=1}^{m} y_i x_i^1 \\
&\vdots \\
a_0 \sum_{i=1}^{m} x_i^n + a_1 \sum_{i=1}^{m} x_i^{n+1} + a_2 \sum_{i=1}^{m} x_i^{n+2} + \cdots + a_n \sum_{i=1}^{m} x_i^{2n} &= \sum_{i=1}^{m} y_i x_i^n
\end{aligned}
$$

Solución única: $x_i \neq x_j \quad \forall i \neq j$.

Ejemplo:

i	x_i	y_i
1	0	1.0000
2	0.25	1.2840
3	0.50	1.6487
4	0.75	2.1170
5	1.00	2.7183

Ejemplo:

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$n = 2, m = 5$

$$
\begin{aligned}
5a_0 + 2.5a_1 + 1.875a_2 &= 8.7680 \\
2.5a_0 + 1.875a_1 + 1.5625a_2 &= 5.4514 \\
1.875a_0 + 1.5625a_1 + 1.3828a_2 &= 4.4015
\end{aligned}
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Solución:

$$a_0 = 1.005,\ a_1 = 0.8642,\ a_2 = 0.8437$$

$$P_2(x) = 1.005 + 0.8642x + 0.8437x^2$$

Error total:

$$E = \sum_{i=1}^{5} [y_i - P_2(x_i)]^2 = 2.74 \times 10^{-4}$$

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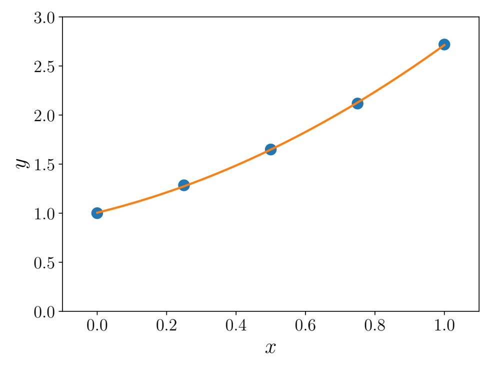


Relación exponencial:

$$y = b\, e^{ax}$$

Minimizar:

$$E = \sum_{i=1}^{m} (y_i - be^{ax_i})^2$$

Ecuaciones normales:

$$0 = \frac{\partial E}{\partial b} = 2 \sum_{i=1}^{m} (y_i - be^{ax_i})(-e^{ax_i})$$

$$0 = \frac{\partial E}{\partial a} = 2 \sum_{i=1}^{m} (y_i - be^{ax_i})(-bx_i e^{ax_i})$$

Alternativa:

$$\ln y = \ln b + ax$$

Relación potencial:

$$y = b\, x^a$$

Minimizar:

$$E = \sum_{i=1}^{m} (y_i - bx_i^a)^2$$

Ecuaciones normales:

$$0 = \frac{\partial E}{\partial b} = 2 \sum_{i=1}^{m} (y_i - bx_i^a)(-x_i^a)$$

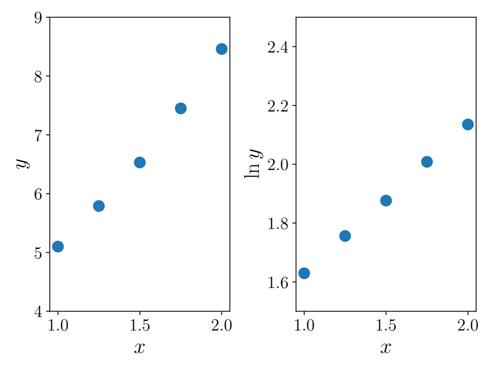
$$0 = \frac{\partial E}{\partial a} = 2 \sum_{i=1}^{m} (y_i - bx_i^a)[-b \ln(x_i) x_i^a]$$

Alternativa:

$$\ln y = \ln b + a \ln x$$

Ejemplo:

i	x_i	y_i	$\ln y_i$	x_i^2	$x_i \ln y_i$
1	1.00	5.10	1.63	1.00	1.63
2	1.25	5.79	1.76	1.56	2.20
3	1.50	6.53	1.88	2.25	2.81
4	1.75	7.45	2.01	3.06	3.51
5	2.00	8.46	2.14	4.00	4.27
	7.50		9.41	11.88	14.42



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3	1.50	6.53	1.88	2.25	2.81
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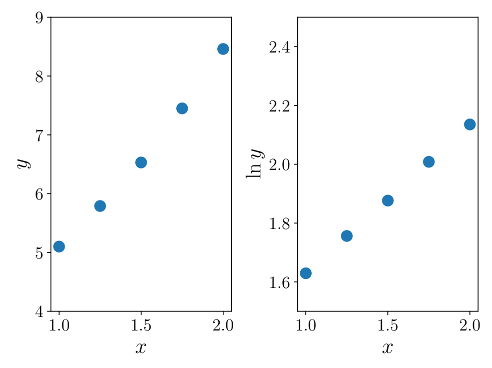

