Introducción a la variable compleja

Números complejos (repaso). Funciones de variable compleja. Límite y continuidad. Diferenciabilidad y funciones analíticas. Integración en el campo complejo. Sucesiones y series.

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Cálculo Avanzado • 2024

· XƎLATEX ·

Los números complejos

Sistema de enteros:

$$2x = 3$$
$$x = ?$$

Números "reales": $\{x : x^2 \geq 0\}$

$$x^2 = -1$$
$$x = ?$$

Motivación: $x^2 + 1 = 0$ ¿tiene solución?

Ejemplo: usar $y = e^{rx}$ para resolver:

$$y'' + y = 0$$

$$r^2 e^{rx} + e^{rx} = 0$$
$$\therefore r^2 + 1 = 0 \therefore r = \pm\sqrt{-1} = \pm i$$
$$\therefore y = e^{ix} \text{ o } y = e^{-ix}$$
$$\boxed{y = \cos x \text{ o } y = \operatorname{sen} x}$$

De "alguna manera" i debe existir y e^{ix} debe estar relacionado a $\operatorname{sen} x$ y $\cos x$.

El sistema de números complejos:

$$\mathbb{C} = \{x + iy : x \text{ y } y \text{ son reales.}\}$$

con la estructura:

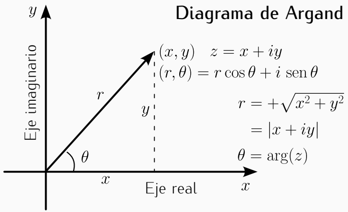
$$(1)\ x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow$$
$$x_1 = x_2 \text{ y } y_1 = y_2$$

$$(2)\ (x_1 + iy_1) + (x_2 + iy_2) =$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

$$(3)\ r(x + iy) = rx + iry$$

r real.

$\therefore$ los números complejos son un espacio vectorial por definición.



Estructura adicional de $\mathbb{C}$:

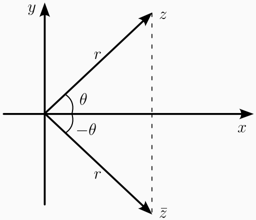
$$(4)\ (a+ib)(c+id) = $$
$$= (ac-bd) + i(bc+ad)$$

Caso especial:

$$(a+ib)(a-ib) = a^2 + b^2 \geq 0$$
$$= |a+ib|^2$$

Definición: el complejo conjugado de $z = x + yi$ es

$$\bar{z} = x - yi$$



$$\frac{c+di}{a+bi} = \left(\frac{c+di}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right)$$
$$= \frac{(ac+bd)+(ad-bc)i}{a^2+b^2}$$

$$\frac{3+2i}{4+i} = \frac{(3+2i)(4-i)}{(4+i)(4-i)}$$
$$= \frac{14+5i}{17}$$
$$= \frac{14}{17} + \frac{5}{17}i$$

$$\therefore \frac{\text{complejo}}{\text{complejo}} = \text{complejo}$$

(excepto para división por cero).

Producto en coordenadas polares:

$$(r_1, \theta_1)(r_2, \theta_2) = (r_1 \cos\theta_1 + ir_1 \operatorname{sen}\theta_1)$$
$$(r_2 \cos\theta_2 + ir_2 \operatorname{sen}\theta_2) =$$
$$r_1 r_2 (\cos\theta_1 \cos\theta_2 - \operatorname{sen}\theta_1 \operatorname{sen}\theta_2) +$$
$$ir_1 r_2 (\operatorname{sen}\theta_1 \cos\theta_2 + \operatorname{sen}\theta_2 \cos\theta_1) =$$
$$r_1 r_2 \cos(\theta_1+\theta_2) + ir_1 r_2 \operatorname{sen}(\theta_1+\theta_2) =$$
$$\boxed{(r_1 r_2, \theta_1+\theta_2)}$$

Por inducción:

$$(r_1, \theta_1) \cdots (r_n, \theta_n) =$$
$$(r_1 \cdots r_n, \theta_1 + \cdots + \theta_n)$$

