

INTRODUCCIÓN A LA VARIABLE COMPLEJA

NÚMEROS COMPLEJOS (REPASO). FUNCIONES DE VARIABLE COMPLEJA. LÍMITE Y CONTINUIDAD.
DIFERENCIABILIDAD Y FUNCIONES ANALÍTICAS. INTEGRACIÓN EN EL CAMPO COMPLEJO. SUCESSIONES Y SERIES.

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LOS NÚMEROS COMPLEJOS

Sistema de enteros:

$$2x = 3$$

$$x = ?$$

Números "reales": $\{x : x^2 \geq 0\}$

$$x^2 = -1$$

$$x = ?$$

Motivación: $x^2 + 1 = 0$ ¿tiene solución?

Ejemplo: usar $y = e^{rx}$ para resolver:

$$y'' + y = 0$$

$$r^2 e^{rx} + e^{rx} = 0$$

$$\therefore r^2 + 1 = 0 \therefore r = \pm\sqrt{-1} = \pm i$$

$$\therefore y = e^{ix} \text{ o } y = e^{-ix}$$

$$y = \cos x \text{ o } y = \sin x$$

De "alguna manera" i debe existir y e^{ix} debe estar relacionado a $\sin x$ y $\cos x$.

El sistema de números complejos:

$$\mathbb{C} = \{x + iy : x \text{ y } y \text{ son reales.}\}$$

con la estructura:

$$(1) x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow$$

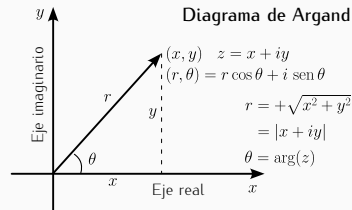
$$x_1 = x_2 \text{ y } y_1 = y_2$$

$$(2) (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$(3) r(x + iy) = rx + iry$$

r real.

\therefore los números complejos son un **espacio vectorial** por definición.



Estructura adicional de \mathbb{C} :

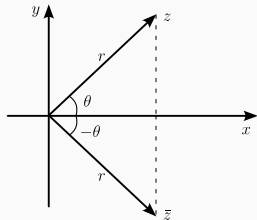
$$(4) (a + ib)(c + id) = \\ = (ac - bd) + i(bc + ad)$$

Caso especial:

$$(a + ib)(a - ib) = a^2 + b^2 \geq 0 \\ = |a + ib|^2$$

Definición: el complejo conjugado de $z = x + yi$ es

$$\bar{z} = x - yi$$



$$\frac{c + di}{a + bi} = \left(\frac{c + di}{a + bi} \right) \left(\frac{a - bi}{a - bi} \right) \\ = \frac{(ac + bd) + (ad - bc)i}{a^2 + b^2}$$

$$\frac{3 + 2i}{4 + i} = \frac{(3 + 2i)(4 - i)}{(4 + i)(4 - i)} \\ = \frac{14 + 5i}{17} \\ = \frac{14}{17} + \frac{5}{17}i$$

$$\therefore \frac{\text{complejo}}{\text{complejo}} = \text{complejo}$$

(excepto para división por cero).

Producto en coordenadas polares:

$$(r_1, \theta_1)(r_2, \theta_2) = (r_1 \cos \theta_1 + ir_1 \sin \theta_1) \\ (r_2 \cos \theta_2 + ir_2 \sin \theta_2) = \\ r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + \\ ir_1 r_2 (\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1) = \\ r_1 r_2 \cos(\theta_1 + \theta_2) + ir_1 r_2 \sin(\theta_1 + \theta_2) = \\ \boxed{(r_1 r_2, \theta_1 + \theta_2)}$$

Por inducción:

$$(r_1, \theta_1) \cdots (r_n, \theta_n) = \\ (r_1 \cdots r_n, \theta_1 + \cdots + \theta_n)$$

Caso especial:

$$(r, \theta)^n = (r^n, n\theta) \\ \therefore r = 1 \rightarrow (1, \theta)^n = (1, n\theta)$$

Teorema de De Moivre:

$$(\cos \theta + i \operatorname{sen} \theta)^n = \cos n\theta + i \operatorname{sen} n\theta$$