$$y = be^{ax} \Rightarrow \ln y = \ln b + ax$$

$$a = \frac{5(14.42) - (7.5)(9.41)}{5(11.88) - (7.5)^2} = 0.5057$$

$$\ln b = \frac{(11.88)(9.41) - (14.42)(7.5)}{5(11.88) - (7.5)^2} = 1.122$$

$$\ln b = 1.122 \rightarrow b = e^{1.122} = 3.071$$

$$y = 3.071e^{0.5056x}$$

Ejemplo:

i	x_i	y_i	$\ln y_i$	x_i^2	$x_i \ln y_i$
1	1.00	5.10	1.63	1.00	1.63
2	1.25	5.79	1.76	1.56	2.20
3	1.50	6.53	1.88	2.25	2.81
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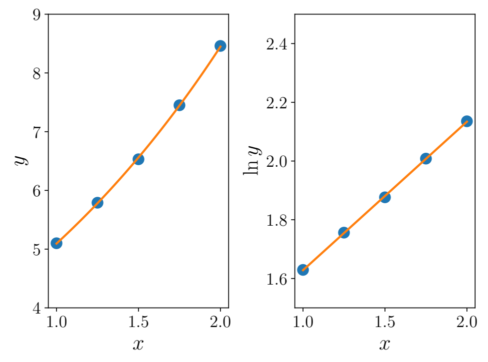

$$y = be^{ax} \Rightarrow \ln y = \ln b + ax$$

$$a = \frac{5(14.42) - (7.5)(9.41)}{5(11.88) - (7.5)^2} = 0.5057$$

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$$\ln b = 1.122 \rightarrow b = e^{1.122} = 3.071$$

$$y = 3.071e^{0.5056x}$$

Python: ejemplo 1

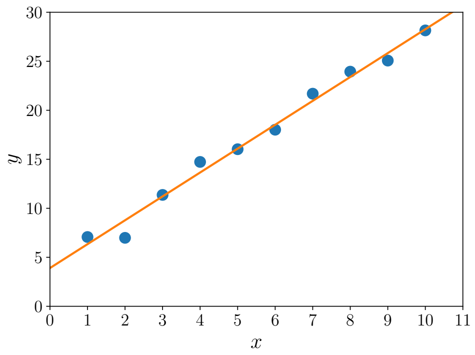
```python
import numpy as np

rng = np.random.default_rng(14)

delta = 5.5
x = np.linspace(1, 10, 10)
y = 2.5 * x + delta * rng.random(x.size)
z = np.polyfit(x, y, 1)
p = np.poly1d(z)
print(p)
print(f"a_0 = {p[0]:.5f}, a_1 = {p[1]:.5f}")
```

```
$ ./ejemplo-01.py

2.437 x + 3.903
a_0 = 3.90314, a_1 = 2.43661
```



Python: ejemplo 2

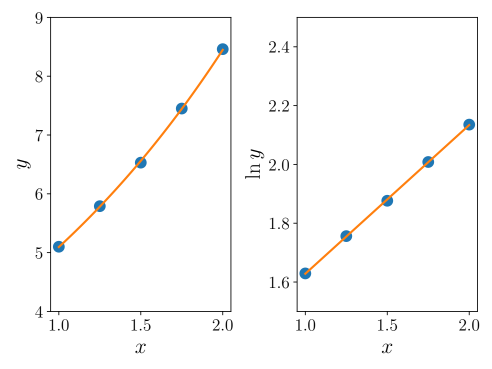
```python
from math import exp
import numpy as np

x = np.array([1.00, 1.25, 1.50, 1.75, 2.00])
y = np.array([5.10, 5.79, 6.53, 7.45, 8.46])
ly = np.log(y)

z = np.polyfit(x, ly, 1)
p = np.poly1d(z)
print(p)
print(f"ln(b) = {p[0]:.5f}, a = {p[1]:.5f}")
```

```
$ ./ejemplo-02.py

0.5057 x + 1.122
ln(b) = 1.12249, a = 0.50572
```



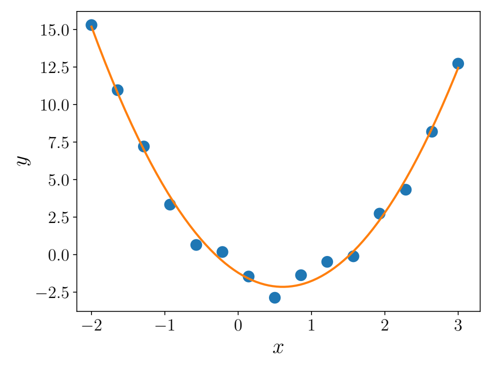
Python: ejemplo 3

```python
import numpy as np

rng = np.random.default_rng(13)

delta = 1.5
x_0, x_1, n = -2, 3, 15
x = np.linspace(x_0, x_1, n)
y = 2.5 * x**2 - 3 * x - 2 + delta * rng.random(x.size)
z = np.polyfit(x, y, 2)
p = np.poly1d(z)
print(p)
print(f"a_0 = {p[0]:.5f}, a_1 = {p[1]:.5f}, a_2 = {p[2]:.5f}")
```

```
$ ./ejemplo-03.py
      2
2.55 x - 3.099 x - 1.205
a_0 = -1.20479, a_1 = -3.09868, a_2 = 2.54970
```