Caso especial:

$$(r, \theta)^n = (r^n, n\theta)$$

$$\therefore r = 1 \to (1, \theta)^n = (1, n\theta)$$

Teorema de De Moivre:

$$(\cos\theta + i\operatorname{sen}\theta)^n = \cos n\theta + i\operatorname{sen} n\theta$$

Ejemplo:

$$\begin{aligned}(\cos\theta + i\operatorname{sen}\theta)^2 &= \cos 2\theta + i\operatorname{sen} 2\theta \\ &= (\cos^2\theta - \operatorname{sen}^2\theta) + i2\operatorname{sen}\theta\cos\theta\end{aligned}$$

$$\begin{aligned}\therefore\ \operatorname{sen} 2\theta &= 2\operatorname{sen}\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \operatorname{sen}^2\theta\end{aligned}$$

Raíces: encontrar $\sqrt{i}$

$$\sqrt[6]{i} = x + iy \to i = (x+iy)^6 = 0 + 1i$$

$$\begin{aligned}\therefore x^6 + 15x^4(iy)^2 + 15x^2(iy)^4 + (iy)^6 \\ + 6x^5(iy) + 20x^3(iy)^3 + 6x(iy)^5\end{aligned}$$

Sistema complicado a resolver:

$$\begin{cases} x^6 + 15x^4(iy)^2 + 15x^2(iy)^4 + (iy)^6 = 0 \\ 6x^5(iy) + 20x^3(iy)^3 + 6x(iy)^5 = 1 \end{cases}$$

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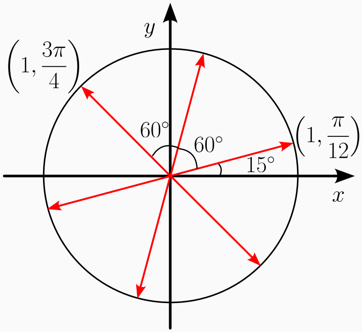
En coordenadas polares:

$$i = (1, \pi/2) \therefore \sqrt[6]{i} = (r, \theta)$$
$$\rightarrow i = (r, \theta)^6 = (r^6, 6\theta)$$
$$\therefore r = 1, \quad 6\theta = \frac{\pi}{2} + 2\pi k = \frac{1+4k}{2}\pi$$

$$r = 1,$$
$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}, \frac{25\pi}{12}, \ldots$$

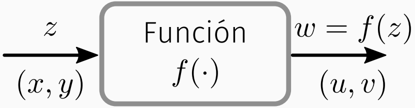
$$\left(1, \frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} + i \operatorname{sen}\frac{3\pi}{4} = \frac{1}{\sqrt{2}}(-1+i)$$

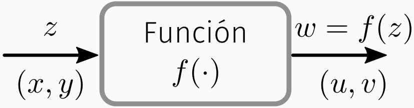


Sistema de números complejos:

Los números complejos son **cerrados** respecto de la radicación.

Pausa para resolver problemas: 1 – 8.


z
(x, y)
Función
$f(\cdot)$
$w = f(z)$
(u, v)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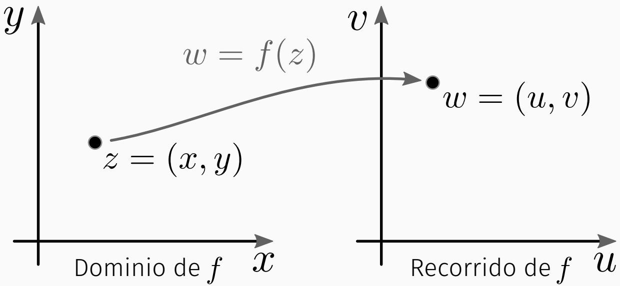
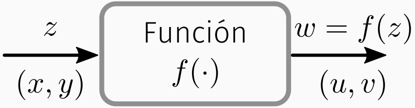
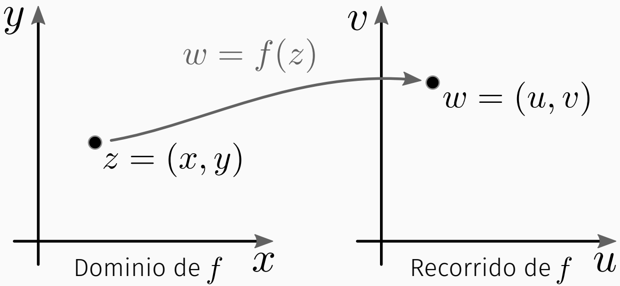

