Introducción a la variable compleja

Números complejos (repaso). Funciones de variable compleja. Límite y continuidad. Diferenciabilidad y funciones analíticas.

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· XeLaTeX ·

Sistema de enteros:

$$2x = 3$$
$$x = ?$$

Números "reales": $\{x : x^2 \geq 0\}$

$$x^2 = -1$$
$$x = ?$$

Motivación: $x^2 + 1 = 0$ ¿tiene solución?

Ejemplo: usar $y = e^{rx}$ para resolver:

$$y'' + y = 0$$

$$r^2 e^{rx} + e^{rx} = 0$$
$$\therefore r^2 + 1 = 0 \therefore r = \pm\sqrt{-1} = \pm i$$
$$\therefore y = e^{ix} \text{ o } y = e^{-ix}$$
$$\boxed{y = \cos x \text{ o } y = \operatorname{sen} x}$$

De "alguna manera" i debe existir y e^{ix} debe estar relacionado a $\operatorname{sen} x$ y $\cos x$.

El sistema de números complejos:

$$\mathbb{C} = \{x + iy : x \text{ y } y \text{ son reales.}\}$$

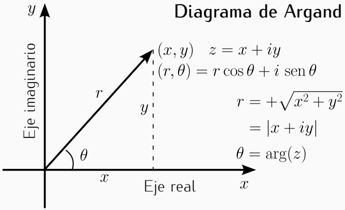
con la estructura:

$$(1)\ x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow$$
$$x_1 = x_2 \text{ y } y_1 = y_2$$

$$(2)\ (x_1 + iy_1) + (x_2 + iy_2) =$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$
$$(3)\ r(x + iy) = rx + iry$$

r real.

$\therefore$ los números complejos son un espacio vectorial por definición.



Estructura adicional de $\mathbb{C}$:

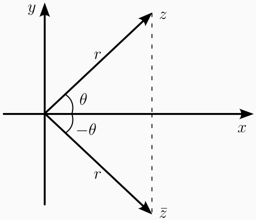
$$(4)\ (a+ib)(c+id) = = (ac-bd) + i(bc+ad)$$

Caso especial:

$$(a+ib)(a-ib) = a^2+b^2 \geq 0 = |a+ib|^2$$

Definición: el complejo conjugado de $z = x+yi$ es

$$\bar{z} = x - yi$$



$$\frac{c+di}{a+bi} = \left(\frac{c+di}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right) = \frac{(ac+bd)+(ad-bc)i}{a^2+b^2}$$

$$\frac{3+2i}{4+i} = \frac{(3+2i)(4-i)}{(4+i)(4-i)} = \frac{14+5i}{17} = \frac{14}{17} + \frac{5}{17}i$$

$$\therefore \frac{\text{complejo}}{\text{complejo}} = \text{complejo}$$

(excepto para división por cero).

Producto en coordenadas polares:

$$(r_1,\theta_1)(r_2,\theta_2) = (r_1\cos\theta_1 + ir_1\operatorname{sen}\theta_1)$$
$$(r_2\cos\theta_2 + ir_2\operatorname{sen}\theta_2) =$$
$$r_1r_2(\cos\theta_1\cos\theta_2 - \operatorname{sen}\theta_1\operatorname{sen}\theta_2) +$$
$$ir_1r_2(\operatorname{sen}\theta_1\cos\theta_2 + \operatorname{sen}\theta_2\cos\theta_1) =$$
$$r_1r_2\cos(\theta_1+\theta_2) + ir_1r_2\operatorname{sen}(\theta_1+\theta_2) =$$
$$\boxed{(r_1r_2, \theta_1+\theta_2)}$$

Por inducción:

$$(r_1,\theta_1)\cdots(r_n,\theta_n) = (r_1\cdots r_n, \theta_1+\cdots+\theta_n)$$