Python: ejemplo 4

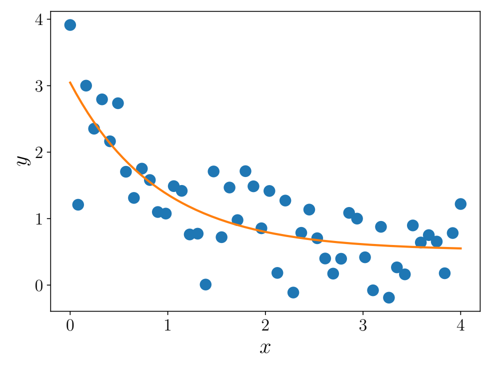
```python
import numpy as np
from scipy.optimize import curve_fit

def modelo(x, a, b, c):
    return a * np.exp(-b * x) + c

rng = np.random.default_rng(13)
x_datos = np.linspace(0, 4, 50)
y = modelo(x_datos, 2.5, 1.3, 0.5)
y_ruido = 0.5 * rng.normal(size=x_datos.size)
y_datos = y + y_ruido

popt, pcov = curve_fit(modelo, x_datos, y_datos)
print(popt)
```

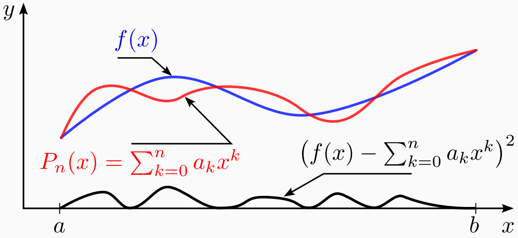
```
> ./ejemplo-04.py
[2.50685815 1.21831291 0.51137751]
```



Ajuste continuo

$f(x) \in \mathbf{C}[a, b]$, hallar $P_n(x)$ que minimize:

$$\int_a^b [f(x) - P_n(x)]^2 \, dx$$



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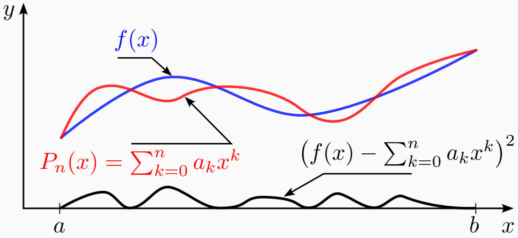

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$$

$$E \equiv E_2(a_0, a_1, \cdots, a_n) = \int_a^b \left(f(x) - \sum_{k=0}^{n} a_k x^k \right)^2 dx$$

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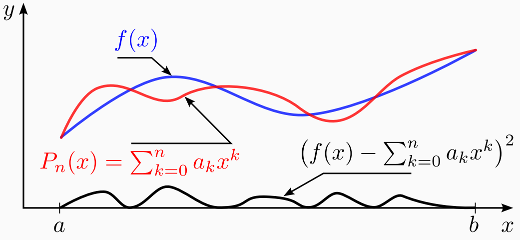

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$$E \equiv E_2(a_0, a_1, \cdots, a_n) = \int_a^b \left(f(x) - \sum_{k=0}^{n} a_k x^k\right)^2 dx$$

$$\frac{\partial E}{\partial a_j} = 0 \quad \text{para cada} \quad j = 0, 1, \cdots, n$$

$$E = \int_a^b [f(x)]^2 dx - 2 \sum_{k=0}^{n} a_k \int_a^b x^k f(x) \, dx + \int_a^b \left(\sum_{k=0}^{n} a_k x^k\right)^2 dx$$

$$\frac{\partial E}{\partial a_j} = -2 \int_a^b x^j f(x) \, dx + 2 \sum_{k=0}^{n} a_k \int_a^b x^{j+k} \, dx$$

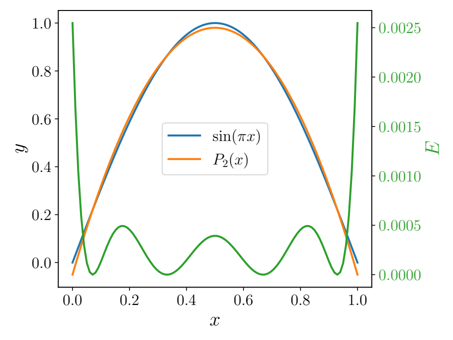
Ecuaciones normales lineales $(n+1)$:

$$\sum_{k=0}^{n} a_k \int_a^b x^{j+k} \, dx = \int_a^b x^j f(x) \, dx$$

para cada $j = 0, 1, \cdots, n$.