y
v
$w = f(z)$
$w = (u, v)$
$z = (x, y)$
Dominio de f
x
Recorrido de f
u

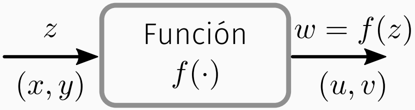
Funciones de variable compleja

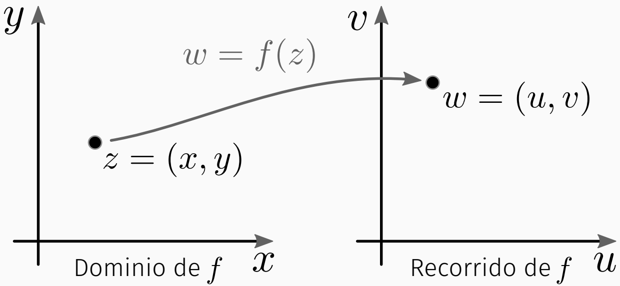




Ejemplo:

$$
\begin{aligned}
f(z) = z^2 &= (x + iy)^2 \\
&= x^2 + 2xiy + i^2y^2 = x^2 - y^2 + 2ixy \\
\therefore f(x, y) &= (x^2 - y^2, 2xy)
\end{aligned}
$$





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\therefore f(x,y) &= (x^2 - y^2, 2xy)
\end{aligned}
$$

$\therefore f(z) = z^2$ es equivalente al sistema real:

$$
\begin{cases}
u = x^2 + y^2 \\
v = 2xy
\end{cases}
$$

$\mathbb{C}$: números complejos

$f : \mathbb{C} \mapsto \mathbb{C}, a \in \mathbb{C}$

Definición:

$$\lim_{z \to a} f(z) = L$$

dado $\epsilon > 0$ existe $\delta > 0$ tal que

$$0 < |z - a| < \delta \rightarrowtail |f(z) - L| < \epsilon$$

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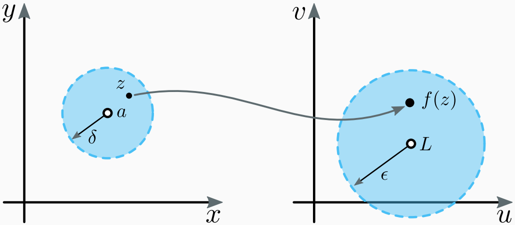

Límites

$\mathbb{C}$: números complejos

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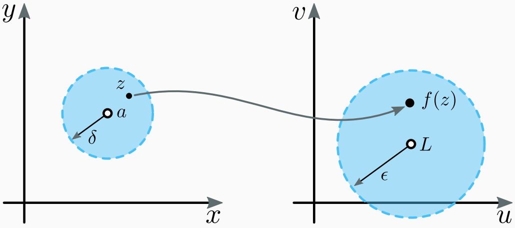

Los teoremas usuales sobre límites son válidos. En particular:

Si

$$f(z) = u(x, y) + iv(x, y)$$
$$L = L_1 + iL_2$$
$$a = a_1 + ia_2$$

Entonces:

$$\lim_{z \to a} f(z) = L \longleftrightarrow \begin{cases} \lim_{(x,y)\to(a_1,a_2)} u(x, y) = L_1 \\ \lim_{(x,y)\to(a_1,a_2)} v(x, y) = L_2 \end{cases}$$

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$$f'(z_0) = \lim_{\Delta z \to 0} \left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right]$$

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$$\begin{aligned} f'(z_0) &= \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left[\frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \right] \end{aligned}$$

