- <u>function in one variable</u>

optimality conditions of first order and second order

first order $\downarrow$ necessary condition for local extrema

second order $\downarrow$ Sufficient condition for local Extrema.

- <u>functions in several variables</u>

$$f : T \to \mathbb{R}, \quad T \subseteq \mathbb{R}^n, \; n > 1$$

$$f(x,y) = xy + x^2 \quad (\text{function in two variables})$$

<u>Necessary Condition</u>

$$f(x_1, x_2, -, x_n) =$$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, -, \frac{\partial f}{\partial x_n} \right)$$

$\frac{\partial f}{\partial x_2}$ $\to$ diff x_2,

$\frac{\partial f}{\partial x_1}$ $\to$ diff w.r.t x_1, considering all other variables $x_2, -, x_n$ as constants

$(x_1^*, x_2^*, -, x_n^*)$ is a candidate for local maxima or minima

if $\nabla f \big|_{(x_1^*, x_2^*, -, x_n^*)} = (0 - 0)$

Example :- $\quad f(x,y) = \boxed{x - y + 2x^2 + 2xy + y^2}$

Example :- $f(x,y) = \underbrace{x - y + 2x^2 + 2xy + y^2}$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

$$\nabla f = \left(1 + 4x + 2y \ , \ -1 + 2x + 2y \right)$$

$$\nabla f = (0,0) \qquad \overset{\shortparallel}{(0,0)}$$

$$1 + 4x + 2y = 0$$
$$-1 + 2x + 2y = 0$$

$$\left.\begin{aligned} 4x + 2y &= -1 \\ 2x + 2y &= 1 \end{aligned}\right) \longrightarrow x = -1, \quad y = 3/2$$

$(-1, 3/2)$ is candidate for local maxima or local minima.

Hessian matrix of function $f(x,y)$

$$H_f(x,y) = \begin{array}{c} \underline{\underline{1,1}} \qquad\qquad\qquad 1,2 \\ \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}, & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \end{array} \qquad \left(\underline{\underline{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}} \right)$$

$$f(x,y) = x - y + 2x^2 + 2xy + y^2$$

$$\frac{\partial f}{\partial x} = \underline{\underline{1 + 4x + 2y}} \qquad \frac{\partial f}{\partial y} = -1 + 2x + 2y$$

$\partial^2 f$.

$$H_f(x,y) = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x^2} = 4 \qquad \frac{\partial^2 f}{\partial y\,\partial x} = 2$$

$f(x_1, x_2, -, x_n)$

$$H_f(x_1, x_2, -, x_n) = \begin{array}{c} \quad 1,1 \qquad\qquad 1,2 \qquad\qquad\qquad\qquad 1,n \\ \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \text{---} & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \text{---} & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ | & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \text{---} & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \end{array}$$

row no. $\leftarrow$ 2,1 col. no.

Example $f(x,y,z) = x^2 + 2y^2 + 3z^2 + 2xy + 2xz$

$$\frac{\partial f}{\partial x} = 2x + 2y + 2z$$

$$\frac{\partial f}{\partial y} = 4y + 2x$$

$$\frac{\partial f}{\partial z} = 6z + 2x$$

$$H_f(x,y,z) = \begin{array}{c} \frac{\partial^2 f}{\partial x^2} \quad \frac{\partial^2 f}{\partial y \partial x} \quad \frac{\partial^2 f}{\partial z \partial x} \\ \begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix} \end{array}$$

$$\boxed{\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}}$$

Symmetric matrix :- A be a matrix (square)

$$A = \begin{bmatrix} a_{11} & a_{12} & \text{---} & a_{1n} \\ a_{21} & a_{22} & \text{---} & a_{2n} \\ | & | & \ddots & | \\ a_{n1} & a_{n2} & \text{---} & a_{nn} \end{bmatrix}$$

$$\begin{bmatrix} | & | & \diagdown & | \\ a_{n1} & a_{n2} & \text{---} & a_{nn} \end{bmatrix}$$

A is called symmetric matrix if $a_{ij} = a_{ji}$ $\forall i, \forall j$

a_{ij} → i^{th} row and j^{th} column

a_{ji} → j^{th} row i^{th} column

$$\begin{array}{cccc} & & a_{12} & a_{13} \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \\ a_{21} & a_{22} & & \\ a_{31} & & \diagdown & \\ & & & a_{nn} \end{bmatrix}$$

$A^t = A$

symmetric

- Hessian matrix is always a symmetric matrix.

