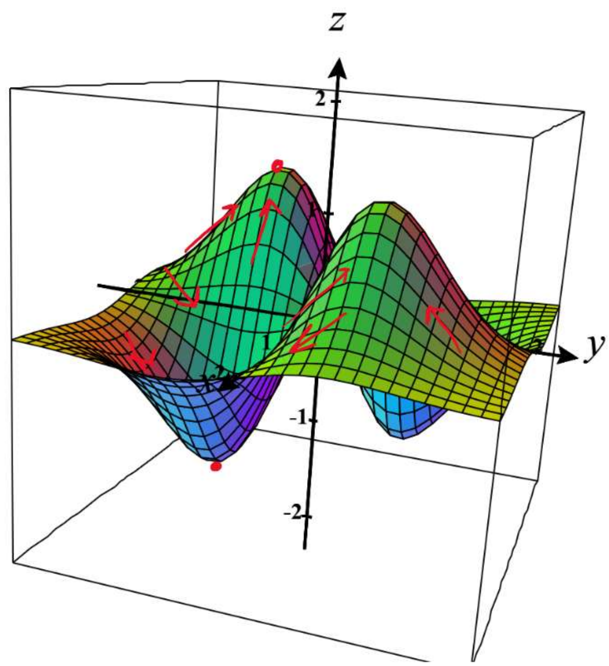
Descent Method: An iterative method for obtaining a sequence
$x_1, x_2, \dots \dots$ of successive approximations to a solution being generated by a relation
$x_{k+1} = x_k + \alpha_k d_k$ such that
$f(x_1) > f(x_2) > f(x_3) > \cdots \dots$



The analysis of behaviour of function in certain directions is very important for optimization Methods.

Directional Derivatives: Let $f: \mathbb{R}^n \to \mathbb{R}$ be a continuous function Consider $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$
If $\nabla f(x)$ exists, then the direction derivative of f in the direction of d is $\underline{d^t \cdot \nabla f(x)}$

Example: find the directional derivative of function
$f(x, y, z) = x^2 z - xyz$ *in the direction of* $(1,0,1)$

$$\nabla f(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \begin{pmatrix} 2xz - yz \\ -xz \\ x^2 - xy \end{pmatrix}$$

Directional derivative in the direction of $(1,0,1)$ is
$(1,0,1) \cdot (2xz - yz, -xz, x^2 - xy) = 2xz - yz + x^2 - xy.$

1. A function at a particular point X increases in the direction of d if its directional derivative is positive and decreases if it is negative

a) Descent direction: $d^t \cdot \nabla f(x) < 0$

b) Ascent direction : $d^t \cdot \nabla f(x) > 0$

c) Direction of gradient is that in which function has steepest ascent. $\nabla f(x_i)$

d) Direction opposite to gradient is that in which function has steepest descent. $-\nabla f(x_i)$

Steepest Descent Method To find min (objective function)

(1) Let X_1 be initial point. Set the iteration number $i=1$.

(2) Find the search direction d_i as $d_i = -\nabla f(X_i)$

(3) Determine the optimal length α_i in the direction d_i by minimizing $f(x_i+\alpha_i d_i)$ and set $X_{i+1} = X_i + \alpha_i d_i$

(4) Test X_{i+1} for optimality. If X_{i+1} is optimum, then stop, otherwise set $i=i+1$, and go to step (2).

Question :- Use the steepest descent method to minimize $f(x,y) = x^2 - xy + y^2$ such that

$X_{k+1} = (x_{k+1}, y_{k+1})$

$X_k = (x_k, y_k)$

$$\| f(X_{k+1}) - f(X_k) \| < 0.05$$

Take initial point $X_1 = (1, \frac{1}{2})$.