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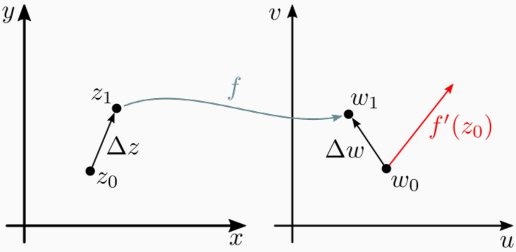

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Caso 1: $\Delta y \equiv 0$.

$$
\begin{aligned}
\therefore f'(z_0) &= \lim_{\Delta x \to 0} \left[\frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right] \\
&= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \bigg|_{z_0 = (x_0, y_0)}
\end{aligned}
$$

Derivada: casos especiales

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Caso 2: $\Delta x \equiv 0$.

$$\begin{aligned} \therefore f'(z_0) &= \lim_{\Delta y \to 0} \left[\frac{\Delta u}{i \Delta y} + \frac{\Delta v}{\Delta y} \right] = \left. \frac{\partial v}{\partial y} + \frac{1}{i} \frac{\partial u}{\partial y} \right|_{z_0 = (x_0, y_0)} \\ &= \left. \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \right|_{z_0 = (x_0, y_0)} \end{aligned}$$

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Ecuaciones de Cauchy-Riemann

Si $f = u + iv$ es diferenciable (analítica), entonces:

$$u_x = v_y$$

$$u_y = -v_x$$

Derivada: ejemplo 1

Ecuaciones de Cauchy-Riemann:

$$\begin{aligned} f(z) = z^2 &= (x+iy)^2 \\ &= (x^2-y^2) + i(2xy) \end{aligned}$$

$$\left.\begin{aligned} u_x &= 2x, & v_x &= 2y \\ u_y &= -2y, & v_y &= 2x \end{aligned}\right\} \Rightarrow \begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

$f(z) \mapsto$ diferenciable

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$f(z) \mapsto$ diferenciable

Derivada por definición:

$$\begin{aligned} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} &= \frac{(z_0+\Delta z)^2 - z_0^2}{\Delta z} \\ &= \frac{2z_0\Delta z + \Delta z^2}{\Delta z} \quad (\Delta z \neq 0) \\ &= 2z_0 + \Delta z \end{aligned}$$

$$\therefore \quad \boxed{f'(z_0) = 2z_0}$$

Derivada: ejemplo 2

$$f(z) = \bar{z} = x - iy$$
$$u = x,\ v = -y \Rightarrow u_x \neq v_y$$

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$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}\\ &= \frac{1 - i\dfrac{\Delta y}{\Delta x}}{1 + i\dfrac{\Delta y}{\Delta x}} \to \frac{1 - \dfrac{dy}{dx}}{1 + i\dfrac{dy}{dx}}\end{aligned}$$

$$f(z) = \bar{z} = x - iy$$
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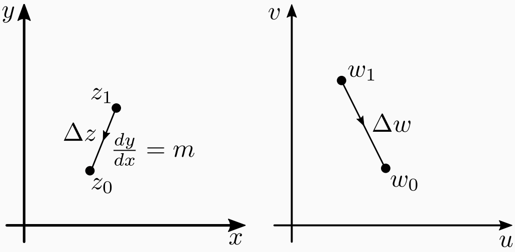

$$\frac{\Delta w}{\Delta z} = \frac{1 - im}{1 + im} = g(m)$$

Aplicación: ecuación de Laplace y ejemplo

$u(x, y)$ satisface la ecuación de Laplace si:

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

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Si $u + iv$ es **analítica**:

$$u_x = v_y \;\therefore\; u_{xx} = v_{yx}$$

$$u_y = -v_x \;\therefore\; u_{yy} = -v_{xy}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Ejemplo:

$$f(z) = z^2 \longrightarrow \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

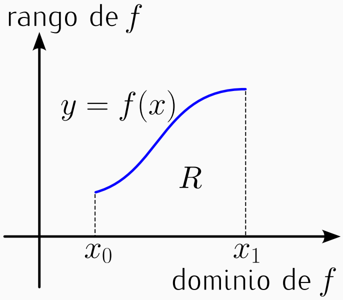
$$\therefore \quad \left. \begin{aligned} u_{xx} + u_{yy} &= 0 \\ v_{xx} + v_{yy} &= 0 \end{aligned} \right\}$$

Pausa para resolver problemas: 9 – 13.

