Ecuaciones diferenciales parciales de segundo orden

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Ecuación diferencial parcial (EDP) lineal de segundo orden:

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} + F\phi = S$$

donde A, B, C, D, E, F y S son funciones de x y y en $D \in \mathbf{R}^2$.

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Tipos:

- Parabólica: $B^2 - 4AC = 0$
- Elíptica: $B^2 - 4AC < 0$
- Hiperbólica: $B^2 - 4AC > 0$

para todo $(x, y) \in D$.

Casos:

- Conducción de calor en sólidos, flujo de fluidos
- Ejemplos:
  - Conducción de calor:

$$\rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T(x,t)}{\partial x^2} + Q(x)$$

  - Transporte convectivo:

$$\frac{\partial\phi}{\partial t} = -\frac{\partial}{\partial x}u(x)\phi + D\frac{\partial^2\phi}{\partial x^2}$$

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Casos:

- Problemas estacionarios de 2 y 3 dimensiones
- Conducción de calor en sólidos, vibración de membranas
- Ejemplos:
  - Ecuación de Poisson:
    $$-\nabla^2\phi(x, y) = S(x, y)$$
  - Ecuación de Laplace:
    $$-\nabla^2\phi(x, y) = 0$$

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Casos:

- Problemas oscilatorios, propagación de ondas, fluidos
- Ejemplos:
  - Ecuación de onda:

$$\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2\nabla^2 u(x,y,z,t)$$

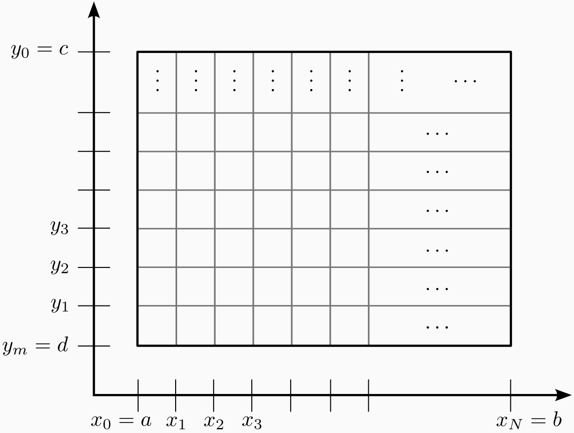
  - Navier-Stokes (incompresible):

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\nabla^2\boldsymbol{u} = -\nabla\left(\frac{p}{p_0}\right) + \boldsymbol{g}$$

Ecuación de Poisson (elíptica):

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y)$$

en $R = \{(x,y) | a < x < b, c < y < d\}$, con $u(x,y) = g(x,y)$ para $(x,y) \in S$, siendo S la frontera de R.



Malla:

- División $[a,b]$ y $[c,d]$ en n y m partes iguales
- $h = (b-a)/n$, $k = (d-c)/m$
- $x_i = a + ih, \ i = 0, 1, \ldots, n$
- $y_j = c + jk, \ j = 0, 1, \ldots, m$

Aproximación en diferencias finitas (serie de Taylor):

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + \mathcal{O}(h^2)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{k^2} + \mathcal{O}(k^2)$$

Con $u(x_i, y_j) \mapsto u_{i,j}$:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = f_{i,j} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{k^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j)$$

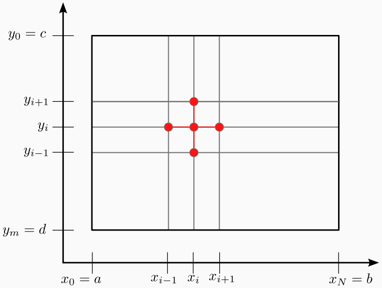
para $i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, m-1$ y condiciones de contorno:

$$u_{0,j} = g_{0,j} \quad \text{y} \quad u_{n,j} = g_{n,j}, \quad j = 0, 1, \ldots m;$$

$$u_{i,0} = g_{i,0} \quad \text{y} \quad u_{i,m} = g_{i,m}, \quad i = 0, 1, \ldots n$$

Resulta:

$$2\left[\left(\frac{h}{k}\right)^2 + 1\right] u_{i,j} - (u_{i+1,j} + u_{i-1,j}) - \left(\frac{h}{k}\right)^2 (u_{i,j+1} + u_{i,j-1}) = -h^2 f_{i,j},\ i \in [1, n-1],\ j \in [1, m-1]$$



Con $u(x_i, y_j) \mapsto u_{i,j}$:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = f_{i,j} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{k^2}{12}\frac{\partial^4 u}{\partial y^4}(x_i, \eta_j)$$

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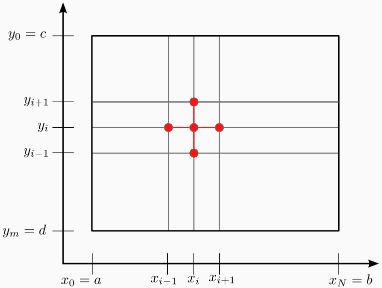
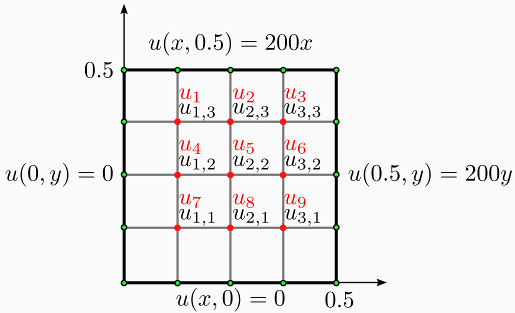


Ejemplo: Determinar la distribución estacionaria de temperaturas en una placa de 0.5×0.5 m usando $n = m = 4$. Dos bordes adyacentes se mantienen a 0 ℃ y la temperatura se incrementa linealmente en los otros bordes hasta llegar a 100 ℃ en la esquina de unión.