Solution :- $f(x,y) = x^2 - xy + y^2$

$$\nabla f(x,y) = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix}$$

$X_1 = (1, \frac{1}{2})$, $\nabla f(1, \frac{1}{2}) = \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}$

$$d_1 = \begin{pmatrix} -3/2 \\ 0 \end{pmatrix} = -\nabla f(1, \tfrac{1}{2})$$

- find the optimal length α_1 in the direction d_i by minimising function

$$f(x_1 + \alpha_1 d_1)$$

$$f\left((1, \tfrac{1}{2}) + \alpha_1 \left(-\tfrac{3}{2}, 0\right)\right) = f\left(1 - \tfrac{3}{2}\alpha_1, \tfrac{1}{2}\right)$$

$$f(x,y) = x^2 - xy + y^2$$

$$f\left(1-\tfrac{3}{2}\alpha_1, \tfrac{1}{2}\right) = \left(1-\tfrac{3}{2}\alpha_1\right)^2 - \left(1-\tfrac{3}{2}\alpha_1\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2$$

$$\text{Min}\left(\left(1-\tfrac{3}{2}\alpha_1\right)^2 - \left(1-\tfrac{3}{2}\alpha_1\right)\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2\right)$$

$$\frac{d f\left(1-\tfrac{3}{2}\alpha_1, \tfrac{1}{2}\right)}{d\alpha_1} = \cancel{2}\left(1-\tfrac{3}{2}\alpha_1\right)\left(\tfrac{-3}{\cancel{2}}\right) - \tfrac{1}{2}\left(\tfrac{-3}{2}\right)$$

$$= -3 + \frac{9}{2}\alpha_1 + \frac{3}{4}$$

$$\frac{d f\left(1-\tfrac{3}{2}\alpha_1, \tfrac{1}{2}\right)}{d\alpha_1} = 0 \quad , \quad -3 + \frac{9}{2}\alpha_1 + \frac{3}{4} = 0$$

$$\frac{9}{2}\alpha_1 = -\frac{3}{4} + 3 \Rightarrow \alpha_1 = \frac{1}{2}$$

$$\frac{d^2 f}{d\alpha_1} = \frac{9}{2} > 0 \quad \Rightarrow \quad \underline{\alpha_1 = \tfrac{1}{2} \text{ is local minima}}$$

set $X_2 = X_1 + \alpha_1 d_1 = \left(1, \frac{1}{2}\right) + \frac{1}{2}\left(\frac{-3}{2}, 0\right)$

$$X_2 = \left(\frac{1}{4}, \frac{1}{2}\right) \qquad f(x,y) = x^2 - xy + y^2$$

$$f\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{16} - \frac{1}{8} + \frac{1}{4} = \frac{3}{16}$$

$$f\left(1, \frac{1}{2}\right) = 1 - \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$\left\| f\left(\frac{1}{4}, \frac{1}{2}\right) - f\left(1, \frac{1}{2}\right) \right\| = \left\| \frac{3}{16} - \frac{3}{4} \right\| = \left\| \frac{9}{16} \right\| = \frac{9}{16} = 0.56$$

$$\underline{> 0.05}$$

Iteration 2

$$X_2 = \left(\frac{1}{4}, \frac{1}{2}\right)$$

$$\nabla f(X_2) = \left(0, 3/4\right) \qquad d_2 = \begin{pmatrix} 0 \\ -3/4 \end{pmatrix}$$

To find α_2, $\min f(X_2 + \alpha_2 d_2)$

$$f\left(\left(\frac{1}{4}, \frac{1}{2}\right) + \alpha_2 \left(0, -3/4\right)\right) = f\left(\frac{1}{4}, \frac{1}{2} - \frac{3}{4}\alpha_2\right)$$

$$f(x,y) = x^2 - xy + y^2$$

$$f' = f\left(\frac{1}{4}, \frac{1}{2} - \frac{3}{4}\alpha_2\right) = \frac{1}{16} - \frac{1}{4}\left(\frac{1}{2} - \frac{3}{4}\alpha_2\right) + \left(\frac{1}{2} - \frac{3}{4}\alpha_2\right)^2$$

$$\frac{df'}{d\alpha_2} = \frac{3}{16} + 2\left(\frac{1}{2} - \frac{3}{4}\alpha_2\right)\left(\frac{-3}{4}\right)$$

$$= \frac{3}{16} - \frac{3}{4} + \frac{9}{8}\alpha_2$$

$$\rightarrow df' = 0$$

$$16 \quad 4 \quad 8$$

$$\frac{d^2 f'}{d\alpha_2^2} = \frac{9}{8} > 0 \qquad \to \frac{df'}{d\alpha_2} = 0$$