Revisión:

$$\int_{x_0}^{x_1} f(x)\,dx = \lim_{\substack{\text{máx}\\ \Delta x \to 0}} \sum_{k=1}^{n} f(c_k^*)\Delta x_k$$

$$= F(x_1) - F(x_0),\ F' = f$$



Integración de funciones complejas

Revisión:

$$\int_{x_0}^{x_1} f(x)\,dx = \lim_{\max \Delta x \to 0} \sum_{k=1}^{n} f(c_k^*)\Delta x_k$$

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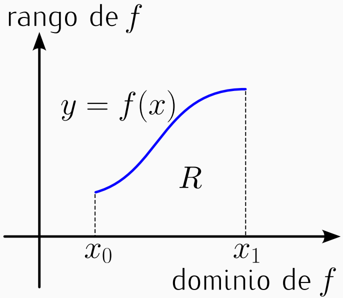
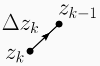
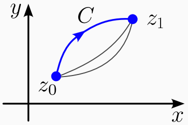


$$\int_{z_0}^{z_1} f(z)\,dz \overset{?}{=} \lim_{\max \Delta z \to 0} \sum_{k=1}^{n} f(c_k^*)\overset{?}{\Delta} z_k$$



$$C : \begin{cases} x = x(t) \\ y = y(y) \end{cases} = \begin{cases} \vec{R} = x(t)\hat{\imath} + y(t)\hat{\jmath} \\ z = x(t) + iy(t) \end{cases}$$

$$t_0 \leq t \leq t_1$$



$$\int_{C:z_0}^{z_1} f(z)\,dz = \lim_{\max \Delta z \to 0} \sum_{k=1}^{n} f(c_k(t_k))\frac{\Delta z_k}{\Delta t_k}\Delta t_k$$

$$\therefore \int_{C:z_0}^{z_1} f(z)\,dz = \int_{t_0}^{t_1} f(z(t))z'(t)\,dt$$

En términos de u y v: $f(z) = u + iv$,
$\Delta z = \Delta x + i\Delta y$:

$$\int_{\substack{C\\ z_0}}^{z_1} f(z)\,dz = \int_{\substack{C\\ (x_0,y_0)}}^{(x_1,y_1)} (u+iv)(dx+idy) =$$

$$\int_{\substack{C\\ (x_0,y_0)}}^{(x_1,y_1)} (u\,dx - v\,dy) + i\int_{\substack{C\\ (x_0,y_0)}}^{(x_1,y_1)} (v\,dx + u\,dy)$$

$$C : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

Si $u + iv$ es analítica: $u_x = v_y,\ u_y = -v_x$.

$$\therefore \left.\begin{matrix} u\,dx - v\,dy \\ v\,dx + u\,dy \end{matrix}\right\} \text{ es diferencial exacta.}$$

$\therefore$ Si $f = u + iv$ en analítica:

$$\int_{z_0}^{z_1} f(z)\,dz$$

es **independiente** de C, y

$$\boxed{\oint_C f(z)\,dz = 0, \quad \forall C}$$

f analítica $\rightarrow$

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0), \quad F' = f$$

Nota:

$$\oint_C f(z)\,dz$$

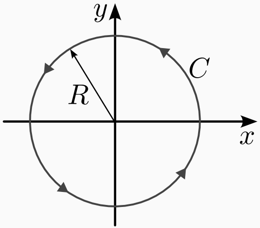
no necesariamente es 0 si f no es analítica.

Ejemplo

Calcular:

$$\oint_C \frac{dz}{z}$$

donde



El integrando es analítico en $\mathbb{C}$ excepto en $z = 0$.