Ejemplo:

$$\begin{aligned}(\cos \theta + i \operatorname{sen} \theta)^2 &= \cos 2\theta + i \operatorname{sen} 2\theta \\ (\cos^2 \theta - \operatorname{sen}^2 \theta) + i 2 \operatorname{sen} \theta \cos \theta\end{aligned}$$

$$\begin{aligned}\therefore \operatorname{sen} 2\theta &= 2 \operatorname{sen} \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \operatorname{sen}^2 \theta\end{aligned}$$

Raíces: encontrar

$$\sqrt[6]{i} = x + iy \rightarrow i = (x + iy)^6 = 0 + 1i$$

$$\begin{aligned}\therefore x^6 + 15x^4(iy)^2 + 15x^2(iy)^4 + (iy)^6 \\ + 6x^5(iy) + 20x^3(iy)^3 + 6x(iy)^5\end{aligned}$$

Sistema complicado a resolver:

$$\begin{cases} x^6 + 15x^4(iy)^2 + 15x^2(iy)^4 + (iy)^6 = 0 \\ 6x^5(iy) + 20x^3(iy)^3 + 6x(iy)^5 = 1 \end{cases}$$

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En coordenadas polares:

$$i = (1, \pi/2) \therefore \sqrt[6]{i} = (r, \theta)$$

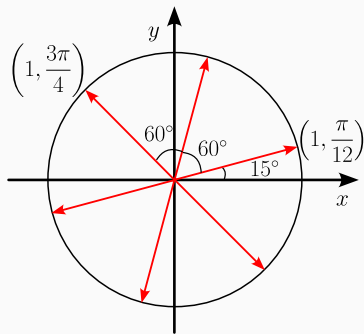
$$\rightarrow i = (r, \theta)^6 = (r^6, 6\theta)$$

$$\therefore r = 1, \quad 6\theta = \frac{\pi}{2} + 2\pi k = \frac{1 + 4k}{2}\pi$$

$$r = 1,$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}, \frac{25\pi}{12}, \dots$$

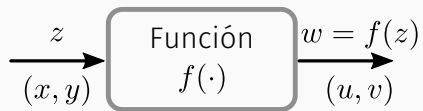
$$\begin{aligned} \left(1, \frac{3\pi}{4}\right) &= \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \\ &= \frac{1}{\sqrt{2}}(-1 + i) \end{aligned}$$

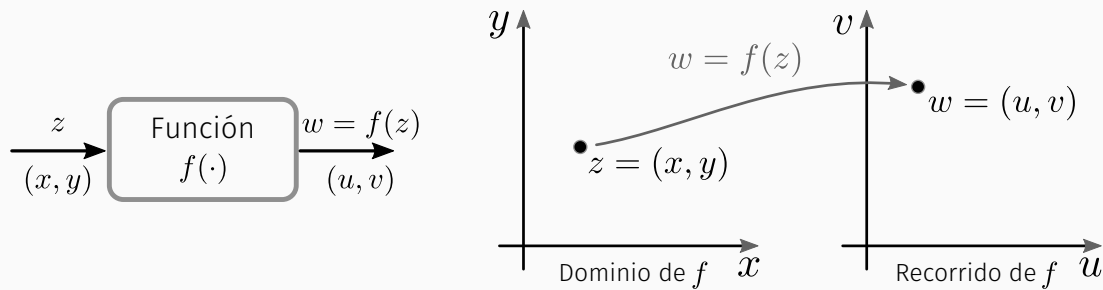


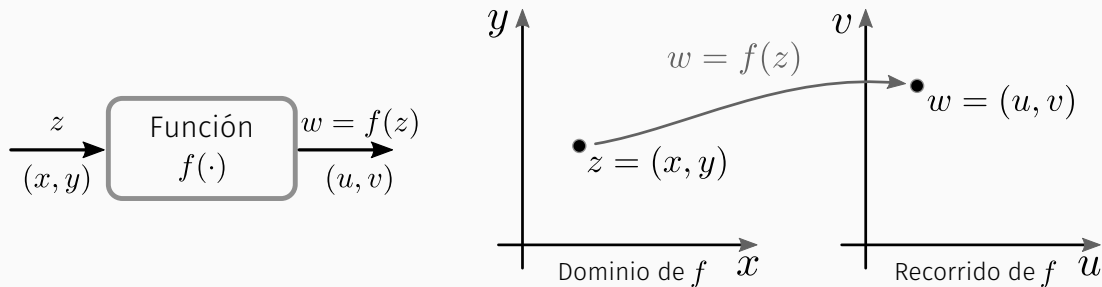
Sistema de números complejos:

Los números complejos son **cerrados** respecto de la radicación.

