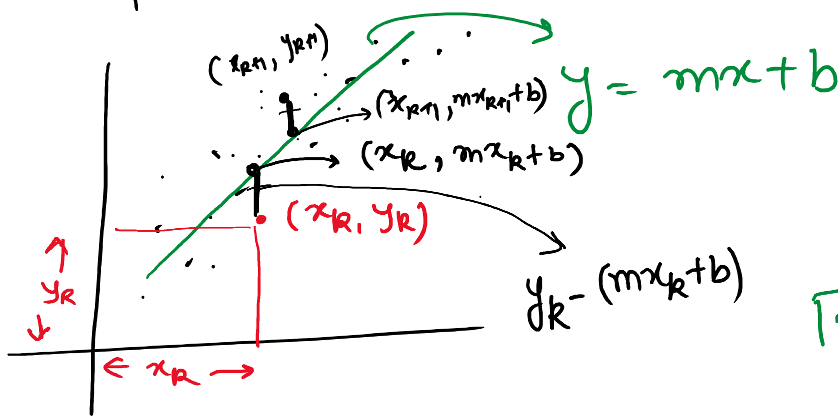
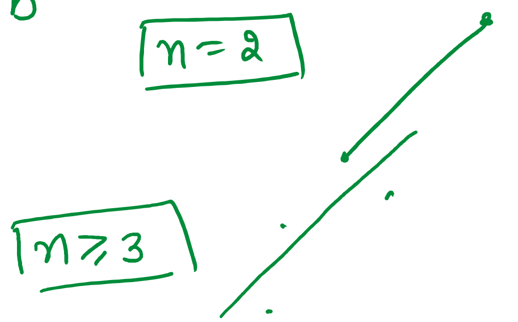
Least Squares Approximations

n points, Not colinear, fit a straight line to this Data

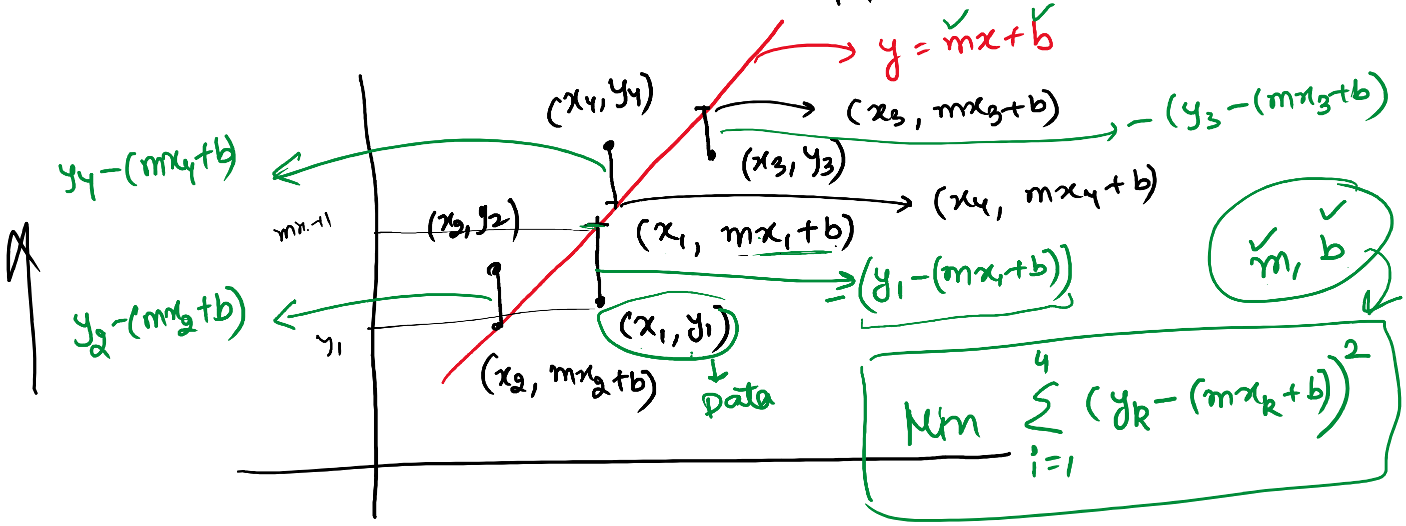




To find best fit line

$$y = mx+b$$

To minimize the error function $\sum_{i=1}^{n} \left(y_k - (mx_k+b)\right)^2$



$$\boxed{\text{Min} \sum_{i=1}^{4} \left(y_k - (mx_k+b)\right)^2}$$

Example:- Find the closest line to the point

$$(0,6),\ (1,0),\ (2,0)$$

Solution:-

$$y = mx+b \quad (\text{To find } m \text{ and } b)$$

error function $\sum^{3} \left[y_k - (mx_k+b)\right]^2$

error function $\sum_{k=1}^{3}\left[y_k - (mx_k + b)\right]^2$

$= (y_1 - mx_1 - b)^2 + (y_2 - mx_2 - b)^2 + (y_3 - mx_3 - b)^2$

$(0,6) = (x_1, y_1)$

$(1,0) = (x_2, y_2)$

$(2,0) = (x_3, y_3)$

$= [6-b]^2 + [0-m-b]^2 + [0-2m-b]^2$

error function $= (b-6)^2 + (m+b)^2 + (2m+b)^2$

$$\text{Min } (b-6)^2 + (m+b)^2 + (2m+b)^2$$

$$\nabla e = \begin{pmatrix} \partial e/\partial b \\ \partial e/\partial m \end{pmatrix} = \begin{pmatrix} \underbrace{2(b-6) + 2(b+m) + 2(2m+b)}_{6b+6m-12} \\ \underbrace{2(b+m) + 2(2m+b)2}_{10m+6b} \end{pmatrix}$$