Método #1:

$$\oint_C \frac{dz}{z} = \oint_C \frac{dx + idy}{x + iy}$$

$$= \oint_C \frac{(x - iy)(dx + idy)}{x^2 + y^2} =$$

$$\oint_C \frac{x\,dx + y\,dy}{x^2 + y^2} + i \oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

En C: $x = R\cos\theta$, $y = R\operatorname{sen}\theta$,

$dx = -R\operatorname{sen}\theta\,d\theta$,

$dy = R\cos\theta\,d\theta$,

$x^2 + y^2 = R^2$, $0 \leq \theta \leq 2\pi$.

$$\therefore \oint_C \frac{dz}{z} = 0$$

$$+\,i\int_0^{2\pi} \frac{R^2(\operatorname{sen}^2\theta + \cos^2\theta)\,d\theta}{R^2}$$

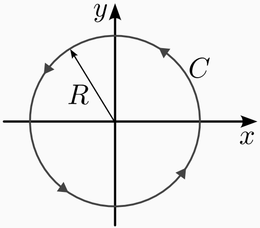
$$= \boxed{2\pi i}$$

Ejemplo

Calcular:

$$\oint_C \frac{dz}{z}$$

donde



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Método #2:
$C : z = Re^{i\theta}$, $0 \le \theta \le 2\pi$.

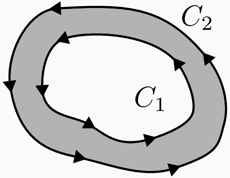
$$\frac{dz}{d\theta} = iRe^{i\theta}$$

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{z(\theta)} \frac{dz}{d\theta} d\theta$$

$$= \int_0^{2\pi} \frac{iRe^{i\theta}}{Re^{i\theta}} d\theta$$

$$= \boxed{2\pi i}$$

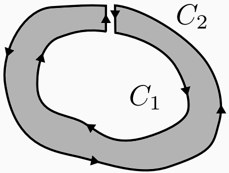
Geometría elástica (topología):



$$\oint_{\not{C}} f(z)\,dz = \oint_{C_2} f(z)\,dz - \oint_{C_1} f(z)\,dz = 0$$

Si f es analítica en C_1 y C_2, y en la región entre ellas, entonces:

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$



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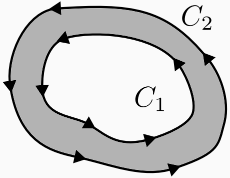

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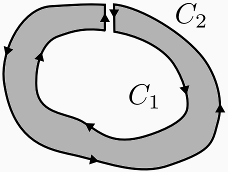
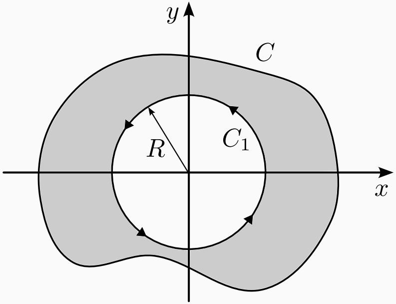


$$\oint_{\not{C}} f(z)\,dz = \oint_{C_2} f(z)\,dz - \oint_{C_1} f(z)\,dz = 0$$

Ejemplo: calcular

$$\oint_C \frac{dz}{z}$$

donde



$$\oint_C \frac{dz}{z} = \oint_{C_1} \frac{dz}{z} = 2\pi i$$

Pausa para resolver problemas: 14 – 17.

Secuencias y series

$$e^z = ?, \qquad \operatorname{sen} z = ?$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = ?$$

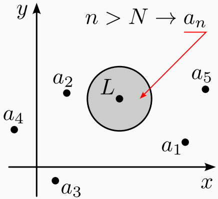
Definición : .

$$\lim_{n\to\infty} a_n = L \text{ significa que}$$

dado $\varepsilon > 0, (\varepsilon \in \mathbb{R})$, existe N, tal que

$$n > N \to |a_n - L| < \varepsilon$$

Gráficamente:



En forma similar podemos definir:

$$\sum_{n=1}^{\infty} c_n = \lim_{n\to\infty} (c_1 + \cdots + c_n)$$

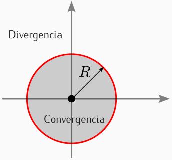
Por su estructura, **son válidos** todos los teoremas usuales.