Ejemplo: aproximar $f(x) = \operatorname{sen} \pi x$ por un polinomio de grado 2 en $[0, 1]$.
Ecuaciones normales:

$$
\begin{aligned}
a_0 \int_0^1 1\, dx + a_1 \int_0^1 x\, dx + a_2 \int_0^1 x^2\, dx &= \int_0^1 \operatorname{sen} \pi x\, dx \\
a_0 \int_0^1 x\, dx + a_1 \int_0^1 x^2\, dx + a_2 \int_0^1 x^3\, dx &= \int_0^1 x \operatorname{sen} \pi x\, dx \\
a_0 \int_0^1 x^2\, dx + a_1 \int_0^1 x^3\, dx + a_2 \int_0^1 x^4\, dx &= \int_0^1 x^2 \operatorname{sen} \pi x\, dx
\end{aligned}
$$



$$
\begin{aligned}
a_0 + \frac{1}{2} a_1 + \frac{1}{3} a_2 &= \frac{2}{\pi} \\
\frac{1}{2} a_0 + \frac{1}{3} a_1 + \frac{1}{4} a_2 &= \frac{1}{\pi} \\
\frac{1}{3} a_0 + \frac{1}{4} a_1 + \frac{1}{5} a_2 &= \frac{\pi^2 - 4}{\pi^3}
\end{aligned}
$$

Solución

$$a_0 = \frac{12\pi^2 - 120}{\pi^3} \approx -0.050465$$

$$a_1 = -a2 = \frac{720 - 60\pi^2}{\pi^3} \approx 4.12251$$

Problemas:

- matriz de Hilbert

$$H_{ij} = \int_a^b x^{j+k}\,dx = \frac{b^{j+k+1} - a^{j+k+1}}{j+k+1}$$

$$\boldsymbol{H} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$\mathrm{cond}(\boldsymbol{H}) \approx 524.05678$$

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Definición : Funciones linealmente independientes .

Se dice que el conjunto de funciones $\{\phi_0, \cdots, \phi_n\}$ es **linealmente independiente** (LI) en $[a, b]$ si

$$P(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) = 0, \forall x \in [a, b]$$

entonces $c_0 = c_1 = \cdots = c_n = 0$. De lo contrario, se dice que el conjunto de funciones es **linealmente dependiente**.

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Teorema : Polinomios LI.

Si para cada $j = 0, 1, \cdots, n$, $\phi_j(x)$ es un polinomio de grado j, entonces el conjunto $\{\phi_0, \cdots, \phi_n\}$ es LI en cualquier intervalo $[a, b]$.

Ejemplo. Si $\phi_0(x) = 2, \phi_1(x) = x - 3, \phi_2(x) = x^2 + 2x + 7$ y $Q(x) = a_0 + a_1x + a_2x^2$, mostrar que existen constantes c_0, c_1, c_2 tales que $Q(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x)$.

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Por el teorema anterior, $\{\phi_0, \phi_1, \phi_2\}$ es LI en cualquier $[a, b]$. Además:

$$
\begin{aligned}
1 &= \frac{1}{2}\phi_0(x) \\
x &= \phi_1(x) + 3 = \phi_1(x) + \frac{3}{2}\phi_0(x) \\
x^2 &= \phi_2(x) - 2x - 7 \\
&= \phi_2(x) - 2\left[\phi_1(x) + \frac{3}{2}\phi_0(x)\right] - 7\left[\frac{1}{2}\phi_0(x)\right] \\
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\end{aligned}
$$

Entonces:

$$
\begin{aligned}
Q(x) &= a_0\left[\frac{1}{2}\phi_0\right] + a_1\left[\phi_1(x) + \frac{3}{2}\phi_0(x)\right] \\
&\quad + a_2\left[\phi_2(x) - 2\phi_1(x) - \frac{13}{2}\phi_0(x)\right] \\
&= \left(\frac{1}{2}a_0 + \frac{3}{2}a_1 - \frac{13}{2}a_2\right)\phi_0(x) \\
&\quad + [a_1 - 2a_2]\phi_1(x) + a_2\phi_2(x)
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$$

Teorema : .

*Si Π_n denota el conjunto de todos los polinomios de grado a lo sumo n, y $\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}$ es un conjunto de polinomios LI en Π_n, entonces **cualquier** polinomio en Π_n se puede escribir como combinación lineal de $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$.*

Definición : Función de peso.

Una función integrable w se denomina **función de peso** en el intervalo I si $w(x) \geq 0, \forall x \in I$, pero $w(x) \not\equiv 0$ en cualquier subintervalo de I

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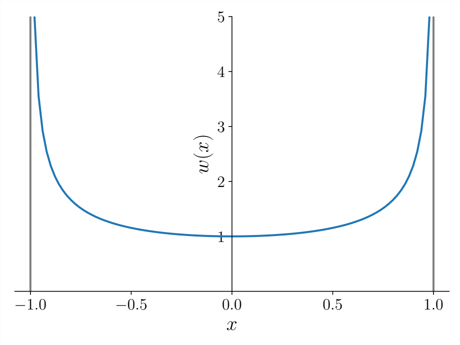
Ejemplo:

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$



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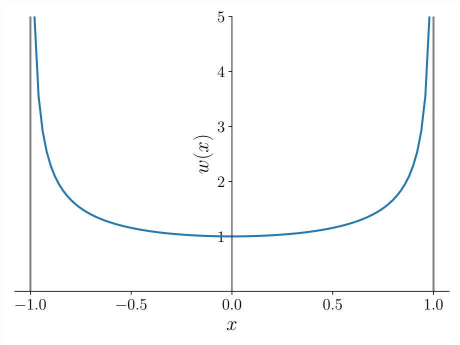

Definición : Funciones ortogonales.