Caso especial:

$$(r,\theta)^n = (r^n, n\theta)$$

$$\therefore r = 1 \rightarrow (1,\theta)^n = (1, n\theta)$$

Teorema de De Moivre:

$$(\cos\theta + i \operatorname{sen}\theta)^n = \cos n\theta + i \operatorname{sen} n\theta$$

Ejemplo:

$$(\cos\theta + i \operatorname{sen}\theta)^2 = \cos 2\theta + i \operatorname{sen} 2\theta$$
$$(\cos^2\theta - \operatorname{sen}^2\theta) + i2 \operatorname{sen}\theta \cos\theta$$

$$\begin{aligned} \therefore\ \operatorname{sen} 2\theta &= 2 \operatorname{sen}\theta \cos\theta \\ \cos 2\theta &= \cos^2\theta - \operatorname{sen}^2\theta \end{aligned}$$

Raíces: encontrar

$$\sqrt[6]{i} = x + iy \rightarrow i = (x+iy)^6 = 0 + 1i$$

$$\begin{aligned} \therefore x^6 + 15x^4(iy)^2 + 15x^2(iy)^4 + (iy)^6 \\ + 6x^5(iy) + 20x^3(iy)^3 + 6x(iy)^5 \end{aligned}$$

Sistema complicado a resolver:

$$\begin{cases} x^6 + 15x^4(iy)^2 + 15x^2(iy)^4 + (iy)^6 = 0 \\ 6x^5(iy) + 20x^3(iy)^3 + 6x(iy)^5 = 1 \end{cases}$$

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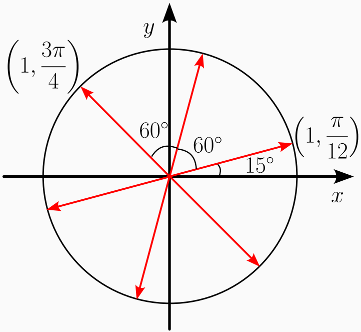
En coordenadas polares:

$$i = (1, \pi/2) \therefore \sqrt[6]{i} = (r, \theta)$$

$$\rightarrow i = (r, \theta)^6 = (r^6, 6\theta)$$

$$\therefore r = 1, \quad 6\theta = \frac{\pi}{2} + 2\pi k = \frac{1+4k}{2}\pi$$



$$r = 1,$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}, \frac{25\pi}{12}, \ldots$$

$$\left(1, \frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} + i \operatorname{sen}\frac{3\pi}{4}$$

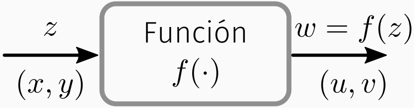
$$= \frac{1}{\sqrt{2}}(-1+i)$$

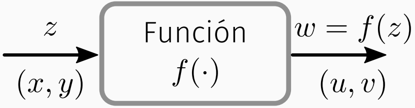
Sistema de números complejos:

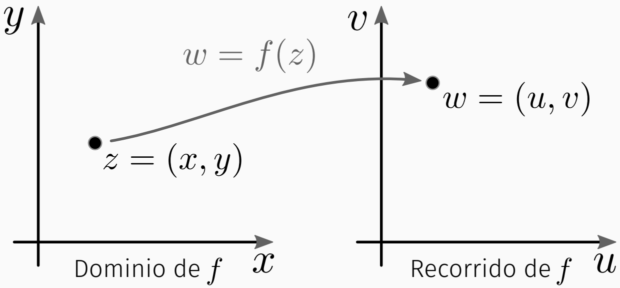
Los números complejos son **cerrados** respecto de la radicación.

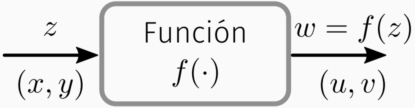
Pausa para resolver problemas: 1 – 8.