En particular, si
$S = \{z : \sum a_n z^n \text{ converge }\}$,
entonces pueden darse los siguientes casos:

I) $S = \{0\}$.

II) $S = \mathbb{C}$ (todos los números complejos).

III) Existe un $R > 0$ tal que $S = \{z : |z| < R\}$, y la convergencia es absoluta e uniforme para $|z| \leq r < R$.



Entonces podemos **definir**:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\operatorname{sen} z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\operatorname{sen} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Se puede entonces probar que:

$$e^{iz} = \cos z + i \operatorname{sen} z$$

o

$$e^{ix} = \cos x + i \operatorname{sen} x$$

$$\therefore (r, \theta) = r\cos\theta + ir\operatorname{sen}\theta = re^{i\theta}$$

Tres observaciones:

1. $z = re^{i\theta} = re^{i(\theta+2\pi k)}$

   $$\therefore \log z = \log r + \log e^{i(\theta+2\pi k)} = \ln r + i(\theta + 2\pi k)$$

   $\therefore \log z$ es multivaluada, el **valor principal** es $-\pi < \theta \leq \pi$.

2. $$\cosh ix = \frac{e^{ix}+e^{-ix}}{2} = \frac{(\cos x + i\operatorname{sen} x)}{2} + \frac{\cos(-x) + i\operatorname{sen}(-x)}{2} = \cos x$$

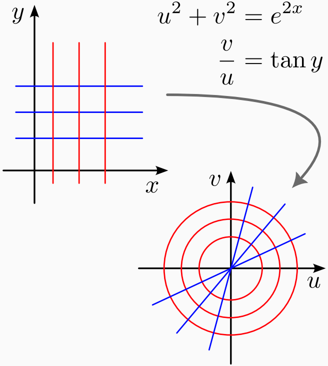
3. $e^z = e^{x+iy} = e^x e^{iy} =$

   $$e^x(\cos y + i\operatorname{sen} y) = e^x\cos y + ie^x\operatorname{sen} y \quad u(x,y) + iv(x,y)$$

u y v representan un **mapeo conforme** real:

$$u^2 + v^2 = e^{2x}$$

$$\frac{v}{u} = \tan y$$



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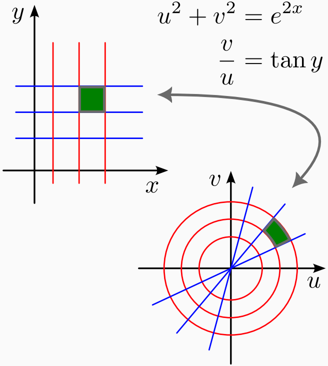


$$\frac{1}{1-u} = 1 + u + u^2 + \cdots$$
$$= \sum_{n=0}^{\infty} u^2$$

converge para $|u| < 1$.

$$\therefore \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$
$$= \sum_{n}^{\infty} (-1)^n x^{2n}$$

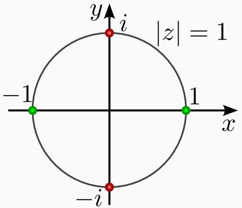
converge para $|x| < 1$.

¿Qué pasa en $x = \pm 1$?

$$\sum_{n=0}^{\infty} (-1)^n z^{2n} = \frac{1}{1+z^2}, \quad (|z| < 1)$$
$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$

$\therefore z = \pm i \leftarrow$ ¡problema!

Gráficamente:



Los puntos problemáticos están **sobre** $|z| = 1$, pero no en $z = 1$ o en $z = -1$.

Pausa para resolver problemas: 18 – 19.

Lecturas recomendadas I

- E. Kreyszig, H. Kreyszig y E.J. Norminton. ***Advanced Engineering Mathematics.*** Hoboken, USA: John Wiley & Sons, Inc, 2011. Capítulo 13 - 16.
- M.R. Spiegel et al. ***Variable compleja.*** Mexico: McGraw-Hill, 1991. Capítulos 1, 2, 6, 14.