Se dice que $\{\phi_0, \phi_1, \cdots, \phi_n\}$ es un **conjunto ortogonal de funciones** en el intervalo $[a, b]$ respecto de la función de peso $w(x)$ si

$$\langle \phi_k, \phi_j \rangle_w = \int_a^b w(x)\phi_k(x)\phi_j(x)\,dx = \begin{cases} 0, & j \neq k, \\ \alpha_j > 0, & j = k \end{cases}$$

Si además $\alpha_j = 1$ para cada $j = 0, 1, 2, \cdots, n$, se dice que el conjunto es **ortonormal**.

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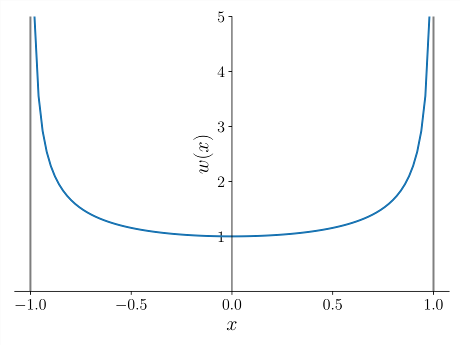

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Si además $\alpha_j = 1$ para cada $j = 0, 1, 2, \cdots, n$, se dice que el conjunto es **ortonormal**.

Ejemplo: $\{\cos nx, \operatorname{sen} mx\}, n, m = 0, 1, \cdots$ es ortogonal en $[-\pi, \pi]$ con $w(x) = 1$:

$$\langle \cos nx, \cos mx \rangle_w = 0 \qquad \langle \cos nx, \cos nx \rangle_w = \pi$$

$$\langle \operatorname{sen} nx, \operatorname{sen} mx \rangle_w = 0 \qquad \langle \operatorname{sen} nx, \operatorname{sen} nx \rangle_w = \pi$$

$$\langle \cos nx, \operatorname{sen} mx \rangle_w = 0 \qquad n \neq m$$

Teorema : .

Si $\{\phi_0, \cdots, \phi_n\}$ *es un conjunto ortogonal de funciones en un intervalo* $[a, b]$ *respecto de la función de peso* $w(x)$*, entonces la aproximación por mínimos cuadrados para* f *en* $[a, b]$ *respecto de* w *es:*

$$P(x) = \sum_{j=0}^{n} a_j \phi_j(x)$$

donde para cada $j = 0, 1, \cdots, n$*:*

$$a_j = \frac{1}{\alpha_j} \langle f, \phi_j \rangle_w$$

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Teorema : .

El conjunto de polinomios $\{\phi_0, \phi_1, \cdots, \phi_n\}$ *definido de la siguiente forma es ortogonal en* $[a,b]$ *respecto de la función de peso* $w(x)$*:*

$$\phi_0(x) \equiv 1,\ \phi_1(x) = x - B_1, \forall x \in [a,b]$$

donde

$$B_1 = \frac{\langle x\phi_0, \phi_0 \rangle_w}{\langle \phi_0, \phi_0 \rangle_w}$$

y cuando $k \geq 2$*:*

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k \phi_{k-2}(x),\ \forall x \in [a,b]$$

donde

$$B_k = \frac{\langle x\phi_{k-1}, \phi_{k-1} \rangle_w}{\langle \phi_{k-1}, \phi_{k-1} \rangle_w} \qquad C_k = \frac{\langle x\phi_{k-1}, \phi_{k-2} \rangle_w}{\langle \phi_{k-2}, \phi_{k-2} \rangle_w}$$

(Proceso de Gram-Schmidt)

Polinomios de Legendre.

$\{P_n(x)\}$ es ortogonal en $[-1,1]$ con $w(x) \equiv 1$. Usando Gram-Schmidt con $P_0(x) \equiv 1$:

$$B_1 = \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} dx} = 0,\ P_1(x) = (x - B_1)P_0(x) = x$$

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Luego:

$$B_2 = \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} = 0,\ C_2(x) = \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx} = \frac{1}{3}$$

$$\begin{aligned} P_2(x) &= (x - B_2)P_1(x) - C_2 P_0(x) \\ &= (x-0)x - \frac{1}{3}1 \\ &= x^2 - \frac{1}{3} \end{aligned}$$

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$$P_3(x) = xP_2(x) - \frac{4}{15}P_1(x) = x^3 - \frac{3}{5}x$$

$$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35},\ P_5(x) = x^5 - \frac{10}{9}x^3 + \frac{5}{21}x$$