$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0;\ (x,y) \in [0,0.5]^2$$

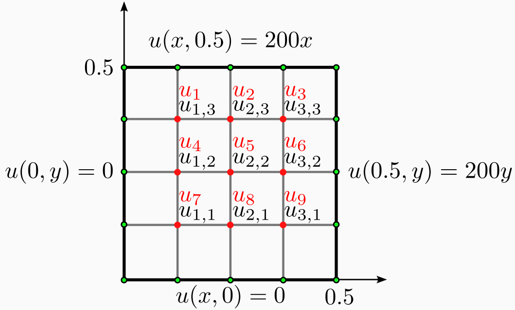
$$h = k = 1/8$$

$$u_{i,j} \mapsto u_l,\ l = i + (m-1-j)(n-1)$$

$$u_{1,1} = u_7,\ u_{2,1} = u_8,\ u_{3,1} = u_9$$

$$u_{1,2} = u_4,\ u_{2,2} = u_5,\ u_{3,2} = u_6$$

$$u_{1,3} = u_1,\ u_{2,3} = u_2,\ u_{3,3} = u_2$$



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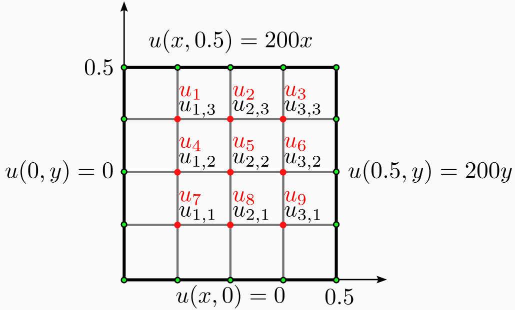
Ecuaciones:

$$4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j-1} - u_{i,j+1} = 0$$

para $i = 1,2,3;\ j = 1,2,3$.



$$\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)=0;\ (x,y)\in[0,0.5]^2$$

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Ecuaciones:

$$4u_{i,j}-u_{i+1,j}-u_{i-1,j}-u_{i,j-1}-u_{i,j+1}=0$$

para $i=1,2,3;\ j=1,2,3.$

Condiciones de borde:

$$u_{0,0}=u_{0,1}=u_{0,2}=u_{0,3}=u_{0,4}=0$$
$$u_{1,0}=u_{2,0}=u_{3,0}=u_{4,0}=0$$
$$u_{1,4}=u_{4,1}=25;\quad u_{2,4}=u_{4,2}=50$$
$$u_{3,4}=u_{4,3}=75;\quad u_{4,4}=100$$

$$u_{i,j}\mapsto u_l,\ l=i+(m-1-j)(n-1)$$

$$u_{1,1}=u_7,\ u_{2,1}=u_8,\ u_{3,1}=u_9$$
$$u_{1,2}=u_4,\ u_{2,2}=u_5,\ u_{3,2}=u_6$$
$$u_{1,3}=u_1,\ u_{2,3}=u_2,\ u_{3,3}=u_2$$

$$
\begin{bmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}
=
\begin{bmatrix} u_{0,3} + u_{1,4} \\ u_{2,4} \\ u_{3,4} + u_{4,3} \\ u_{0,2} \\ 0 \\ u_{4,2} \\ u_{0,1} \\ u_{2,0} \\ u_{3,0} + u_{4,1} \end{bmatrix}
=
\begin{bmatrix} 25 \\ 50 \\ 150 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 25 \end{bmatrix}
\rightarrow
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}
=
\begin{bmatrix} 18.75 \\ 37.50 \\ 56.25 \\ 12.50 \\ 25.00 \\ 37.50 \\ 6.25 \\ 12.50 \\ 18.75 \end{bmatrix}
$$

$$
\begin{bmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
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$$

Solución exacta: $u(x, y) = 400xy$ por lo que la aproximación en diferencias finitas no tiene error:

$$
\frac{\partial^4 u}{\partial x^4} = \frac{\partial^4 u}{\partial y^4} = 0
$$

Ecuación de calor (parabólica):

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

(más condiciones de borde/inicial)

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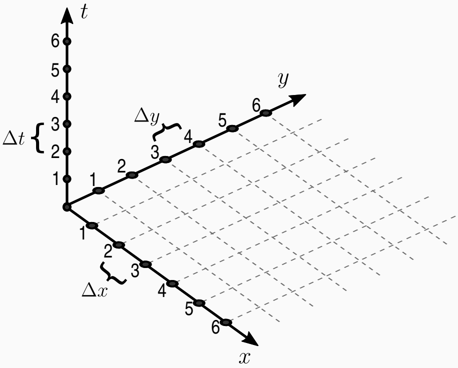

Grilla:

$$x_i = i\Delta x,\ y_j = j\Delta y,\ t_k = k\Delta t,\ u(x,y,t) = u_{i,j}^k$$

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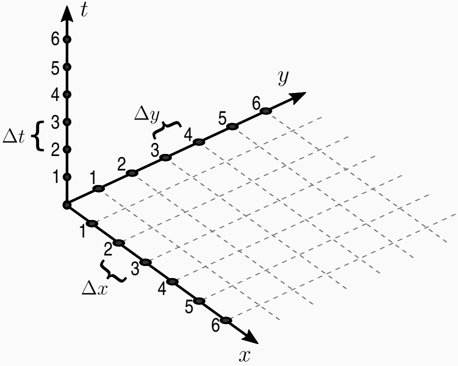

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Diferencias finitas (hacia adelante - explícito):

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\Delta t} = \alpha \left(\frac{u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k}{\Delta x^2} + \frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k}{\Delta y^2} \right)$$

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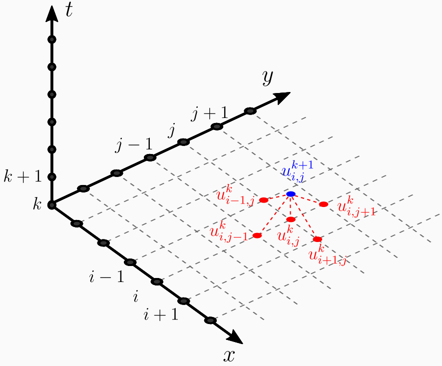

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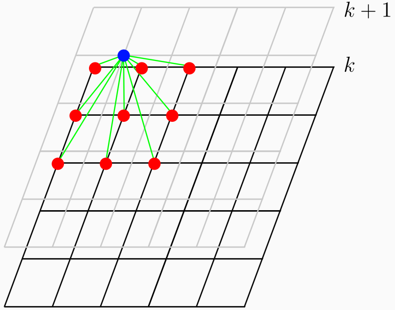
Hacemos $\Delta x = \Delta y$, $\gamma = \alpha\dfrac{\Delta t}{\Delta x^2}$:

$$u_{i,j}^{k+1} = \gamma\left(u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^k\right) + u_{i,j}^k$$

Método explícito: $\Delta t \le \dfrac{\Delta x^2}{4\alpha} \longleftarrow$ estabilidad numérica.