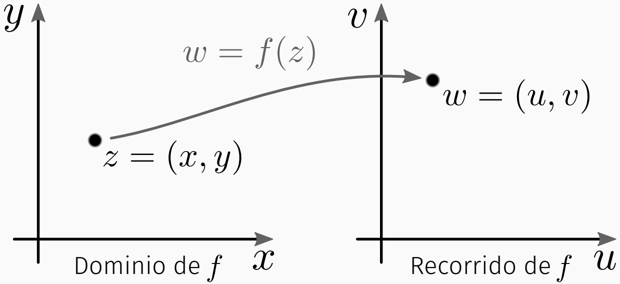
Funciones de variable compleja




z
(x,y)
Función
$f(\cdot)$
$w=f(z)$
(u,v)


y
$w=f(z)$
$z=(x,y)$
x
Dominio de f
v
$w=(u,v)$
u
Recorrido de f





Ejemplo:

$$
\begin{aligned}
f(z) &= z^2 = (x + iy)^2 \\
&= x^2 + 2xiy + i^2y^2 = x^2 - y^2 + 2ixy \\
\therefore f(x, y) &= (x^2 - y^2, 2xy)
\end{aligned}
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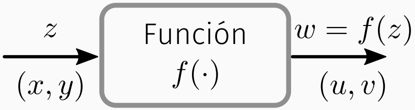

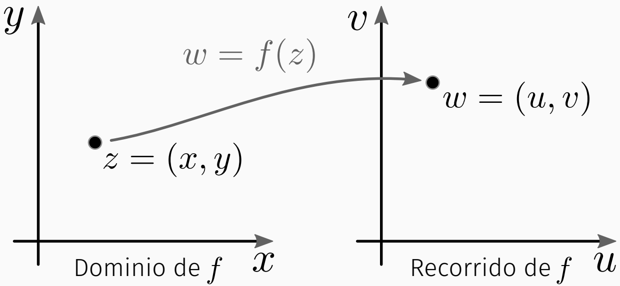

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f(z) = z^2 &= (x + iy)^2 \\
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\end{aligned}
$$

$\therefore f(z) = z^2$ es equivalente al sistema real:

$$
\begin{cases}
u = x^2 + y^2 \\
v = 2xy
\end{cases}
$$

Límites

$\mathbb{C}$: números complejos

$f : \mathbb{C} \mapsto \mathbb{C}, a \in \mathbb{C}$

Definición:

$$\lim_{z \to a} f(z) = L$$

dado $\epsilon > 0$ existe $\delta > 0$ tal que

$$0 < |z - a| < \delta \rightarrowtail |f(z) - L| < \epsilon$$

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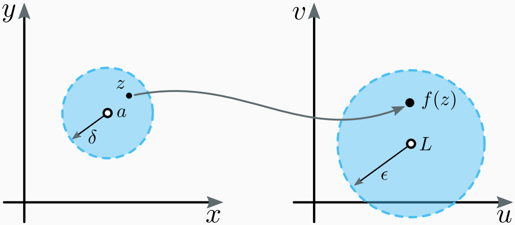

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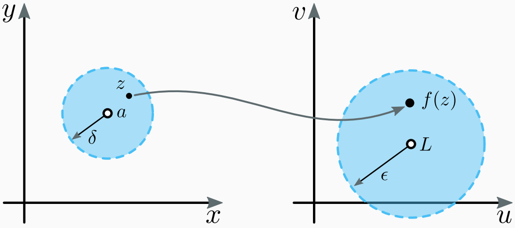

Los teoremas usuales sobre límites son válidos. En particular:

Si

$$f(z) = u(x, y) + iv(x, y)$$
$$L = L_1 + iL_2$$
$$a = a_1 + ia_2$$

Entonces:

$$\lim_{z \to a} f(z) = L \longleftrightarrow \begin{cases} \lim_{(x,y) \to (a_1,a_2)} u(x, y) = L_1 \\ \lim_{(x,y) \to (a_1,a_2)} v(x, y) = L_2 \end{cases}$$

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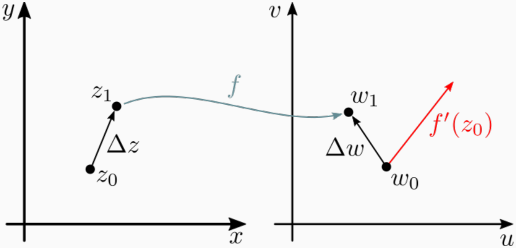

