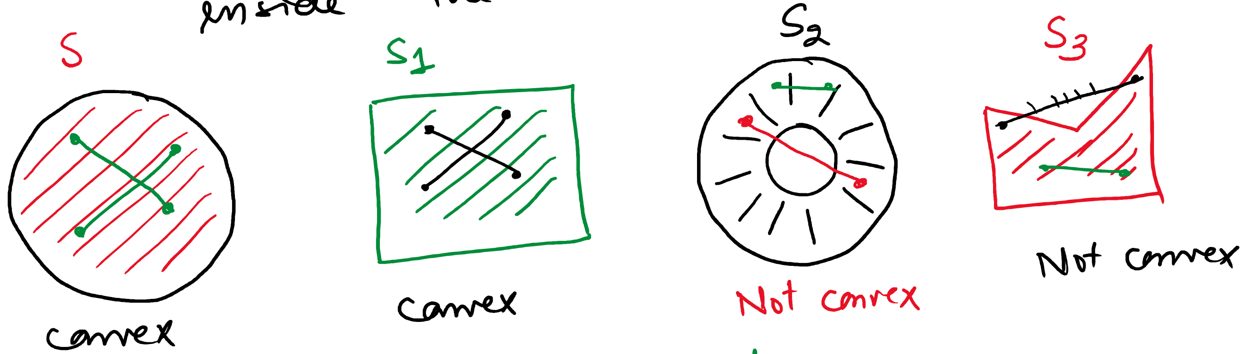
18 January 2025 11:31

$\mathbb{R}^n = \{ (x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \}$

Linear combination, Basis, dimension of V.S.

Convex set :- A subset $S \subseteq \mathbb{R}^n$ is called convex set if the straight line segment joining any two points of S lie entirely inside the set S.



Correction :- ϕ is convex set.

Singalton set is convex set.

$\{a\}$

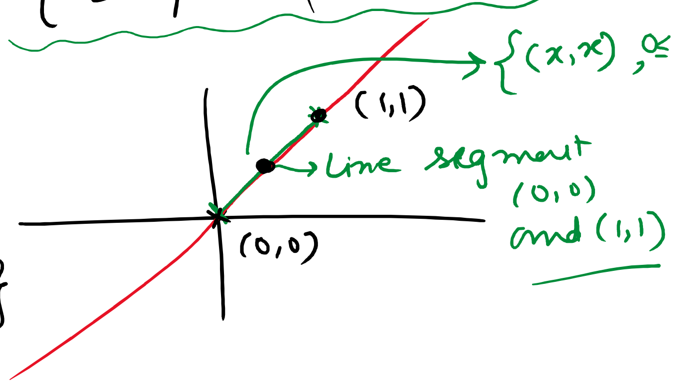
- **Line segment joining two points in $\mathbb{R}^n$**

$X \in \mathbb{R}^n$, $Y \in \mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \}$

line segment joining X and Y

$$= \{ Z \mid Z = (\lambda X + (1-\lambda) Y) \mid 0 \le \lambda \le 1 \}$$

$\mathbb{R}^2 = \{ (x, y) \mid x, y \in \mathbb{R} \}$



$\{ (x, x), 0 \le x \le 1 \}$

$\lambda = 1$, $z = (0,0)$

$z = \lambda(0,0) + (1-\lambda)(1,1)$

$0 \le \lambda \le 1$

$\lambda = 0$ $\quad z = (1,1)$

$\lambda = 0.5 \Rightarrow (0.5, 0.5)$

Convex set :- Let $S \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. Then ... convex set if

Convex set :- Let $S \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. Then

S is a convex set if

$x, y \in S$

$\Rightarrow \underline{\lambda x + (1-\lambda) y \in S} \quad \forall \quad 0 \le \lambda \le 1.$

$\downarrow$

line segement of joining x and y

Convex Combination of vectors :- A vector $x \in \mathbb{R}^n$

is called convex combination of

$\{x_1, x_2\}$ $\{x_1, x_2, \dots, x_n\}$ if $\exists$ scalars

λ_1, λ_2 $\lambda_1, \lambda_2, \dots, \lambda_n$ s.t.