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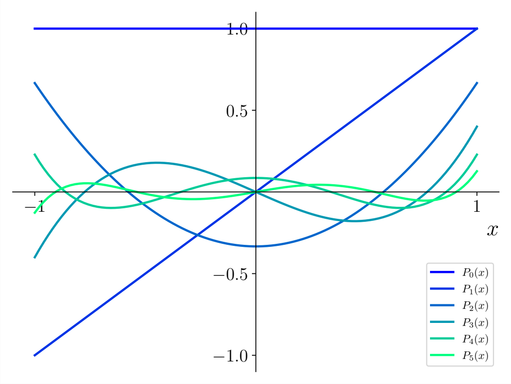
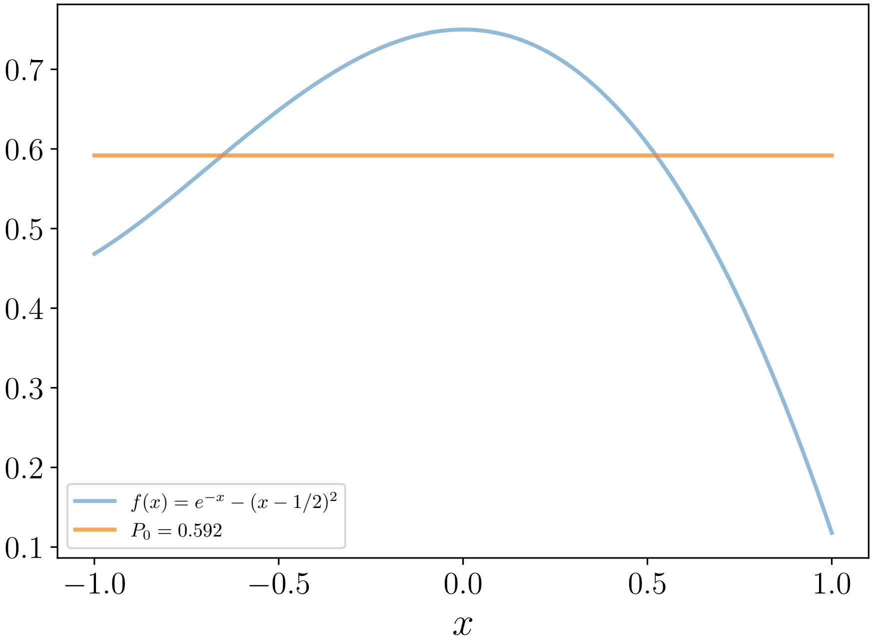
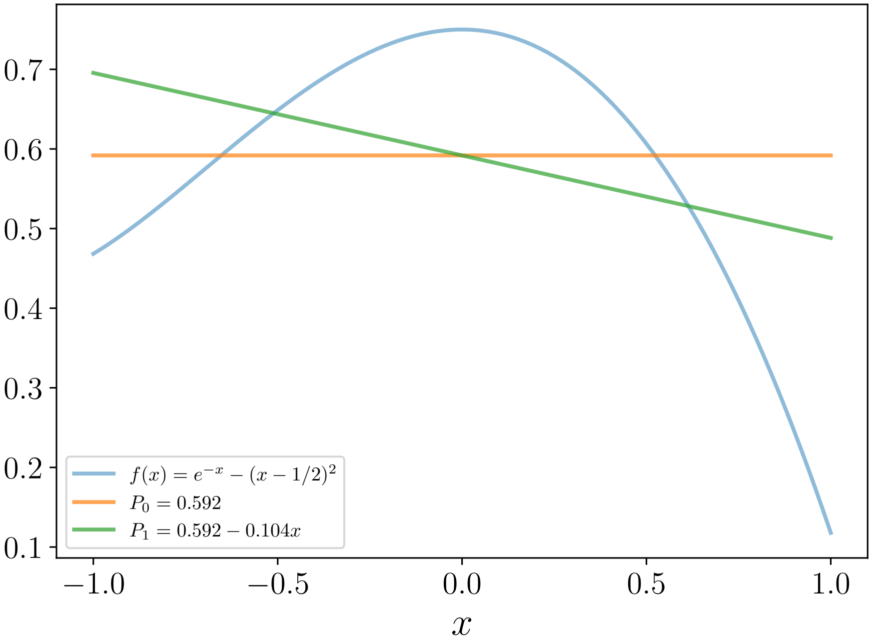