$$\frac{3}{16} - \frac{3}{4} + \frac{9}{8}\alpha_2 = 0$$

$$\alpha_2 = \frac{1}{2}$$

$$X_3 = X_2 + \alpha_2 d_2$$
$$= \left(\frac{1}{4}, \frac{1}{2}\right) + \frac{1}{2}\left(0, -3/4\right) = \left(\frac{1}{4}, \frac{1}{8}\right)$$

$$\| f(X_3) - f(X_2) \| = \left\| f\left(\frac{1}{4}, \frac{1}{8}\right) - f\left(\frac{1}{4}, \frac{1}{2}\right) \right\|$$
$$= \left\| \cancel{\left(\frac{1}{4}\right)^2} - \frac{1}{4} \times \frac{1}{8} + \left(\frac{1}{8}\right)^2 - \cancel{\left(\frac{1}{4}\right)^2} + \frac{1}{4} \times \frac{1}{2} - \left(\frac{1}{2}\right)^2 \right\|$$
$$= \left\| \frac{1}{8} + \left(\frac{1}{8}\right)^2 - \frac{1}{32} - \frac{1}{4} \right\| = \frac{9}{64} = 0.14 > 0.05.$$

Iteration 3

$$X_3 = \left(\frac{1}{4}, \frac{1}{8}\right)$$

$$d_3 = -\nabla f(X_3) = -\left(\frac{3}{8}, 0\right) = \left(-\frac{3}{8}, 0\right)$$

Optimal length :- α_3

$$\min f(X_3 + \alpha_3 d_3)$$

$$f\left(\left(\frac{1}{4}, \frac{1}{8}\right) + \alpha_3\left(-\frac{3}{8}, 0\right)\right)$$

$$f\left(\frac{1}{4} - \frac{3}{8}\alpha_3, \frac{1}{8}\right)$$

$$\left(\left(\frac{1}{4} - \frac{3}{8}\alpha_3\right)^2 - \left(\frac{1}{4} - \frac{3}{8}\alpha_3\right)\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2\right)$$

$$\min g = \min\left(\left(\frac{1}{4} - \frac{3}{8}\alpha_3\right)^2 - \left(\frac{1}{4} - \frac{3}{8}\alpha_3\right)\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2\right)$$

$$\frac{dg}{d\alpha_3} = 2\left(\frac{1}{4} - \frac{3}{8}\alpha_3\right)\left(-\frac{3}{8}\right) - \frac{1}{8}\left(-\frac{3}{8}\right)$$

$$= -\frac{3}{16} + \frac{9}{32}\alpha_3 + \frac{3}{64}$$

$$\frac{dg}{d\alpha_3} = 0 \qquad -\frac{3}{16} + \frac{9}{32}\alpha_3 + \frac{3}{64} = 0$$

$$\alpha_3 = \frac{1}{2}$$

$$\frac{d^2 g}{d\alpha_3^2} = \frac{9}{32} > 0$$

$$X_4 = X_3 + \alpha_3 d_3 = \left(\frac{1}{4}, \frac{1}{8}\right) + \frac{1}{2}\left(-\frac{3}{8}, 0\right)$$

$$= \left(\frac{1}{16}, \frac{1}{8}\right)$$

$$\|f(X_4) - f(X_3)\| = \left\| f\left(\frac{1}{16}, \frac{1}{8}\right) - f\left(\frac{1}{4}, \frac{1}{8}\right)\right\|$$

$$= \left\| \left(\frac{1}{16}\right)^2 - \frac{1}{16} \times \frac{1}{8} + \cancel{\left(\frac{1}{8}\right)^2} - \left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{1}{8} - \cancel{\left(\frac{1}{8}\right)^2} \right\|$$

$$= \left\| \left(\frac{1}{16}\right)^2 - \frac{2}{(16)^2} - \frac{16}{(16)^2} + \frac{8}{(16)^2} \right\| = \frac{9}{(16)^2} = 0.03 < 0.05.$$

$X_4 = \underline{\left(\frac{1}{16}, \frac{1}{8}\right) \text{ is minima.}}$