$X = \lambda_1 X_1 + \lambda_2 X_2$, $(1-\lambda_1)$

$\lambda_1 + \lambda_2 = 1$

$\lambda_2 = 1 - \lambda_1$

(1) $X = \lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_n X_n$

(2) $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$ (3) $\lambda_i \ge 0, \quad 1 \le i \le n.$

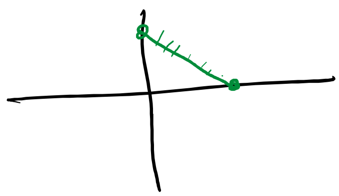
$0 \le \lambda_1 \le 1$

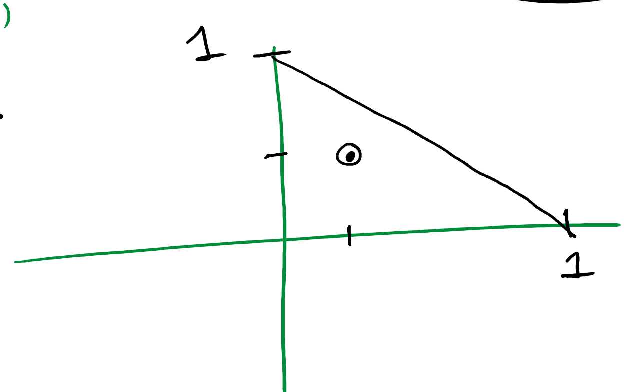
Question :- $(2,5)$ is ~~linear~~ convex combination of $(1,0)$ and $(0,1)$? Not

$(2,5) = \underset{\lambda_1}{2}(1,0) + \underset{\lambda_2}{5}(0,1)$!

$2+5=7$

$(2,1)$





- what is set of convex combinations of $(1,0)$ and $(0,1)$?

λ_1, λ_2, $\lambda_1 \ge 0, \lambda_2 \ge 0$

s.t. $\lambda_1 + \lambda_2 = 1 \Rightarrow \lambda_2 = 1 - \lambda_1$

Convex combination $\lambda_1(1,0) + \lambda_2(0,1)$

$\lambda_1(1,0) + (1-\lambda_1)(0,1)$

Convex combination $\lambda_1(1,0) + \lambda_2(0,1)$

$$\lambda_1(1,0) + (1-\lambda_1)(0,1)$$

$$0 \le \lambda_1 \le 1$$

$\lambda_1 \geq 0$

$\lambda_2 \geq 0 \Rightarrow 1-\lambda_1 \geq 0 \Rightarrow \lambda_1 \leq 1$

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functions:- function in one variable

$$f(x) = x^2 + 2x$$

function in two variable

$$f(x,y) = x^2 + xy + 2y^2$$

Optimization problem :- Max $f(x)$, Max $f(x,y)$ __

Min $f(x)$, Min $f(x,y)$ __

$f: A \to B$ → codomain

↑ Domain

Max $f(x)$

$x^* \in A$ s.t.

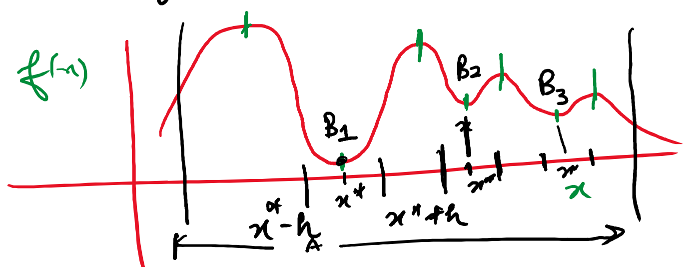
$f(x^*) \geq f(x)$

$\forall x \in A$.

Functions in one variable

A function of one variable $f(x)$ is such that have a local ~~minima~~ maxima at x^* if

$f(x^*) \leq f(x^* + h)$ [$\geq$], for all sufficiently small positive and negative values of h.



Global minima :- A function $f(x)$ is s.t.b attain

Global minima :- A function $f(x)$ is s.t.b attain global minima x^* if

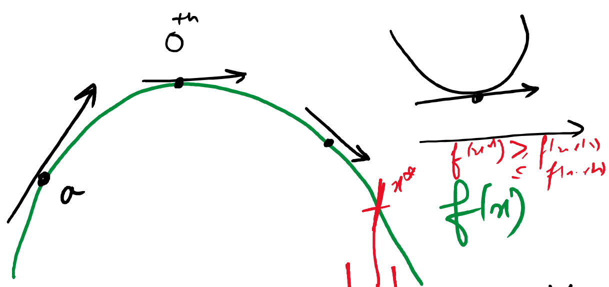
(B1)

$f(x^*) \leq f(x)$ for all x in the domain of f.