Positive (semi) definite matrix :- A symmetric matrix A is called positive (semi) definite matrix if its all leading minors are positive. (Non-negative) → zero + positive

n^{th} leading minor = determinant of $n \times n$ submatrix in upper left corner

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a_{11} → first leading minor.

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ → second leading minor.

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ → third leading minor.

$n \times n$ matrix

↳ total n leading minors.

Example :- $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ & & \end{bmatrix}$

first leading minor $= 2 > 0$

second leading minor

$$\begin{bmatrix} 2 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$

Second leading minor

$$\begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 8-4 = 4 > 0$$

Third leading minor

$$\begin{vmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 4 & 0 \\ 0 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 2 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ 2 & 0 \end{vmatrix}$$

$$= 2(24) - 2(12) + 2(-8) = 8 > 0$$

- This matrix is a positive definite matrix.

Negative semi definite matrix :- k^{th} leading minor has sign $(-1)^k$, (or it is zero)

Example
$$\begin{pmatrix} -4 & 2 & 0 \\ 2 & -5 & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

1st leading minor $\longrightarrow$ $-ve$
second leading minor $\longrightarrow$ $+ve$
third leading minor $\longrightarrow$ $-ve$

$\rightarrow$ Negative definite matrix.

Sufficient Condition :- Let $f: D \to \mathbb{R}$ be multivariable function, $D \subseteq \mathbb{R}^n$.

A vector $X_0 \in D$ is local ~~minima~~ maxima minima if

① $\nabla f(X_0) = (0 — 0)$ $\rightarrow$ $(\alpha_1, \alpha_2, \ldots \alpha_n)$

$x_1, x_2, \ldots x_n$ ② Hessian matrix at X_0 is positive semi-definite. (Negative semi-definite)

all minors must be Non-negative i.e. zero or positive.

all minors ...

zero is positive.

Convexity By Hessian matrix

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$ be a twice diff function on convex set D. If H_f is positive semi-definite for all X in D, then f is a convex function in D.

- If H_f is a Negative semi-definite matrix $\forall x \in D$, then f is a concave function in D.

* Convex function: local minima $\Rightarrow$ Global minima
* Concave function: local maxima $\Rightarrow$ Global maxima.

Example :- Min $x^2+y^2+z^2$

subject to $x^2 - z^2 = 1$

$x^2 = z^2 + 1$

* min $(z^2+1) + y^2 + z^2$

$$f(y,z) = y^2 + 2z^2 + 1$$

$$\nabla f(y,z) = \left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2y, 4z)$$

$$\nabla f(y,z) = (0,0) \qquad \begin{matrix} y=0 \\ z=0 \end{matrix}$$

$(0,0)$ is candidate local minima / local maxima

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

first leading minor $= 2 > 0$

second leading minor $= \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8 > 0$

positive ~~semi~~ definite matrix.

- $f(y,z)$ is a convex function

$$H_f\Big|_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$ is also positive definite matrix

$(0,0)$ is a local minima

$\Downarrow$

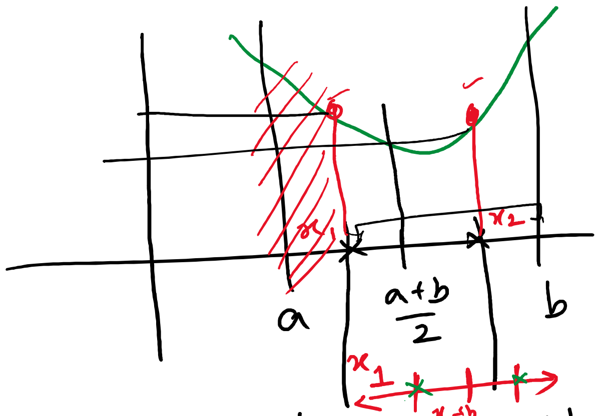
$(0,0)$ is a Global minima

Line search method $\rightarrow$ maxima or minima of single variable objective Unimodal function in [a,b]

- Golden section method
- Fibonacci method

Unimodal in [a,b] :- The objective function $f(x)$ has only one local minima or local maxima (but not both) in this region [a,b].