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Derivada: casos especiales

Caso 1: $\Delta y \equiv 0$.

$$\begin{aligned} \therefore f'(z_0) &= \lim_{\Delta x \to 0} \left[\frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right] \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \bigg|_{z_0 = (x_0, y_0)} \end{aligned}$$

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Caso 2: $\Delta x \equiv 0$.

$$\therefore f'(z_0) = \lim_{\Delta y \to 0} \left[\frac{\Delta u}{i\Delta y} + \frac{\Delta v}{\Delta y} \right] = \frac{\partial v}{\partial y} + \frac{1}{i} \frac{\partial u}{\partial y} \bigg|_{z_0 = (x_0, y_0)}$$
$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \bigg|_{z_0 = (x_0, y_0)}$$

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Ecuaciones de Cauchy-Riemann

Si $f = u + iv$ es diferenciable (analítica), entonces:

$$u_x = v_y$$

$$u_y = -v_x$$

Derivada: ejemplo 1

Ecuaciones de Cauchy-Riemann:

$$\begin{aligned} f(z) &= z^2 = (x+iy)^2 \\ &= (x^2 - y^2) + i(2xy) \end{aligned}$$

$$\left.\begin{array}{ll} u_x = 2x, & v_x = 2y \\ u_y = -2y, & v_y = 2x \end{array}\right\} \Rightarrow \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}$$

$f(z) \mapsto$ diferenciable

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$f(z) \mapsto$ diferenciable

Derivada por definición:

$$\begin{aligned} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} &= \frac{(z_0+\Delta z)^2 - z_0^2}{\Delta z} \\ &= \frac{2z_0\Delta z + \Delta z^2}{\Delta z} \quad (\Delta z \neq 0) \\ &= 2z_0 + \Delta z \end{aligned}$$

$$\therefore \quad \boxed{f'(z_0) = 2z_0}$$

Derivada: ejemplo 2

$$f(z) = \bar{z} = x - iy$$
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$$\begin{aligned} \frac{\Delta w}{\Delta z} &= \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \\ &= \frac{1 - i\frac{\Delta y}{\Delta x}}{1 + i\frac{\Delta y}{\Delta x}} \to \frac{1 - \frac{dy}{dx}}{1 + i\frac{dy}{dx}} \end{aligned}$$

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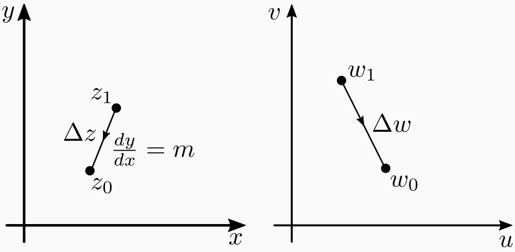

$$\frac{\Delta w}{\Delta z} = \frac{1 - im}{1 + im} = g(m)$$

Aplicación: ecuación de Laplace y ejemplo

$u(x, y)$ satisface la ecuación de Laplace si:

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Si $u + iv$ es **analítica**:

$$u_x = v_y \;\therefore\; u_{xx} = v_{yx}$$

$$u_y = -v_x \;\therefore\; u_{yy} = -v_{xy}$$

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$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Ejemplo:

$$f(z) = z^2 \longrightarrow \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

$$\therefore \left. \begin{aligned} u_{xx} + u_{yy} &= 0 \\ v_{xx} + v_{yy} &= 0 \end{aligned} \right\}$$

Lecturas recomendadas I

- E. Kreyszig, H. Kreyszig y E.J. Norminton. ***Advanced Engineering Mathematics.*** Hoboken, USA: John Wiley & Sons, Inc, 2011. Capítulo 13.
- M.R. Spiegel et al. ***Variable compleja.*** Mexico: McGraw-Hill, 1991. Capítulos 1 y 2.