Ejemplo: Python

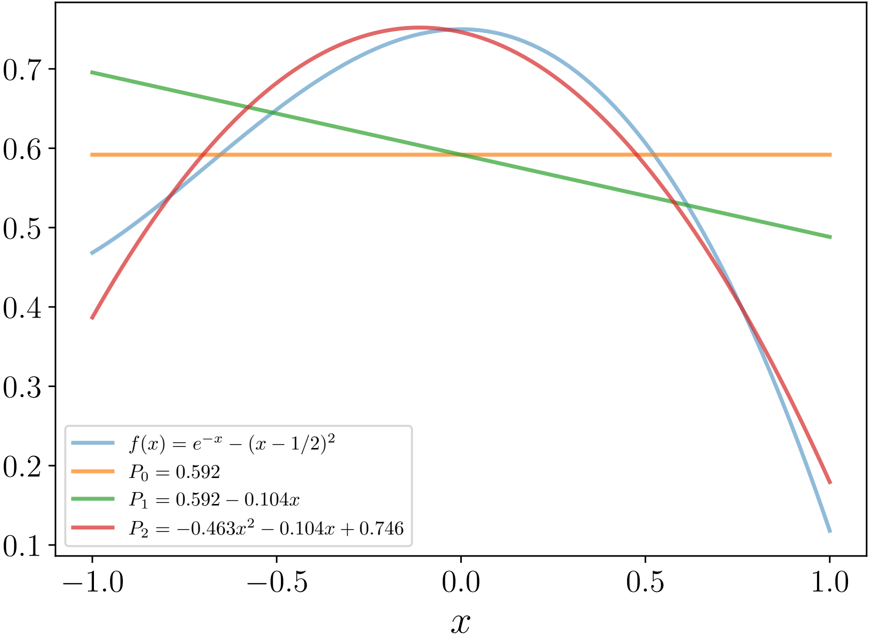
```python
#!/usr/bin/env python3

from math import exp
import matplotlib.pyplot as plt
plt.style.use('../../utils/clases.mplstyle')
import numpy as np

x = np.array([1.00, 1.25, 1.50, 1.75, 2.00])
y = np.array([5.10, 5.79, 6.53, 7.45, 8.46])
ly = np.log(y)

for i in range(x.size):
    s = f"{i+1:2d} & "
    s += f"{x[i]:.2f} & "
    s += f"{y[i]:.2f} & "
    s += f"{ly[i]:.2f} & "
    s += f"{(x[i]**2):.2f} & "
    s += f"{(x[i]*ly[i]):.2f} & "
    print(s)

xs = x.sum()
lys = ly.sum()
```

```python
xlys = (x * ly).sum()

print(f" & {xs:.2f} & & {lys:.2f} & {x2s:.2f} & {xlys:.2f}")

xp = np.linspace(1, 2, 100)
z = np.polyfit(x, ly, 1)
p = np.poly1d(z)
print(p)
print(p[0], p[1])

ye = exp(p[0]) * np.exp(p[1] * xp)

fig, (ax1, ax2) = plt.subplots(1, 2)
ax1.plot(x, y, 'o')
ax1.plot(xp, ye)
ax2.plot(x, ly, 'o')
ax2.plot(xp, p(xp))
ax1.set_ylim([4, 9])
ax2.set_ylim([1.5, 2.5])
ax1.set_xlabel(r"$x$")
ax2.set_xlabel(r"$x$")
ax1.set_ylabel(r"$y$")
ax2.set_ylabel(r"$\ln y$")
# plt.plot(xp, p(xp), '-')
# plt.xlabel(r'$x$')
# plt.ylabel(r'$y$')
```

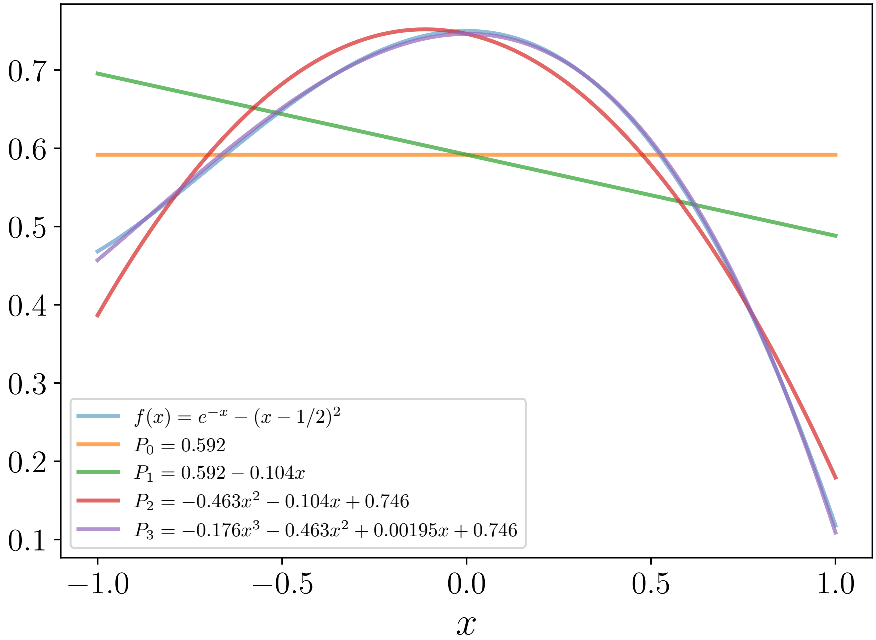

0.7
0.6
0.5
0.4
0.3
0.2
0.1
$f(x) = e^{-x} - (x - 1/2)^2$
$P_0 = 0.592$
−1.0
−0.5
0.0
0.5
1.0
x



0.7
0.6
0.5
0.4
0.3
0.2
0.1
−1.0
−0.5
0.0
0.5
1.0
x
$f(x) = e^{-x} - (x - 1/2)^2$
$P_0 = 0.592$
$P_1 = 0.592 - 0.104x$



0.7
0.6
0.5
0.4
0.3
0.2
0.1
$f(x) = e^{-x} - (x - 1/2)^2$
$P_0 = 0.592$
$P_1 = 0.592 - 0.104x$
$P_2 = -0.463x^2 - 0.104x + 0.746$
−1.0
−0.5
0.0
0.5
1.0
x



0.7
0.6
0.5
0.4
0.3
0.2
0.1
$f(x) = e^{-x} - (x - 1/2)^2$
$P_0 = 0.592$
$P_1 = 0.592 - 0.104x$
$P_2 = -0.463x^2 - 0.104x + 0.746$
$P_3 = -0.176x^3 - 0.463x^2 + 0.00195x + 0.746$
−1.0
−0.5
0.0
0.5
1.0
x

Polinomios de Chebyshev.