PAUSA PARA RESOLVER PROBLEMAS: 1 – 8.

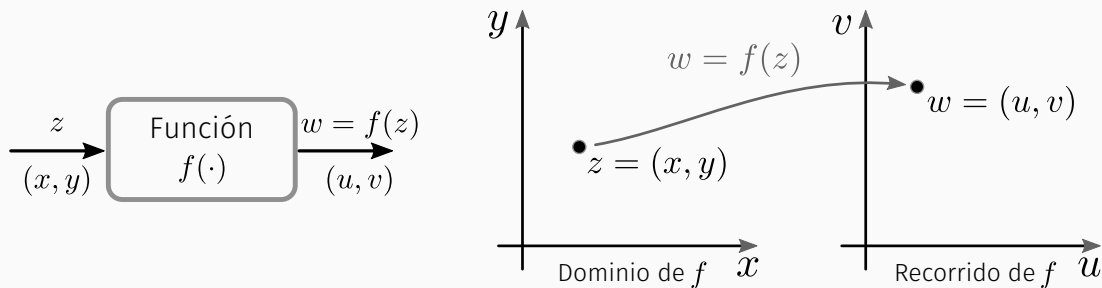






Ejemplo:

$$\begin{aligned}
 f(z) &= z^2 = (x + iy)^2 \\
 &= x^2 + 2xiy + i^2 y^2 = x^2 - y^2 + 2ixy \\
 \therefore f(x, y) &= (x^2 - y^2, 2xy)
 \end{aligned}$$



Ejemplo:

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$\therefore f(z) = z^2$ es equivalente al sistema real:

$$\begin{cases} u = x^2 + y^2 \\ v = 2xy \end{cases}$$

\mathbb{C} : números complejos

$f : \mathbb{C} \mapsto \mathbb{C}, a \in \mathbb{C}$

Definición:

$$\lim_{z \rightarrow a} f(z) = L$$

dado $\epsilon > 0$ existe $\delta > 0$ tal que

$$0 < |z - a| < \delta \mapsto |f(z) - L| < \epsilon$$

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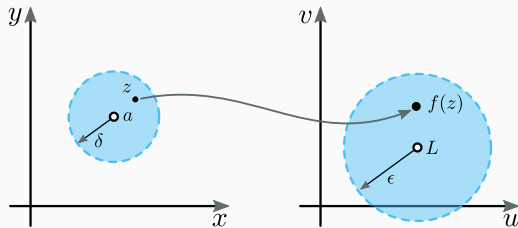
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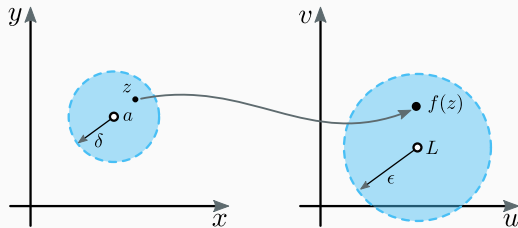
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Los teoremas usuales sobre límites **son válidos**. En particular:

Si

Entonces:

$$f(z) = u(x, y) + iv(x, y)$$

$$L = L_1 + iL_2$$

$$a = a_1 + ia_2$$

$$\lim_{z \rightarrow a} f(z) = L \iff \begin{cases} \lim_{(x,y) \rightarrow (a_1, a_2)} u(x, y) = L_1 \\ \lim_{(x,y) \rightarrow (a_1, a_2)} v(x, y) = L_2 \end{cases}$$

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$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right]$$

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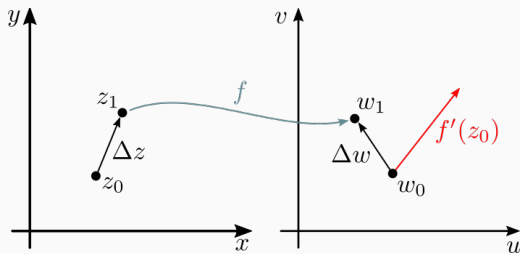
$$\begin{aligned} f'(z_0) &= \frac{dw}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \left[\frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \right] \end{aligned}$$