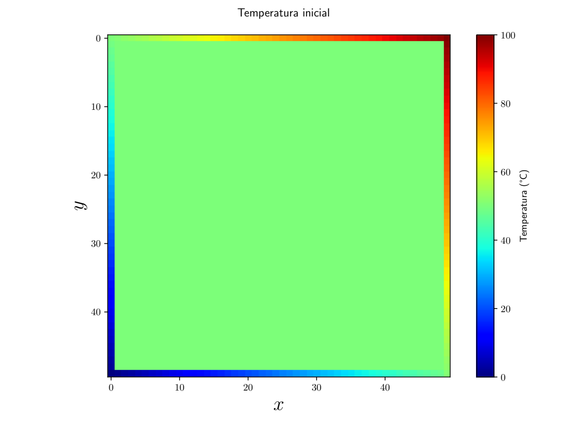


Stencil:

$$u_{i,j}^{k+1} = \gamma \left(u_{i+1,j}^{k} + u_{i-1,j}^{k} + u_{i,j+1}^{k} + u_{i,j-1}^{k} - 4u_{i,j}^{k} \right) + u_{i,j}^{k}$$

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$0 \leq x \leq L_x,\ 0 \leq y \leq L_y$$



Condiciones de borde:

$$u(x,0) = 50 + \frac{x(100-50)}{L_x}$$

$$u(0,y) = 50 - y\frac{50}{L_y}$$

$$u(x,L_y) = 50\frac{x}{L_x}$$

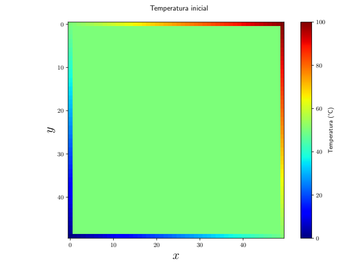
$$u(L_x,y) = 100 - y\frac{50}{L_y}$$

Condición inicial:

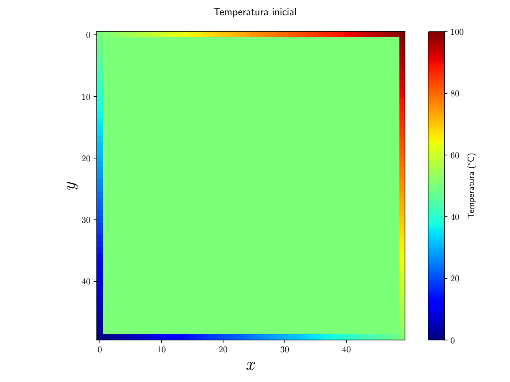
$$u(x,y)_{t=0} = 50$$

```python
#!/usr/bin/env python3

import numpy as np

Lx, Ly = 50, 50
max_iter_tiempo = 750
alpha = 5
delta_x = 1
delta_t = (delta_x ** 2)/(4 * alpha)
gamma = (alpha * delta_t) / (delta_x ** 2)

# Condiciones de borde
u_SO = 0.0
u_NO, u_SE = 50.0, 50.0
u_NE = 100.0

# Condición inicial interior
u_inicial = 50
```

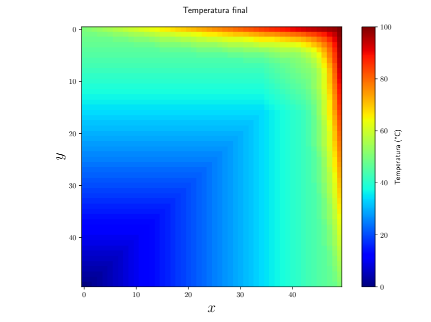
```python
def inicializar_u(max_iter_tiempo, ni=Lx, nj=Ly):
    # Inicializar la solución: u(k, i, j)
    u = np.full((max_iter_tiempo, ni, nj), u_inicial)
    # Establecer condiciones de borde (t, fila, columna)
    u[:, 0, :] = (u_NO + np.arange(ni) * delta_x
                  * (u_NE - u_NO) / Lx)
    u[:, :, 0] = (u_NO - np.arange(nj) * delta_x
                  * u_NO / Ly)
    u[:, -1, :] = np.arange(ni) * delta_x * u_SE / Lx
    u[:, :, -1] = (u_NE - np.arange(nj) * delta_x
                   * u_SE / Ly)
    return u

u = inicializar_u(max_iter_tiempo)
```