$\{T_n(x)\}$ es ortogonal en $(-1,1)$ con función de peso $w(x)=(1-x^2)^{-1/2}$. Para $x\in[-1,1]$:

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Para $n \geq 1$, $\theta = \arccos x$:

$$T_n(\theta(x)) \equiv T_n(\theta) = \cos(n\theta),\ \theta \in [0,\pi]$$

Relación de recurrencia:

$$T_{n+1}(\theta) = \cos(n+1)\theta = \cos\theta\cos(n\theta) - \operatorname{sen}\theta\operatorname{sen}(n\theta)$$
$$T_{n-1}(\theta) = \cos(n-1)\theta = \cos\theta\cos(n\theta) + \operatorname{sen}\theta\operatorname{sen}(n\theta)$$

Sumando:

$$T_{n+1}(\theta) = 2\cos\theta\cos(n\theta) - T_{n-1}(\theta)$$

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Regresando a $x = \cos\theta$, para $n \geq 1$:

$$T_{n+1} = 2x\cos(n \arccos x) - T_{n-1}(x)$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

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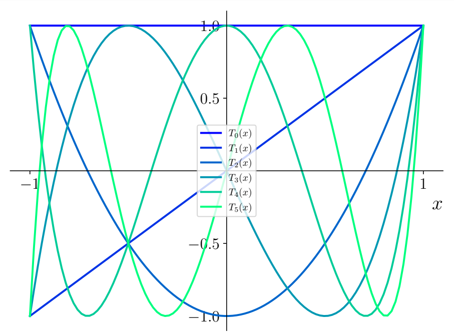
Dado que $T_0(x) = 1$ y $T_1(x) = x$:

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1$$

$$T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 3x$$

$$T_4(x) = 2xT_3(x) - T_2(x) = 8x^4 - 8x^2 + 1$$



Ortogonalidad de polinomios de Chebyshev

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x}}\,dx = \int_{-1}^{1} \frac{\cos(n \arccos x)\cos(m \arccos x)}{\sqrt{1-x^2}}dx$$

Reintroducimos $\theta = \arccos x$:

$$d\theta = -\frac{1}{\sqrt{1-x^2}}dx$$

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = -\int_{\pi}^{0} \cos(n\theta)\cos(m\theta)\,d\theta$$
$$= \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\,d\theta$$

Para $n \neq m$:

$$\cos(n\theta)\cos(m\theta) = \frac{1}{2}[\cos(n+m)\theta + \cos(n-m)\theta]$$

Entonces:

$$\begin{aligned}\langle T_n, T_m\rangle_w &= \frac{1}{2}\int_0^\pi \cos[(n+m)\theta]\,d\theta \\ &+ \frac{1}{2}\int_0^\pi \cos[(n-m)\theta]\,d\theta \\ &= \left[\frac{\operatorname{sen}[(n+m)\theta]}{2(n+m)} + \frac{\operatorname{sen}[(n-m)\theta]}{2(n-m)}\right]_0^\pi \\ &= 0\end{aligned}$$

y

$$\langle T_n, T_n\rangle_w = \begin{cases} \frac{\pi}{2}, & n \geq 1 \\ \pi, & n = 0 \end{cases}$$

Reducción de grado de polinomio:

$$q(x) = x^5 - 4x^4 + x^3 - x - 3$$

Aproximación por:

$$P_4(x) = c_0 + c_1 T_1(x) + c_2 T_2(x) + c_3 T_3(x) + c_4 T_4(x)$$

donde

$$c_j = \frac{\langle q, T_j \rangle_w}{\langle T_j, T_j \rangle_w}$$

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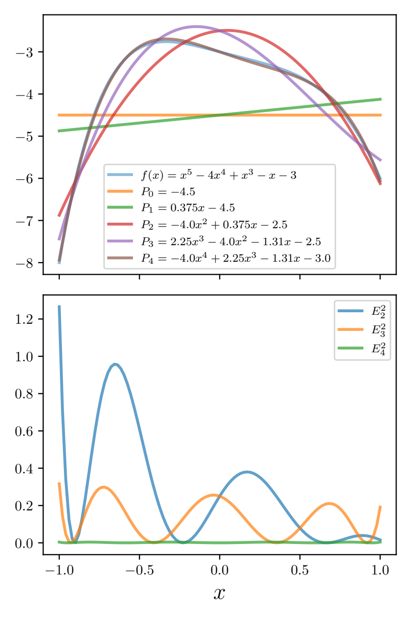
donde

$$c_j = \frac{\langle q, T_j \rangle_w}{\langle T_j, T_j \rangle_w}$$

Resultado: ver `code/plot-12.py`.

$$P(x) = -4.0x^4 + 2.25x^3 - 1.31x - 3.0$$



Lecturas recomendadas

- R.L. Burden, D.J. Faires y A.M. Burden. ***Análisis numérico.*** 10.ª ed. Mexico: Cengage Learning, 2017. Capítulo 8.
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- A. Quarteroni, R. Sacco y F. Saleri. ***Numerical Mathematics.*** New York, United States: Springer-Verlag, 2000. Capítulo 10.
- E. Kreyszig, H. Kreyszig y E.J. Norminton. ***Advanced Engineering Mathematics.*** Hoboken, USA: John Wiley & Sons, Inc, 2011. Capítulo 25.9.