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Caso 1: $\Delta y \equiv 0$.

$$\begin{aligned}\therefore f'(z_0) &= \lim_{\Delta x \rightarrow 0} \left[\frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right] \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \Big|_{z_0=(x_0, y_0)}\end{aligned}$$

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Caso 2: $\Delta x \equiv 0$.

$$\begin{aligned}\therefore f'(z_0) &= \lim_{\Delta y \rightarrow 0} \left[\frac{\Delta u}{i \Delta y} + \frac{\Delta v}{\Delta y} \right] = \frac{\partial v}{\partial y} + \frac{1}{i} \frac{\partial u}{\partial y} \Big|_{z_0=(x_0, y_0)} \\ &= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \Big|_{z_0=(x_0, y_0)}\end{aligned}$$

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Ecuaciones de Cauchy-Riemann

Si $f = u + iv$ es diferenciable (**analítica**), entonces:

$$u_x = v_y$$

$$u_y = -v_x$$

Ecuaciones de Cauchy-Riemann:

$$\begin{aligned} f(z) &= z^2 = (x + iy)^2 \\ &= (x^2 - y^2) + i(2xy) \end{aligned}$$

$$\left. \begin{array}{l} u_x = 2x, \quad v_x = 2y \\ u_y = -2y, \quad v_y = 2x \end{array} \right\} \Rightarrow \begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}$$

$$f(z) \mapsto \text{diferenciable}$$

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$f(z) \mapsto$ **diferenciable**

Derivada por definición:

$$\begin{aligned}\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} \\&= \frac{2z_0\Delta z + \Delta z^2}{\Delta z} \quad (\Delta z \neq 0) \\&= 2z_0 + \Delta z\end{aligned}$$

$$\therefore \boxed{f'(z_0) = 2z_0}$$

$$f(z) = \bar{z} = x - iy$$
$$u = x, v = -y \Rightarrow u_x \neq v_y$$

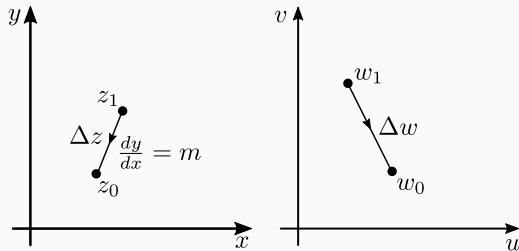
$$f(z) = \bar{z} = x - iy$$
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$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \\ &= \frac{1 - i\frac{\Delta y}{\Delta x}}{1 + i\frac{\Delta y}{\Delta x}} \rightarrow \frac{1 - \frac{dy}{dx}}{1 + i\frac{dy}{dx}}\end{aligned}$$

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$$\frac{\Delta w}{\Delta z} = \frac{1 - im}{1 + im} = g(m)$$

$u(x, y)$ satisface la ecuación de Laplace si:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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$$u_x = v_y \quad \therefore \quad u_{xx} = v_{yx}$$

$$u_y = -v_x \quad \therefore \quad u_{yy} = -v_{xy}$$

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$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Ejemplo:

$$f(z) = z^2 \longrightarrow \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

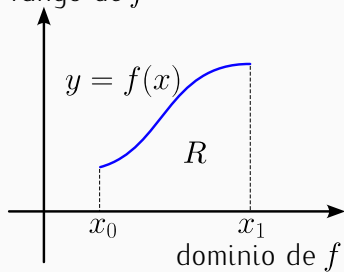
$$\therefore \left. \begin{aligned} u_{xx} + u_{yy} &= 0 \\ v_{xx} + v_{yy} &= 0 \end{aligned} \right\}$$