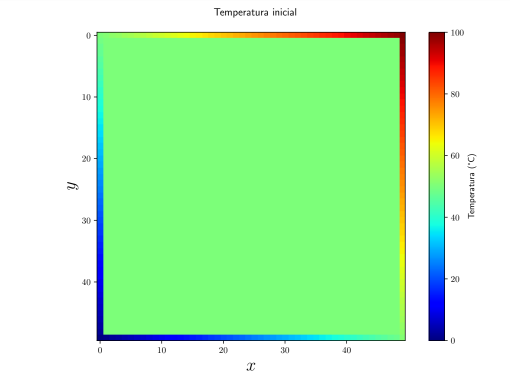
```python
import matplotlib
import matplotlib.pyplot as plt
matplotlib.rcParams.update({"text.usetex": True})
fig, ax = plt.subplots(figsize=(8,6))
mappable = ax.imshow(u[0], interpolation=None,
                     cmap=plt.cm.jet)
fig.colorbar(mappable, label="Temperatura (°C)", ax=ax)
ax.set_xlabel(r"$x$", fontsize=20)
ax.set_ylabel(r"$y$", fontsize=20)
fig.suptitle("Temperatura inicial")
fig.tight_layout()
fig.savefig("temp-inicial.pdf")
plt.close()
```



```python
# Código que aplica el stencil en la grilla (i, j) y en cada tiempo k
def calcular(u):
    nk, ni, nj = u.shape
    for k in range(0, nk-1):
        for i in range(1, ni-1):
            for j in range(1, nj-1):
                u[k + 1, i, j] = gamma * (u[k, i+1, j] + u[k, i-1 ,j]
                                        + u[k, i, j+1] + u[k, i, j-1]
                                    - 4 * u[k, i, j]) + u[k, i, j]
    return u

u = calcular(u)
fig, ax = plt.subplots(figsize=(8,6))
mappable = ax.imshow(u[-1], interpolation=None,
                     cmap=plt.cm.jet)
fig.colorbar(mappable, label="Temperatura (°C)", ax=ax)
ax.set_xlabel(r"$x$", fontsize=20)
ax.set_ylabel(r"$y$", fontsize=20)
fig.suptitle("Temperatura final")
fig.tight_layout()
fig.savefig("temp-final.pdf")
plt.close()
```



```python
def plotheatmap(u_k, k):
    # Limpiamos la figura
    plt.clf()

    plt.title(f"Temperatura en t = {k*delta_t:.2f} u.t.")
    plt.xlabel(r"$x$", fontsize=20)
    plt.ylabel(r"$y$", fontsize=20)

    # Ploteamos u_k (u_{i,j} en `paso de tiempo k`)
    plt.imshow(u_k, cmap=plt.cm.jet,
               interpolation="bicubic", vmin=0, vmax=100)
    plt.colorbar()

    return plt

import matplotlib.animation as animation
from matplotlib.animation import FuncAnimation

def animate(k):
    plotheatmap(u[k], k)

anim = animation.FuncAnimation(plt.figure(),
    animate, interval=50, frames=max_iter_tiempo,
    repeat=False)
anim.save("solucion_ecuacion_calor_t.mp4")
```



```python
def plotheatmap(u_k, k):
    # Limpiamos la figura
    plt.clf()

    plt.title(f"Temperatura en t = {k*delta_t:.2f} u.t.")
    plt.xlabel(r"$x$", fontsize=20)
    plt.ylabel(r"$y$", fontsize=20)

    # Ploteamos u_k (u_{i,j} en `paso de tiempo k`)
    plt.imshow(u_k, cmap=plt.cm.jet,
               interpolation="bicubic", vmin=0, vmax=100)
    plt.colorbar()

    return plt

import matplotlib.animation as animation
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def animate(k):
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    animate, interval=50, frames=max_iter_tiempo,
    repeat=False)
anim.save("solucion_ecuacion_calor_t.mp4")
```
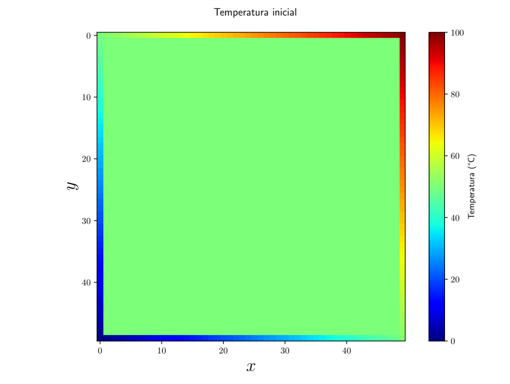



Análisis de estabilidad:

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < l,\ 0 < t$$

Condiciones de frontera:

$$u(0,t) = u(l,t) = 0, t > 0;$$
$$u(x,0) = f(x), 0 \leq x \leq l$$

Diferencia hacia adelante (o progresiva):

$$\frac{\partial u}{\partial t}(x_i,t_j) = \frac{u(x_i,t_j+k) - u(x_i,t_j)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\mu_j)$$
$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) = \frac{u(x_i+h,t_j) - 2u(x_i,t_j) + u(x_i-h,t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j)$$

con $t_j \in (t_j, t_{j+1}), \xi_i \in (x_i, x_{i+1})$

Resulta:

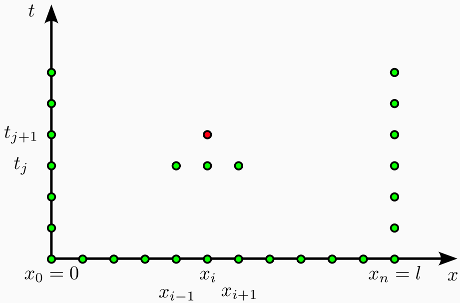
$$\frac{u_{i,j+1} - u_{i,j}}{k} - \alpha^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = 0$$

con un error local de truncación:

$$\epsilon_{i,j} = \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\mu_j) - \alpha^2 \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j)$$

Solución explícita:

$$u_{i,j+1} = \left(1 - \frac{2\alpha k}{h^2}\right) u_{i,j} + \alpha^2 \frac{k}{h^2}(u_{i+1,j} + u_{i-1,j})$$



Matriz $(n-1)\times(n-1)$:

$$\boldsymbol{A}=\begin{bmatrix}(1-2\lambda) & \lambda & 0 & \cdots & 0\\ \lambda & (1-2\lambda) & \lambda & \cdots & 0\\ 0 & \cdots & \cdots & \cdots & 0\\ 0 & \cdots & \cdots & \cdots & \lambda\\ 0 & \cdots & 0 & \lambda & (1-2\lambda)\end{bmatrix}$$

con $\lambda=\alpha^2k/h^2$
Si $\boldsymbol{u}^{\mathbf{(0)}}=(f(x_1),f(x_2),\cdots,f(x_{n-1}))^T$, la solución aproximada es:

$$\boldsymbol{u}^{(j)}=\boldsymbol{A}\boldsymbol{u}^{(j-1)}$$

Supongamos un error $\boldsymbol{e}^{(0)}=(e_1^{(0)},e_2^{(0)},\cdots,e_{n-1}^{(0)})^T$:

$$\boldsymbol{u}^{(1)}=\boldsymbol{A}(\boldsymbol{u}^{(0)}+\boldsymbol{e}^{(0)})=\boldsymbol{A}\boldsymbol{u}^{(0)}+\boldsymbol{A}\boldsymbol{e}^{(0)}$$

Para el paso k, el error en $\boldsymbol{u}^{(k)}=\boldsymbol{A}^k\boldsymbol{e}^{(0)}$. El método es estable si $\|\boldsymbol{A}^k\boldsymbol{e}^{(0)}\|\leq\|\boldsymbol{e}^{(0)}\|$

$$\|\boldsymbol{A}^k\|\leq 1 \implies \rho(\boldsymbol{A}^k)=(\rho(\boldsymbol{A}))^k\leq 1$$