$\nabla e = 0$

$2b - 12 + 2b + 2m + 4m + 2b = 0$

$6b + 6m = 12 \Rightarrow \boxed{b + m = 2}$

$2b + 2m + 8m + 4b = 0 \Rightarrow 10m + 6b = 0$

$\boxed{5m + 3b = 0}$

$\boxed{b = 5, \quad m = -3}$

$$H_f = \begin{bmatrix} 6 & 6 \\ 6 & 10 \end{bmatrix}$$

$6 > 0$

$60 - 36 > 0 \Rightarrow$ positive definite matrix.

minima $b = 5, \quad m = -3$

$(-3)x + 5$

$b = 5,$ ...

minima

Best fit line is $y = (-3)x + 5$

$y + 3x = 5$

- Let $\underbrace{(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)}$, be the given points

If not collinear

If the line $y = mx + b$ passes through these points, Then we have following equations

$$\boxed{\begin{aligned} y_1 &= b + mx_1 \\ y_2 &= b + mx_2 \\ &\vdots \\ y_n &= b + mx_n \end{aligned}}$$

system of linear equations with variables b, m

$\boxed{AX = Y}$ → No solution

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \quad X = \begin{bmatrix} b \\ m \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$AX = Y$ → Unique solution

→ Infinitely many solution

→ No solution (Approximate solution)

↓

Minimizing the error function.

Optimization problem ← $$e = \sum_{k=1}^{n} (b + mx_k - y_k)^2$$

Op problem $\left\{ \; e = \sum_{R=1} (b + mx_R - y_R) \right.$

$\Downarrow$

Problem of solving a system of linear equations.

Thm :- The line $y = b + mx$ minimizes the error function $e = \sum_{i=1}^{n} (b + mx_i - y_i)^2$ when $\underline{A^t A X = A^t y}$

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ | & | \\ 1 & x_n \end{bmatrix} \quad X = \begin{bmatrix} b \\ m \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ | \\ y_n \end{bmatrix}$$

Proof :- Necessary condition to minimize the condition is $\nabla e(b, m) = (0, 0)$.

$$\Updownarrow$$

$$\boxed{A^t A X = A^t y}$$

$$A^t A X = \begin{bmatrix} 1 & 1 & - & 1 \\ x_1 & x_2 & - & x_n \end{bmatrix}_{2\times n} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ | & | \\ 1 & x_n \end{bmatrix}_{n\times 2} \begin{bmatrix} b \\ m \end{bmatrix}$$

$$= \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}_{2\times 2} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} nb + m \sum_{i=1}^{n} x_i \\ b \sum_{i=1}^{n} x_i + m \sum_{i=1}^{n} x_i^2 \end{bmatrix}$$

$$A^t y = \begin{bmatrix} 1 & 1 & — & 1 \\ x_1 & x_2 & — & x_n \end{bmatrix}_{2\times n} \begin{bmatrix} y_1 \\ y_2 \\ | \\ y_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

$A^t A X = A^t y$, we get

$$\boxed{\begin{aligned} nb + m\sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} y_i \quad \text{———} \quad (1) \\ b\sum_{i=1}^{n} x_i + m\sum_{i=1}^{n} x_i^2 &= \sum_{i=1}^{n} x_i y_i \quad \text{———} \quad (2) \end{aligned}}$$

$$\nabla e = \begin{pmatrix} \frac{\partial e}{\partial b} \\ \frac{\partial e}{\partial m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad e = \sum_{i=1}^{n} (b + mx_i - y_i)^2$$

$$\frac{\partial e}{\partial b} = \sum_{i=1}^{n} 2(b + mx_i - y_i) \quad , \quad \frac{\partial e}{\partial m} = \sum_{i=1}^{n} 2(b + mx_i - y_i)\, x_i$$

$$\frac{\partial e}{\partial b} = 0 \Rightarrow \sum_{i=1}^{n} 2\underline{(b + mx_i - y_i)} = 0$$

$$\Rightarrow \sum_{i=1}^{n} (b + mx_i - y_i) = 0$$

$$\boxed{\begin{aligned} &(\underline{b} + mx_1 - y_1) + (\underline{b} + mx_2 - y_2) \\ &= 2b + m\sum_{i=1}^{2} x_i - \sum_{i=1}^{2} y_i \end{aligned}}$$

$$\Rightarrow nb + m\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \quad \left(\text{Same as eqn } (1) \right)$$

$$\frac{\partial e}{\partial m} = 0 \Rightarrow \sum_{i=1}^{n} 2(b + mx_i - y_i)x_i = 0$$

$$\ldots mx_i^2 - y_i x_i) = 0$$

∂m $\quad$ $i=1$

$$\sum_{i=1}^{n}\left(b x_i + m x_i^2 - y_i x_i\right) = 0$$

$$b\sum_{i=1}^{n} x_i + m\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i x_i \rightarrow \text{same as eqn (2).}$$

$(5,16) = (x_1, y_1)$

$(10,19) = (x_2, y_2)$

$(15,23) = (x_3, y_3)$

$(20,26) = (x_4, y_4)$

$(25,30) = (x_5, y_5)$

$$n b + m \boxed{\sum_{i=1}^{n} x_i} = \boxed{\sum_{i=1}^{n} y_i}$$

$$b \sum_{i=1}^{n} x_i + m \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

$$5b + 75m = 114$$
$$75b + 1375m = 1885$$

$\rightarrow$ $y = mx + b$