- Equal interval Method



$\frac{a+b}{2}-\varepsilon$ $\quad$ $\frac{a+b}{2}+\varepsilon$

$f(x_1) < f(x_2)$? $\rightarrow$ No

$f(x_1) > f(x_2)$? $\rightarrow$ Yes

$f(x_1) = f\left(\frac{a+b}{2}-\varepsilon\right)$ $\qquad$ $f\left(\frac{a+b}{2}+\varepsilon\right) = f(x_2)$

Golden section Method $\longrightarrow$ line search method

function must be unimodal (only one local minima in interval of search)

Golden Ratio^{-1} :- $0.618 = g$

$\downarrow$

Golden Ratio -1

$g = 0.618$

Given $f(x) \longrightarrow$ unimodal function in search interval $[a, b]$

$\downarrow$

one local minima

$g = 0.618$

$d = g(b-a)$

$x_1 = a+d$, find $f(x_1)$ and $f(x_2)$

$x_2 = b-d$

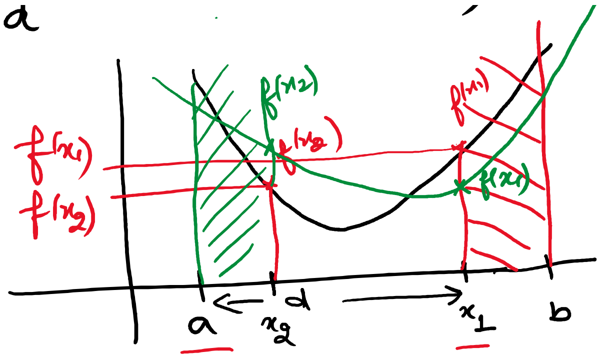
Yes

$f(x_1) > f(x_2)$?

$f(x_1) < f(x_2)$?

No

$x_2 = b - a$



$f(x_1) < f(x_2)$?

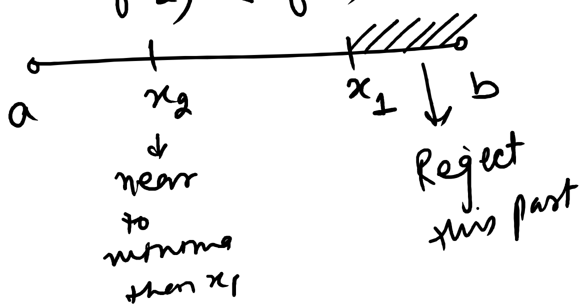
No

Yes

New interval of search becomes $[a, x_1]$

① $f(x_1) > f(x_2)$

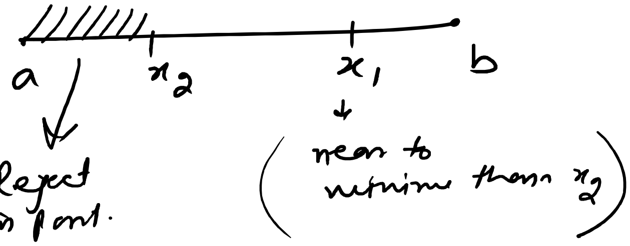
$f(x_2) < f(x_1)$ → $[a, x_1]$ becomes new interval of search.



② $f(x_1) < f(x_2)$

$[x_2, b]$ becomes new interval of search.

$f(x_2) > f(x_1)$



Example $f(x) = x^2 - 6x + 15$ min $f(x)$ using G.S. method $[0, 4]$.

Hand calculation → $[0, 10]$ (a, b)