Matriz $(n-1)\times(n-1)$:

$$\boldsymbol{A} = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \lambda \\ 0 & \cdots & 0 & \lambda & (1-2\lambda) \end{bmatrix}$$

con $\lambda = \alpha^2 k/h^2$

Si $\boldsymbol{u}^{(0)} = (f(x_1), f(x_2), \cdots, f(x_{n-1}))^T$, la solución aproximada es:

$$\boldsymbol{u}^{(j)} = \boldsymbol{A}\boldsymbol{u}^{(j-1)}$$

Supongamos un error $\boldsymbol{e}^{(0)} = (e_1^{(0)}, e_2^{(0)}, \cdots, e_{n-1}^{(0)})^T$:

$$\boldsymbol{u}^{(1)} = \boldsymbol{A}(\boldsymbol{u}^{(0)} + \boldsymbol{e}^{(0)}) = \boldsymbol{A}\boldsymbol{u}^{(0)} + \boldsymbol{A}\boldsymbol{e}^{(0)}$$

Para el paso k, el error en $\boldsymbol{u}^{(k)} = \boldsymbol{A}^k \boldsymbol{e}^{(0)}$. El método es estable si $\|\boldsymbol{A}^k \boldsymbol{e}^{(0)}\| \leq \|\boldsymbol{e}^{(0)}\|$

$$\|\boldsymbol{A}^k\| \leq 1 \implies \rho(\boldsymbol{A}^k) = (\rho(\boldsymbol{A}))^k \leq 1$$

Autovalores de $\boldsymbol{A}$:

$$\mu_i = 1 - 4\lambda \left(\operatorname{sen}\left(\frac{i\pi}{2n}\right)\right)^2$$

Norma L_∞:

$$\rho(\boldsymbol{A}) = \max_{1\leq i\leq n} \left| 1 - 4\lambda \left(\operatorname{sen}\left(\frac{i\pi}{2n}\right)\right)^2 \right|$$

que se simplifica a

$$0 \leq \lambda \left(\operatorname{sen}\left(\frac{i\pi}{2n}\right)\right)^2 \leq \frac{1}{2},\ i = 1, 2, \ldots, n-1$$

Esta desigualdad debe valer cuando $h \to 0, n \to \infty$:

$$\lim_{n\to\infty} \left[\operatorname{sen}\left(\frac{(n-1)\pi}{2n}\right)^2\right] = 1$$

Por lo tanto habrá estabilidad si $0 \leq \lambda \leq 1/2$:

$$\alpha^2 \frac{k}{h^2} \leq \frac{1}{2} \qquad \leftarrow \text{condicionalmente estable.}$$

Ejemplo:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < 1,\ 0 < t$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t;$$

$$u(x,0) = \operatorname{sen}(\pi x), \quad 0 \leq x \leq 1$$

- con $h = 0.1$ y $k = 0.0005$. (1000 pasos)

$$\alpha^2 \frac{k}{h} = 0.05$$

- con $h = 0.1$ y $k = 0.01$. (50 pasos)

$$\alpha^2 \frac{k}{h} = 1$$

Solución exacta:

$$u(x,t) = e^{-\pi^2 t} \operatorname{sen}(\pi x)$$

Solución: `parabolica-progresiva.py`

i	x_i	$u_{i,1000}$	$u(x_i, 0.5)$	$\lvert u_{i,1000} - u(x_i, 0.5)\rvert$
0	0.0	0.00000	0.00000	0.000e+00
1	0.1	0.00229	0.00222	6.411e-05
2	0.2	0.00435	0.00423	1.219e-04
3	0.3	0.00599	0.00582	1.678e-04
4	0.4	0.00704	0.00684	1.973e-04
5	0.5	0.00740	0.00719	2.075e-04
6	0.6	0.00704	0.00684	1.973e-04
7	0.7	0.00599	0.00582	1.678e-04
8	0.8	0.00435	0.00423	1.219e-04
9	0.9	0.00229	0.00222	6.411e-05
10	1.0	0.00000	0.00000	8.808e-19

Ejemplo:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < 1,\ 0 < t$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t;$$

$$u(x,0) = \operatorname{sen}(\pi x), \quad 0 \le x \le 1$$

- con $h = 0.1$ y $k = 0.0005$. (1000 pasos)

$$\alpha^2 \frac{k}{h} = 0.05$$

- con $h = 0.1$ y $k = 0.01$. (50 pasos)

$$\alpha^2 \frac{k}{h} = 1$$

Solución exacta:

$$u(x,t) = e^{-\pi^2 t} \operatorname{sen}(\pi x)$$

Solución: `parabolica-progresiva.py`

i	x_i	$u_{i,50}$	$u(x_i, 0.5)$	$\lvert u_{i,50} - u(x_i, 0.5)\rvert$
0	0.0	0.000e+00	0.00000	0.000e+00
1	0.1	2.637e+05	0.00222	2.637e+05
2	0.2	-5.026e+05	0.00423	5.026e+05
3	0.3	6.938e+05	0.00582	6.938e+05
4	0.4	-8.186e+05	0.00684	8.186e+05
5	0.5	8.643e+05	0.00719	8.643e+05
6	0.6	-8.254e+05	0.00684	8.254e+05
7	0.7	7.047e+05	0.00582	7.047e+05
8	0.8	-5.135e+05	0.00423	5.135e+05
9	0.9	2.704e+05	0.00222	2.704e+05
10	1.0	0.000e+00	0.00000	8.808e-19

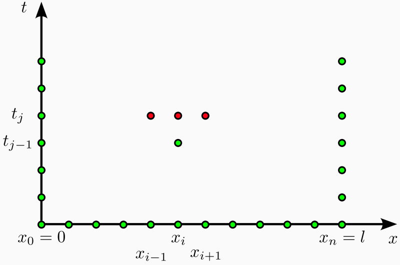
Incondicionalmente estable: diferencias regresivas (implícito).

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_j) - u(x_i, t_{j-1})}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i, \mu_j)$$

con $\mu_j \in (t_{j-1}, t_j)$.
Reemplazando en la ecuación en derivadas parciales:

$$\frac{u_{i,j} - u_{i,j-1}}{k} - \alpha^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = 0$$

para $i = 1, 2, \ldots, n-1$ y $j = 1, 2, \ldots$.



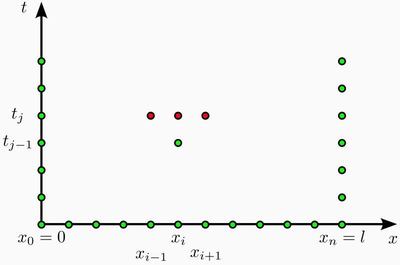
Incondicionalmente estable: diferencias regresivas (implícito).

$$\frac{\partial u}{\partial t}(x_i,t_j)=\frac{u(x_i,t_j)-u(x_i,t_{j-1})}{k}-\frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\mu_j)$$

con $\mu_j \in (t_{j-1},t_j)$.
Reemplazando en la ecuación en derivadas parciales:

$$\frac{u_{i,j}-u_{i,j-1}}{k}-\alpha^2\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}=0$$

para $i=1,2,\ldots,n-1$ y $j=1,2,\ldots$.