Revisión:

$$\int_{x_0}^{x_1} f(x) dx = \lim_{\substack{\max \\ \Delta x \rightarrow 0}} \sum_{k=1}^n f(c_k^*) \Delta x_k$$

$$= F(x_1) - F(x_0), \quad F' = f$$

rango de f



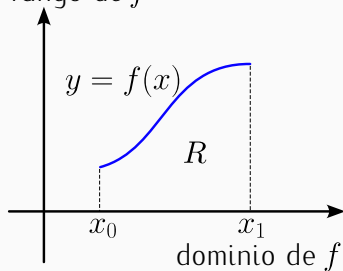
Revisión:

$$\int_{x_0}^{x_1} f(x) dx = \lim_{\substack{\text{máx} \\ \Delta x \rightarrow 0}} \sum_{k=1}^n f(c_k^*) \Delta x_k$$

$$\int_{z_0}^{z_1} f(z) dz \stackrel{?}{=} \lim_{\substack{\text{máx} \\ \Delta z \rightarrow 0}} \sum_{k=1}^n f(c_k^*) \Delta z_k \stackrel{?}{=}$$

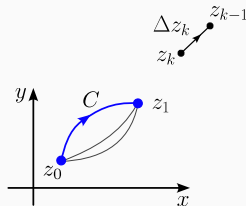
$$= F(x_1) - F(x_0), F' = f$$

rango de f



$$C : \begin{cases} x = x(t) \\ y = y(t) \end{cases} = \begin{cases} \vec{R} = x(t)\hat{i} + y(t)\hat{j} \\ z = x(t) + iy(t) \end{cases}$$

$$t_0 \leq t \leq t_1$$



$$\int_{C: z_0}^{z_1} f(z) dz = \lim_{\substack{\text{máx} \\ \Delta z \rightarrow 0}} \sum_{k=1}^n f(c_k(t_k)) \frac{\Delta z_k}{\Delta t_k} \Delta t_k$$

$$\therefore \int_{C: z_0}^{z_1} f(z) dz = \int_{t_0}^{t_1} f(z(t)) z'(t) dt$$

En términos de u y v : $f(z) = u + iv$,

$$\Delta z = \Delta x + i\Delta y:$$

$$\begin{aligned} \int_C^{z_1} f(z) dz &= \int_C^{(x_1, y_1)} (u + iv)(dx + idy) = \\ &= \int_C^{(x_1, y_1)} (u dx - v dy) + i \int_C^{(x_1, y_1)} (v dx + u dy) \\ C: &\begin{cases} x = x(t) \\ y = y(t) \end{cases} \end{aligned}$$

Si $u + iv$ es **analítica**: $u_x = v_y$, $u_y = -v_x$.

$$\therefore \begin{cases} u dx - v dy \\ v dx + u dy \end{cases} \text{ es diferencial exacta.}$$

\therefore Si $f = u + iv$ en analítica:

$$\int_{z_0}^{z_1} f(z) dz$$

es **independiente** de C , y

$$\oint_C f(z) dz = 0, \quad \forall C$$

f analítica \rightarrow

$$\int_{z_0}^{z_1} f(z) dz = F(z_1) - F(z_0), \quad F' = f$$

Nota:

$$\oint_C f(z) dz$$

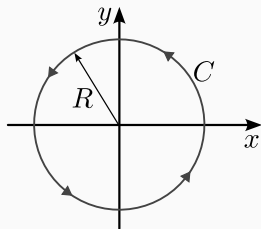
no necesariamente es 0 si f no es analítica.

EJEMPLO

Calcular:

$$\oint_C \frac{dz}{z}$$

donde



El integrando es analítico en \mathbb{C} excepto en $z = 0$.

Método #1:

$$\begin{aligned}\oint_C \frac{dz}{z} &= \oint_C \frac{dx + idy}{x + iy} \\ &= \oint_C \frac{(x - iy)(dx + idy)}{x^2 + y^2} = \\ &= \oint_C \frac{x dx + y dy}{x^2 + y^2} + i \oint_C \frac{-y dx + x dy}{x^2 + y^2}\end{aligned}$$

$$\begin{aligned}\text{En } C: x &= R \cos \theta, y = R \sin \theta, \\ dx &= -R \sin \theta d\theta, \\ dy &= R \cos \theta d\theta, \\ x^2 + y^2 &= R^2, 0 \leq \theta \leq 2\pi.\end{aligned}$$

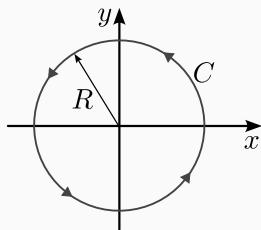
$$\begin{aligned}\therefore \oint_C \frac{dz}{z} &= 0 \\ &+ i \int_0^{2\pi} \frac{R^2(\sin^2 \theta + \cos^2 \theta) d\theta}{R^2} \\ &= \boxed{2\pi i}\end{aligned}$$