$g = 0.618 \sim 0.62$

$d = g(b-a) = 0.62 \times 10 = 6.2$

$6.2 = 6.2$

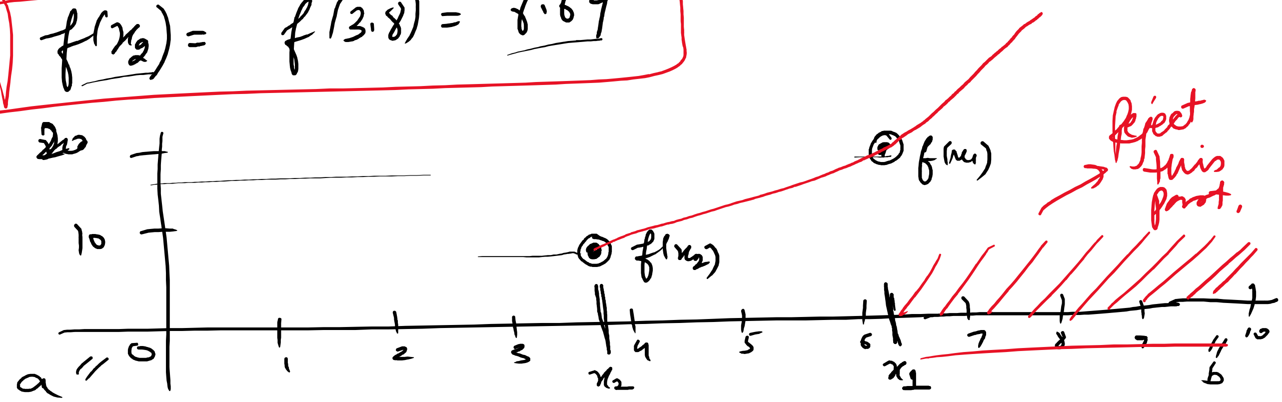
$d = g(b-a) = 0.62 \times 10 = 6.2$

$x_1 = a + d = 0 + 6.2 = 6.2$

$x_2 = b - d = 10 - 6.2 = 3.8$

$f(x_1) = f(6.2) = 16.24$ $\qquad f(x_1) > f(x_2)$

$\boxed{f(x_2) = f(3.8) = 6.64}$



② $[a, x_1] = [0, 6.2]$, with $x_1 = b$

$g = 0.62$

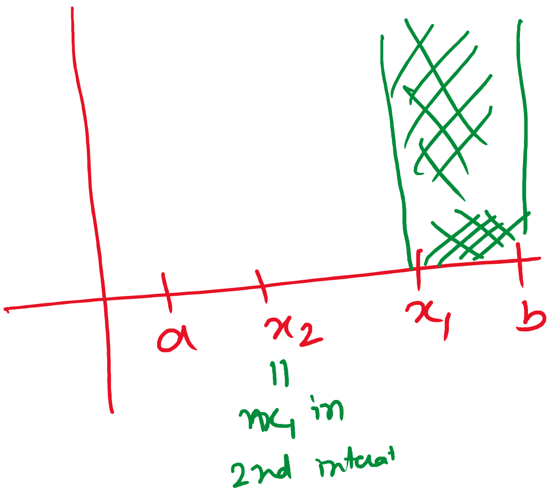
$d = 0.62 \times (b-a) = 0.62 \times 6.2 = 3.8$

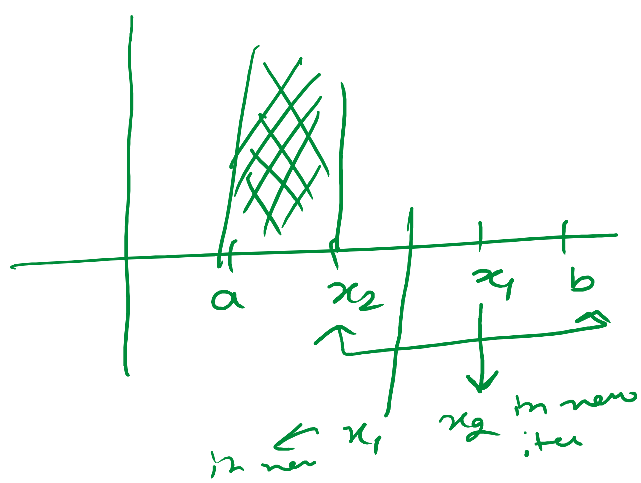
$x_1 = a + d = 0 + 3.8 = 3.8$

$x_2 = b - d = 6.2 - 3.8 = 2.4$

$f(x_1) = f(3.8) = 6.64$ $\qquad f(x_2) = f(2.4) =$

By algorithm of Golden section


a
x_2
x_1
b
||
x_1 in
2nd interval


a
x_2
x_1
b
x_1
in new
x_2 in new
iter