Hacemos $\lambda=\alpha^2k/h^2$:

$$(1+2\lambda)u_{i,j}-\lambda u_{i+1,j}-\lambda u_{i-1,j}=u_{i,j-1}$$

Con las condiciones de frontera:

$$u_{i,0}=f(x_i),\ i=1,2,\ldots,n-1$$
$$u_{0,j}=u_{n,j}=0,\ j=1,2,\ldots$$

Matriz $(n-1)\times(n-1)$:

$$\boldsymbol{A}=\begin{bmatrix}(1+2\lambda) & -\lambda & 0 & \cdots & 0\\ -\lambda & (1+2\lambda) & -\lambda & \cdots & 0\\ 0 & \cdots & \cdots & \cdots & 0\\ 0 & \cdots & \cdots & \cdots & -\lambda\\ 0 & \cdots & 0 & -\lambda & (1+2\lambda)\end{bmatrix}$$

$\boldsymbol{u}^{(j)}=(u_{1,j},u_{2,j},\cdots,u_{n-1,j})^T$,
$\boldsymbol{u}^{(j-1)}=(u_{1,j-1},u_{2,j-1},\cdots,u_{n-1,j-1})^T$

$$\mapsto \boldsymbol{A}\boldsymbol{u}^{(j)}=\boldsymbol{u}^{(j-1)},\ j=1,2,\ldots$$

Ejemplo:

$$\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < 1,\ 0 < t$$

$$u(0,t) = u(1,t) = 0, \quad t > 0;$$

$$u(x,0) = \operatorname{sen}(\pi x), \quad 0 \leq x \leq 1$$

con $h = 0.1$ y $k = 0.01$.

Solución exacta:

$$u(x,t) = e^{-\pi^2 t} \operatorname{sen}(\pi x)$$

Solución: `parabolica-regresiva.py`

i	x_i	$u_{i,50}$	$u(x_i, 0.5)$	$\lvert u_{i,50} - u(x_i, 0.5)\rvert$
0	0.0	0.00000	0.00000	0.000e+00
1	0.1	0.00780	0.00222	5.576e-03
2	0.2	0.01236	0.00423	8.133e-03
3	0.3	0.01591	0.00582	1.010e-02
4	0.4	0.01817	0.00684	1.133e-02
5	0.5	0.01894	0.00719	1.175e-02
6	0.6	0.01817	0.00684	1.133e-02
7	0.7	0.01591	0.00582	1.010e-02
8	0.8	0.01236	0.00423	8.133e-03
9	0.9	0.00780	0.00222	5.576e-03
10	1.0	0.00000	0.00000	8.808e-19

Ejemplo:

$$\frac{\partial u}{\partial t}(x,t) = \alpha\frac{\partial^2 u}{\partial x^2}(x,t),\ 0<x<1,\ 0<t$$

$$u(0,t) = u(1,t) = 0, \quad t>0;$$

$$u(x,0) = \operatorname{sen}(\pi x), \quad 0\le x\le 1$$

con $h = 0.1$ y $k = 0.01$.

Solución exacta:

$$u(x,t) = e^{-\pi^2 t}\operatorname{sen}(\pi x)$$

Solución: `parabolica-regresiva.py`

i	x_i	$u_{i,50}$	$u(x_i,0.5)$	$\lvert u_{i,50} - u(x_i,0.5)\rvert$
0	0.0	0.00000	0.00000	0.000e+00
1	0.1	0.00780	0.00222	5.576e-03
2	0.2	0.01236	0.00423	8.133e-03
3	0.3	0.01591	0.00582	1.010e-02
4	0.4	0.01817	0.00684	1.133e-02
5	0.5	0.01894	0.00719	1.175e-02
6	0.6	0.01817	0.00684	1.133e-02
7	0.7	0.01591	0.00582	1.010e-02
8	0.8	0.01236	0.00423	8.133e-03
9	0.9	0.00780	0.00222	5.576e-03
10	1.0	0.00000	0.00000	8.808e-19

Autovalores de $\boldsymbol{A}$:

$$\mu_i = 1 + 4\lambda\left[\operatorname{sen}\left(\frac{i\pi}{2n}\right)\right]^2$$

para $i = 1,2,\cdots,n-1$. Como $\lambda>0$, $\mu_i>0$.

Entonces $\rho(\boldsymbol{A}^{-1})<1 \mapsto \boldsymbol{A}$ es una matriz convergente:

$$\lim_{j\to\infty}(\boldsymbol{A}^{-1})^j\boldsymbol{e}^{(0)} = \boldsymbol{0} \qquad \leftarrow \text{incondicionalmente estable.}$$

Ejemplo:

$$\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < 1,\ 0 < t$$

$$u(0,t) = u(1,t) = 0, \quad t > 0;$$

$$u(x,0) = \operatorname{sen}(\pi x), \quad 0 \leq x \leq 1$$

con $h = 0.1$ y $k = 0.01$.

Solución exacta:

$$u(x,t) = e^{-\pi^2 t} \operatorname{sen}(\pi x)$$

Solución: `parabolica-regresiva.py`

i	x_i	$u_{i,50}$	$u(x_i, 0.5)$	$\lvert u_{i,50} - u(x_i, 0.5)\rvert$
0	0.0	0.00000	0.00000	0.000e+00
1	0.1	0.00780	0.00222	5.576e-03
2	0.2	0.01236	0.00423	8.133e-03
3	0.3	0.01591	0.00582	1.010e-02
4	0.4	0.01817	0.00684	1.133e-02
5	0.5	0.01894	0.00719	1.175e-02
6	0.6	0.01817	0.00684	1.133e-02
7	0.7	0.01591	0.00582	1.010e-02
8	0.8	0.01236	0.00423	8.133e-03
9	0.9	0.00780	0.00222	5.576e-03
10	1.0	0.00000	0.00000	8.808e-19

Autovalores de $\boldsymbol{A}$:

$$\mu_i = 1 + 4\lambda \left[\operatorname{sen}\left(\frac{i\pi}{2n}\right)\right]^2$$

para $i = 1, 2, \cdots, n-1$. Como $\lambda > 0$, $\mu_i > 0$.