Global maxima :- $f(x^*) \geq f(x)$ for all x in the domain f.

How to find local minima / local maxima?

Necessary condition :- A function $f(x)$ has local minima or local maxima at point x^* if $\frac{d\,f(x)}{dx} = 0$ at x^*.



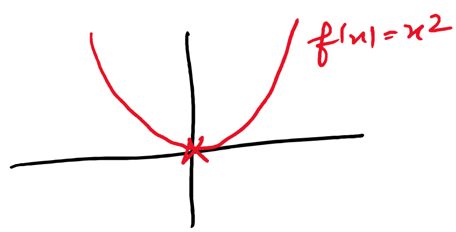
$\left.\frac{d\,f(x)}{dx}\right|_a$ = slope of the tangent at point a.

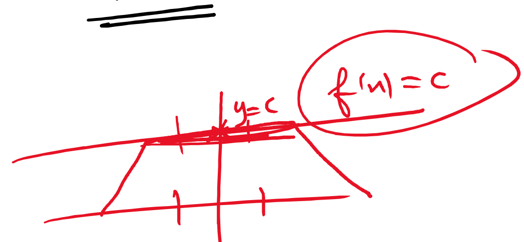
> find possible candidates $f(x) = x^2$, of local minima / local maxima.

$$\frac{d\,f(x)}{dx} = \frac{dx^2}{dx} = 2x$$

$2x = 0$

$x = 0$





Sufficient Condition :- Let x^* be candidate for local minima / local maxima for function $f(x)$.

$$f'(x^*) = \left.\frac{d\,f(x)}{dx}\right|_{x^*} = 0$$

$f'(x) = \left.\frac{df(x)}{dx}\right|_{x^*} = 0$

Let $f'(x^*)=0$, $f''(x^*) = \left.\frac{d^2 f(x)}{dx^2}\right|_{x^*} = 0$, ——

—— $f^{(n-1)}(x^*)=0$, $f^{n}(x^*) \neq 0$

1. If n is even and $f^n(x^*) > 0$, then x^* is a local minima.
2. If n is even and $f^n(x^*) < 0$, then x^* is a local maxima.
3. If n is odd, then x^* is neither local minima nor local maxima (saddle points)

Example :- $f(x) = 12x^5 - 45x^4 + 40x^3 + 5$

$$f'(x) = \frac{d f(x)}{dx} = 60x^4 - 180x^3 + 120x^2$$

$f''(x) = 240x^3 - 540x^2 + 240x$

$$f'(x) = 0$$

$$60x^4 - 180x^3 + 120x^2 = 0$$

$$60x^2(x^2 - 3x + 2) = 0$$

$x^2 = 0$

$\boxed{x = 0}$

$x^2 - 3x + 2 = 0$

$x^2 - 2x - x + 2 = 0$

$x(x-2) - 1(x-2) = 0$

$(x-1)(x-2) = 0$

$\boxed{x=1}$, $\boxed{x=2}$

$f''(0) = 0$

$f''(1) = -60 < 0$

$f''(2) = 240 > 0$

$f'''(x)$

$= (240) \times 3x^2$

$- (540 \times 2)x$

$+ 240$

$f'''(0) = 240 > 0$

$x=1$, $f'(1)=0$, $f''(1)<0$

$x=1$ is a local maxima

$x=2$, $f'(2)=0$, $f''(2)>0$, $x=2$ is a local minima.

$x=0$ $f'(0)=0$, $f''(0)=0$, $f'''(0)>0$,

$x=0$ is a stationary point.