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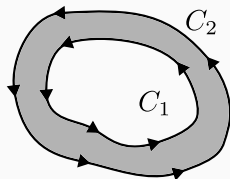
$$\begin{aligned}\therefore \oint_C \frac{dz}{z} &= 0 \\ &+ i \int_0^{2\pi} \frac{R^2(\sin^2 \theta + \cos^2 \theta) d\theta}{R^2} \\ &= \boxed{2\pi i}\end{aligned}$$

Método #2:

$$C : z = Re^{i\theta}, 0 \leq \theta \leq 2\pi.$$

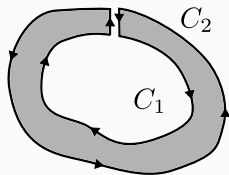
$$\begin{aligned}\frac{dz}{d\theta} &= iRe^{i\theta} \\ \oint_C \frac{dz}{z} &= \int_0^{2\pi} \frac{1}{z(\theta)} \frac{dz}{d\theta} d\theta \\ &= \int_0^{2\pi} \frac{iRe^{i\theta}}{Re^{i\theta}} d\theta \\ &= \boxed{2\pi i}\end{aligned}$$

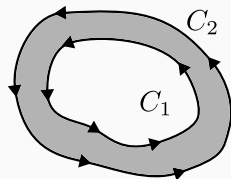
$$\oint_{\phi} f(z) dz = \oint_{C_2} f(z) dz - \oint_{C_1} f(z) dz = 0$$



Si f es analítica en C_1 y C_2 , y en la región entre ellas, entonces:

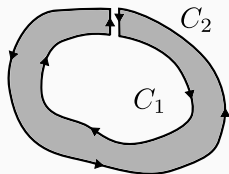
$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$





Si f es analítica en C_1 y C_2 , y en la región entre ellas, entonces:

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$

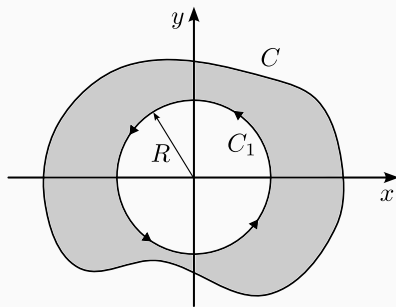


$$\oint_{\emptyset} f(z) dz = \oint_{C_2} f(z) dz - \oint_{C_1} f(z) dz = 0$$

Ejemplo: calcular

$$\oint_C \frac{dz}{z}$$

donde



$$\oint_C \frac{dz}{z} = \oint_{C_1} \frac{dz}{z} = 2\pi i$$

PAUSA PARA RESOLVER PROBLEMAS DE LA PRÁCTICA NRO. 2.

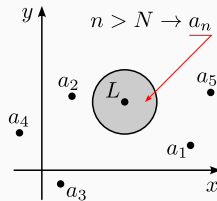
EJERCICIOS 1 – 4.

$$e^z = ?, \quad \text{sen } z = ?$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = ?$$

Gráficamente:



En forma similar podemos definir:

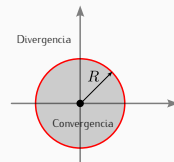
$$\sum_{n=1}^{\infty} c_n = \lim_{n \rightarrow \infty} (c_1 + \dots + c_n)$$

Por su estructura, **son válidos** todos los teoremas usuales.

En particular, si

$S = \{z : \sum a_n z^n \text{ converge}\}$, entonces pueden darse los siguientes casos:

- i) $S = \{0\}$.
- ii) $S = \mathbb{C}$ (todos los números complejos).
- iii) Existe un $R > 0$ tal que $S = \{z : |z| < R\}$, y la convergencia es absoluta e uniforme para $|z| \leq r < R$.