Entonces $\rho(\boldsymbol{A}^{-1}) < 1 \mapsto \boldsymbol{A}$ es una matriz convergente:

$$\lim_{j\to\infty} (\boldsymbol{A}^{-1})^j \boldsymbol{e}^{(0)} = \boldsymbol{0} \qquad \leftarrow \text{incondicionalmente estable.}$$

Precisión: $\mathcal{O}(k + h^2) \leftarrow$

Método de Crank-Nicolson:

Diferencias progresivas:

$$\frac{u_{i,j+1} - u_{i,j}}{k} - \alpha^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = 0$$

con error de truncamiento:

$$\epsilon_p = \frac{k}{2} \frac{\partial^2 u}{\partial t^2}(x_i, \mu_j) + \mathcal{O}(h^2)$$

Diferencias regresivas:

$$\frac{u_{i,j+1} - u_{i,j}}{k} - \alpha^2 \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} = 0$$

con error de truncamiento:

$$\epsilon_r = -\frac{k}{2} \frac{\partial^2 u}{\partial t^2}(x_i, \hat{\mu}_j) + \mathcal{O}(h^2)$$

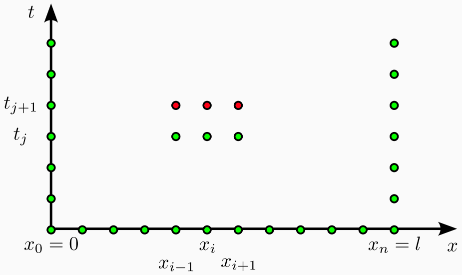
Suponiendo que:

$$\frac{k}{2} \frac{\partial^2 u}{\partial t^2}(x_i, \mu_j) \approx \frac{k}{2} \frac{\partial^2 u}{\partial t^2}(x_i, \hat{\mu}_j)$$

el método de la diferencia promediado:

$$\frac{u_{i,j+1} - u_{i,j}}{k} - \frac{\alpha^2}{2} \left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} \right] = 0$$

tiene un error de truncamiento $\mathcal{O}(k^2 + h^2) \leftarrow$



En forma matricial:

$$\boldsymbol{A}\boldsymbol{u}^{(j+1)} = \boldsymbol{B}\boldsymbol{u}^{(j)}, \quad \text{para cada} \quad j = 0, 1, \ldots, \quad \text{donde} \quad \lambda = \alpha^2 \frac{k}{h^2},\ \boldsymbol{u}^{(j)} = (u_{1,j}, u_{2,j}, \cdots, u_{n-1,j})^T$$

y

$$\boldsymbol{A} = \begin{bmatrix} (1+\lambda) & -\lambda/2 & 0 & \cdots & 0 \\ -\lambda/2 & (1+\lambda) & -\lambda/2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & -\lambda/2 \\ 0 & \cdots & 0 & -\lambda/2 & (1+\lambda) \end{bmatrix} \quad \text{y} \quad \boldsymbol{B} = \begin{bmatrix} (1-\lambda) & \lambda/2 & 0 & \cdots & 0 \\ \lambda/2 & (1-\lambda) & \lambda/2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \lambda/2 \\ 0 & \cdots & 0 & \lambda/2 & (1-\lambda) \end{bmatrix}$$

Ejemplo:

$$
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < 1,\ 0 < t \\
u(0,t) &= u(1,t) = 0, \quad t > 0; \\
u(x,0) &= \operatorname{sen}(\pi x), \quad 0 \leq x \leq 1
\end{aligned}
$$

con $h = 0.1$ y $k = 0.01$.

Solución exacta:

$$
u(x,t) = e^{-\pi^2 t} \operatorname{sen}(\pi x)
$$

Solución: `crank-nicolson.py`

i	x_i	$u_{i,50}$	$u(x_i, 0.5)$	$\lvert u_{i,50} - u(x_i, 0.5)\rvert$
0	0.0	0.00000	0.00000	0.000e+00
1	0.1	0.00488	0.00222	2.660e-03
2	0.2	0.00812	0.00423	3.895e-03
3	0.3	0.01066	0.00582	4.842e-03
4	0.4	0.01228	0.00684	5.436e-03
5	0.5	0.01283	0.00719	5.639e-03
6	0.6	0.01228	0.00684	5.436e-03
7	0.7	0.01066	0.00582	4.842e-03
8	0.8	0.00812	0.00423	3.895e-03
9	0.9	0.00488	0.00222	2.660e-03
10	1.0	0.00000	0.00000	8.808e-19

Ecuación de onda (hiperbólica)

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0,\ 0 < x < l,\ 0 < t$$

Con las ecuaciones de frontera:

$$u(0,t) = u(l,t) = 0, t > 0;$$

$$u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) = g(x);\ 0 \leq x \leq l$$

Malla:

$$x_i = ih,\ t_j = jk$$

para $i = 0, 1, \ldots, n$ y $j = 0, 1, \cdots$

En todos los puntos de la malla interior:

$$\frac{\partial^2 u}{\partial t^2}(x_i,t_j) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x_i,t_j) = 0$$

Diferencias centrales para las derivadas segundas:

$$\frac{\partial^2 u}{\partial t^2}(x_i,t_j) = \frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{k^2} - \frac{k^2}{12}\frac{\partial^4 u}{\partial t^4}(x_i,\mu_j)$$

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) = \frac{u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j)$$

donde $\xi_i \in (x_{i-1}, x_{i+1}), \mu_i \in (t_{j-1}, t_{j+1})$. Resulta:

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - \alpha^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = \epsilon_{i,j}$$

con

$$\epsilon_{i,j} = \frac{1}{12}\left[k^2 \frac{\partial^4 u}{\partial t^4}(x_i,t_j) - \alpha^2 h^2 \frac{\partial^4 u}{\partial x^4}(x_i,t_j)\right]$$

Ignorando $\epsilon_{i,j}$ y haciendo $\lambda = \alpha k/h$ (número de Courant):

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} - \lambda^2 u_{i+1,j} + 2\lambda^2 u_{i,j} - \lambda^2 u_{i-1,j} = 0$$

Resolviendo para el paso temporal:

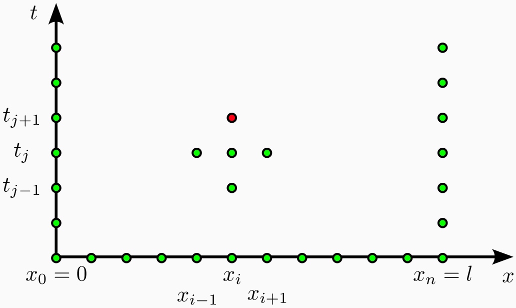
$$u_{i,j+1} = 2(1-\lambda^2)u_{i,j} + \lambda^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

para $i = 1, 2, \ldots, n-1$ y $j = 1, 2, \ldots$.
Las condiciones de frontera resultan:

$$u_{0,j} = u_{n,j} = 0,\ j = 1, 2, \ldots$$

y la condición inicial:

$$u_{i,0} = f(x_i),\ i = 1, 2, \ldots, n-1$$



Enfoque matricial:

$$\boldsymbol{u}^{(j+1)} = \boldsymbol{A}\boldsymbol{u}^{(j)} - \boldsymbol{u}^{(j-1)}$$

$$
\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{n-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1+\lambda^2) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{n-1,j} \end{bmatrix} - \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ \vdots \\ u_{n-1,j-1} \end{bmatrix}
$$

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{n-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1+\lambda^2) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{n-1,j} \end{bmatrix} - \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ \vdots \\ u_{n-1,j-1} \end{bmatrix}$$

Problema: ¿$u_{i,1}$?

Condición de velocidad inicial:

$$\frac{\partial u}{\partial t}(x,0) = g(x),\ 0 \leq x \leq l$$

Aproximación por diferencias progresivas:

$$\frac{\partial u}{\partial t}(x,0) = \frac{u(x_i,t_1) - u(x_i,0)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\mu_i)$$

Resolviendo:

$$u(x_i,t_1) = u(x_i,0) + kg(x_i) + \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\mu_i)$$

Error de truncamiento: $\mathcal{O}(k) \leftarrow$

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{n-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1+\lambda^2) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{n-1,j} \end{bmatrix} - \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ \vdots \\ u_{n-1,j-1} \end{bmatrix}$$

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Error de truncamiento: $\mathcal{O}(k) \leftarrow$

Mejora: expansión de McLaurin ($\sim$ Taylor):

$$u(x_i,t_1) = u(x_i,0) + k\frac{\partial u}{\partial t}(x_i,0) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i,0) + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i,\hat{\mu}_i)$$

Si f'' existe:

$$\frac{\partial^2 u}{\partial t^2}(x_i,0) = \alpha^2\frac{\partial^2 u}{\partial x^2}(x_i,0) = \alpha^2 f''(x_i)$$

entonces:

$$u(x_i,t_1) = u(x_i,0) + kg(x_i) + \frac{\alpha^2 k^2}{2}f''(x_i)$$

con error $\mathcal{O}(k^3) \leftarrow$.

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{n-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1+\lambda^2) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{n-1,j} \end{bmatrix} - \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ \vdots \\ u_{n-1,j-1} \end{bmatrix}$$

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Resolviendo:

$$u(x_i,t_1) = u(x_i,0) + kg(x_i) + \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\mu_i)$$

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entonces:

$$u(x_i,t_1) = u(x_i,0) + kg(x_i) + \frac{\alpha^2 k^2}{2}f''(x_i)$$

con error $\mathcal{O}(k^3) \leftarrow$. ¿Y si no tenemos $f''(x_i)$?

Ecuación en diferencias:

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} - \frac{h^2}{12} f^{(4)}(\xi_i)$$

Usando $\lambda = \alpha k/h$:

$$u(x_i, 1) = (1 - \lambda^2) f(x_i) + \frac{\lambda^2}{2} f(x_{i+1}) + \frac{\lambda^2}{2} f(x_{i-1}) + kg(x_i) + \mathcal{O}(k^3 + h^2 k^2)$$

Entonces, para $i = 1, 2, \ldots, n-1$ usamos:

$$u_{i,1} = (1 - \lambda^2) f(x_i) + \frac{\lambda^2}{2} f(x_{i+1}) + \frac{\lambda^2}{2} f(x_{i-1}) + kg(x_i)$$

y la ecuación matricial para $j = 2, 3, \ldots$

hiperbolica.py

```python
# Parámetros
L, T = 1.0, 1.0    # Dominio
Nx, Nt = 100, 1000 # Grilla
c = 1.0            # Velocidad
# Discretización
h, k = L / (Nx - 1), T / (Nt - 1)
l2 = (c * k / h)**2  # lambda**2
x = np.linspace(0, L, Nx)
t = np.linspace(0, T, Nt)

# Inicialización
u = np.zeros((Nt, Nx))
u[0, :] = np.sin(np.pi * x)  # Condición inicial
u[1, 1:Nx-1] = u[0, 1:Nx-1] + 0.5 * l2 * (u[0, 2:Nx] \
    - 2 * u[0, 1:Nx-1] + u[0, 0:Nx-2])

# Iteración en el tiempo
for n in range(1, Nt - 1):
    for i in range(1, Nx - 1):
        u[n + 1, i] = 2 * (1 - l2) * u[n, i] \
            - u[n - 1, i] + l2 * (u[n, i + 1] \
            + u[n, i - 1])
```

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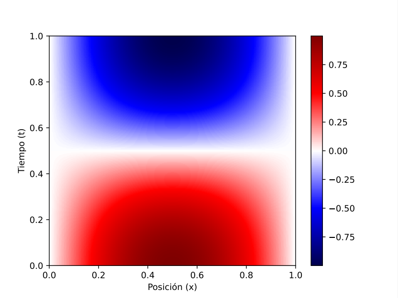
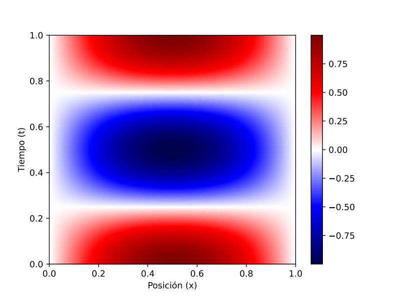
            - u[n - 1, i] + l2 * (u[n, i + 1] \
            + u[n, i - 1])
```





Lecturas recomendadas I

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- H.P. Langtangen y S. Linge. ***Finite Difference Computing with PDEs - A Modern Software Approach.*** https://hplgit.github.io/fdm-book/doc/web/index.html. 2016. Capítulo 3.