Convex function :- Let $D \subseteq \mathbb{R}$ be subset of $\mathbb{R}$, and $f: D \to \mathbb{R}$ be a function

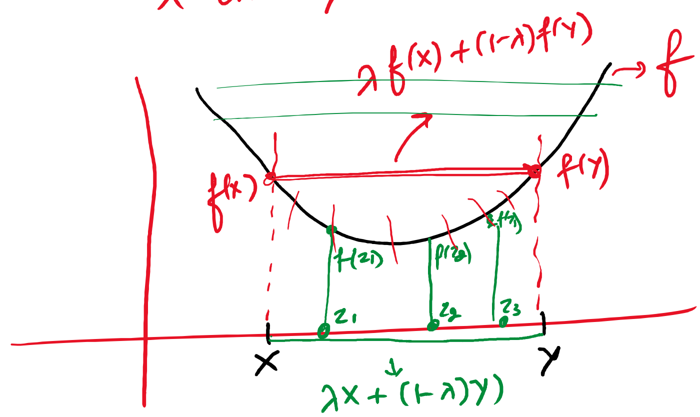
Then f is called convex function

1. D is a convex set

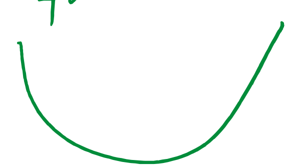
2. $f(\underbrace{\lambda x+(1-\lambda)y}) \leq \underbrace{\lambda f(x) + (1-\lambda) f(y)}$ $\forall$ $0 \leq \lambda \leq 1$, $\forall$ $x, y \in D$.

line segment forming $f(x)$ and $f(y)$

line segment forming x and y



Convex function

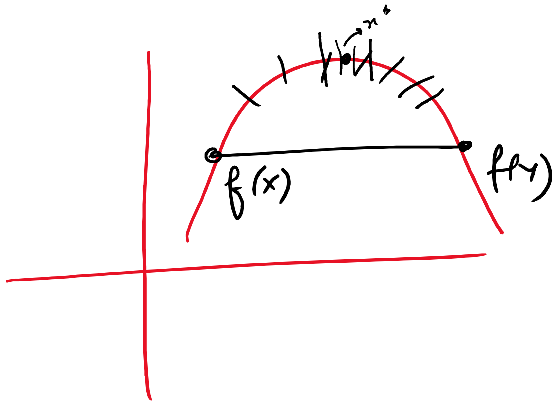


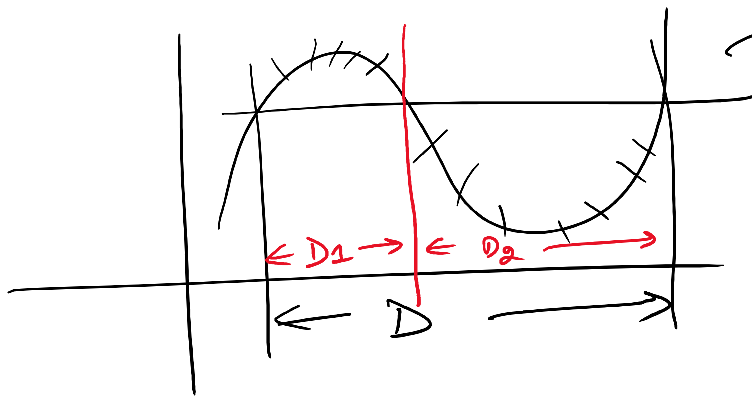
- Graph of a convex function lies below the straight line forming $f(x)$ and $f(y)$.

Concave function :- A function $f(x)$ is called concave function if $-f(x)$ is a convex function.

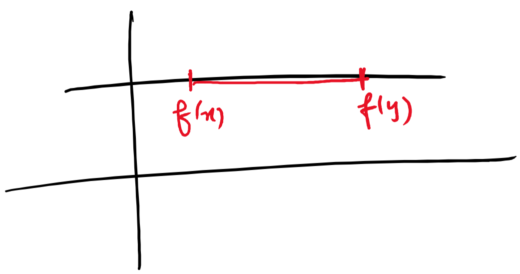
- Graph of a concave function lies above the straight line forming $f(x)$

- Graph of a Concave function lies above the straight line joining f(x) and f(y).





Neither convex Nor concave functions



$f(x) = 2x$ → both convex & concave.

$$
\begin{aligned}
f(\lambda x + (1-\lambda) y) &= 2\,(\lambda x + (1-\lambda) y) \\
&= \lambda\,(2x) + (1-\lambda)\,2y \\
&= \lambda f(x) + (1-\lambda) f(y)
\end{aligned}
$$

- Why convex functions / concave functions are important in optimization.

<u>Convex function</u> :- local minima is always a global minima.

Concave function :- local maxima is always a global maxima.