Definición :

$\lim_{n \rightarrow \infty} a_n = L$ significa que

dado $\varepsilon > 0$, ($\varepsilon \in \mathbb{R}$), existe N , tal que

$$n > N \rightarrow |a_n - L| < \varepsilon$$

Entonces podemos **definir**:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\operatorname{sen} z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\operatorname{sen} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Se puede entonces probar que:

$$e^{iz} = \cos z + i \operatorname{sen} z$$

o

$$e^{ix} = \cos x + i \operatorname{sen} x$$

$$\begin{aligned} \therefore (r, \theta) &= r \cos \theta + i r \operatorname{sen} \theta \\ &= r e^{i\theta} \end{aligned}$$

Tres observaciones:

$$1. z = r e^{i\theta} = r e^{i(\theta+2\pi k)}$$

$$\begin{aligned} \therefore \log z &= \log r + \log e^{i(\theta+2\pi k)} \\ &= \ln r + i(\theta + 2\pi k) \end{aligned}$$

$\therefore \log z$ es multivaluada, el **valor principal** es
 $-\pi < \theta \leq \pi$.

2.

$$\begin{aligned} \cosh ix &= \frac{e^{ix} + e^{-ix}}{2} = \\ &= \frac{(\cos x + i \operatorname{sen} x) + (\cos(-x) + i \operatorname{sen}(-x))}{2} = \cos x \end{aligned}$$

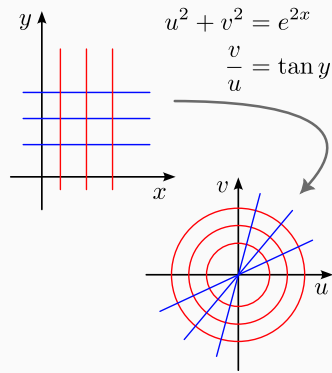
$$3. e^z = e^{x+iy} = e^x e^{iy} =$$

$$e^x (\cos y + i \operatorname{sen} y) =$$

$$e^x \cos y + i e^x \operatorname{sen} y$$

$$u(x, y) + i v(x, y)$$

u y v representan un **mapeo conforme** real:



Entonces podemos **definir**:

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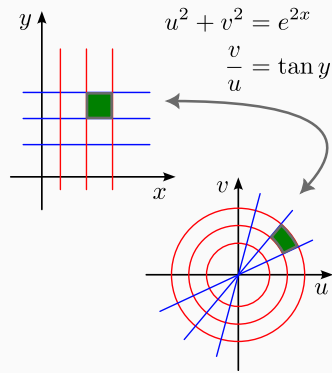
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$$u(x, y) + i v(x, y)$$

u y v representan un **mapeo conforme** real:



$$\frac{1}{1-u} = 1 + u + u^2 + \dots$$

$$= \sum_{n=0}^{\infty} u^n$$

converge para $|u| < 1$.

$$\therefore \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

$$= \sum_n (-1)^n x^{2n}$$

converge para $|x| < 1$.

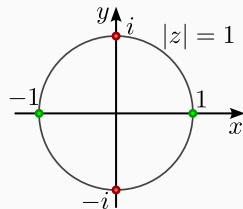
¿Qué pasa en $x = \pm 1$?

$$\sum_{n=0}^{\infty} (-1)^n z^{2n} = \frac{1}{1+z^2}, \quad (|z| < 1)$$

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$

$\therefore z = \pm i \leftarrow$ ¡problema!

Gráficamente:



Los puntos problemáticos están **sobre** $|z| = 1$, pero no en $z = 1$ o en $z = -1$.

PAUSA PARA RESOLVER PROBLEMAS DE LA PRÁCTICA NRO. 2

EJERCICIOS 5 – 6

- ▶ E. Kreyszig, H. Kreyszig y E.J. Norminton. ***Advanced Engineering Mathematics***. Hoboken, USA: John Wiley & Sons, Inc, 2011. Capítulo 13.
- ▶ M.R. Spiegel et al. ***Variable compleja***. Mexico: McGraw-Hill, 1991. Capítulos 1 y 2.
- ▶ E. Kreyszig, H. Kreyszig y E.J. Norminton. ***Advanced Engineering Mathematics***. Hoboken, USA: John Wiley & Sons, Inc, 2011. Capítulo 4 (integración). Capítulo 15 (sucesiones y series).
- ▶ M.R. Spiegel et al. ***Variable compleja***. Mexico: McGraw-Hill, 1991. Capítulo 14 (integración). Capítulo 6 (sucesiones y series)