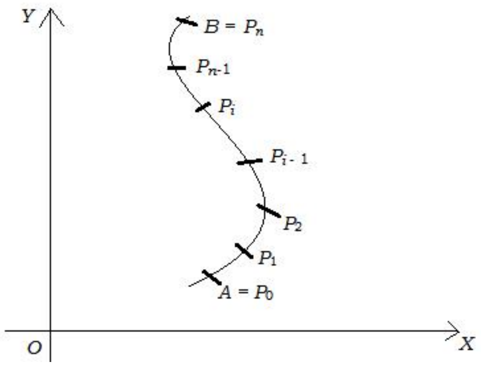
COMPLEX INTEGRATION

2.6 Complex Line Integrals



Let f (z) be a continuous function of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide the curve C into 'n' parts at the points $A = P_0(z_0), P_1(z_1), P_2(z_2), \ldots, P_n(z_n) = B$. Let $\Delta z_i = z_i - z_{i-1}$ and ξ_i be any point arc from p_{i-1} to p_i.

Then $\underset{n \to \infty}{Lt} \left[\sum_{i=1}^{n} f(\xi_i) \Delta z_i \right]$ and each $\Delta z_i \to 0$, if it exists, is called the line integral of f (z) along the curve C. It is denoted by $\int_C f(z)dz$. In case the points $P_0(z_0)$ and $P_n(z_n)$ coincide so that C is a closed curve, then this integral is called contour integral and is denoted by $\oint_C f(z)dz$.

If $f(z) = u(x, y) + iv(x, y)$ and $dz = dx + i\,dy$ then

$$\int_C f(z)dz = \int_C (u + iv)(dx + i\,dy)$$

$$\Rightarrow \qquad \int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

This shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Note:

1. If C denotes the curve traversed from B to A then

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$$

2. If C is a point on the arc joining a and b then

$$\int_a^b f(z)\,dz = \int_a^c f(z)\,dz + \int_c^b f(z)\,dz$$

3. If C is split into a number of parts C_1, C_2, C_3, … then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \ldots$$

WORKED EXAMPLES

Example 2.2.1: Evaluate $\int_0^{1+i} \left(x - y + ix^2\right) dz$ **,**

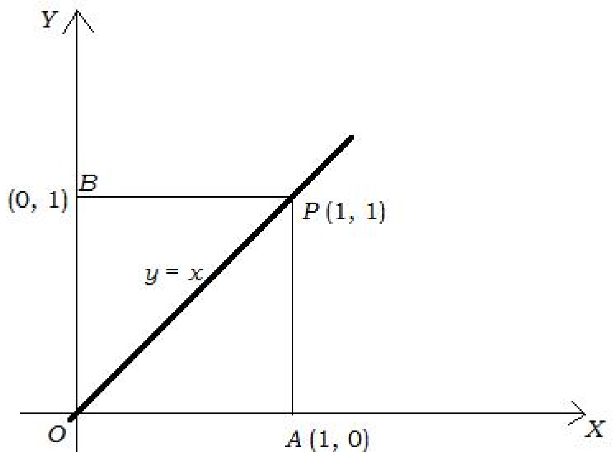
(i) **along the straight line from $z = 0$ to $z = 1 + i$,**

(ii) **along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 0$ to $z = 1 + i$,**

(iii) **along the imaginary axis from $z = 0$ to $z = i$ and then along a line parallel to real axis from $z = i$ to $z = 1 + i$.**

Solution:

Here z varies from 0 to $1 + i$ means that (x, y) varies from (0, 0) to (1, 1). The equation of the line joining these points is $y = x$.



(i) Along the straight line OP

Its equation is $y = x$.

$\Rightarrow$ $dy = dx$ and x varies from 0 to 1.

$$\therefore \int_0^{1+i}\left(x-y+ix^2\right)dz=\int_0^{1+i}\left(x-y+ix^2\right)(dx+i\,dy)$$

$$=\int_0^1\left(x-x+ix^2\right)(dx+i\,dx)$$

$$=\int_0^1\left(ix^2\right)(1+i)dx=i(1+i)\int_0^1 x^2dx$$

$$=\left(i+i^2\right)\left[\frac{x^3}{3}\right]_0^1=(i-1)\left(\frac{1}{3}-0\right)$$

$$=-\frac{1}{3}+\frac{1}{3}i$$

(ii) Along the path OAP

$$\int_0^{1+i}\left(x-y+ix^2\right)dz=\int_{OA}\ +\int_{AP}$$

Along OA

Its equation is $y = 0$.

$\Rightarrow$ $dy = 0$ and x varies from 0 to 1.

$$dz=dx+i\,dy\Rightarrow dz=dx$$

$$\therefore \int_{OA}\left(x-y+ix^2\right)dz=\int_0^1\left(x+ix^2\right)dx$$

$$=\left[\frac{x^2}{2}+i\frac{x^3}{3}\right]_0^1=\frac{1}{2}+\frac{1}{3}i$$

Along AP

Its equation is $x = 1$.

$\Rightarrow$ $dx = 0$ and y varies from 0 to 1.

$$dz=dx+i\,dy\Rightarrow dz=i\,dy$$

$$\therefore \quad \int_{AP}\left(x-y+i\,x^2\right)dz=\int_0^1(1-y+i)(idy)$$

$$=i\left[y-\frac{y^2}{2}+iy\right]_0^1=i\left(1-\frac{1}{2}+i\right)=-1+\frac{1}{2}i$$

$$\therefore \int_0^{1+i}\left(x-y+i\,x^2\right)dz=\frac{1}{2}+\frac{1}{3}i-1+\frac{1}{2}i=-\frac{1}{2}+\frac{5}{6}i$$

(iii) Along the path *OBP*

$$\int_0^{1+i}\left(x-y+i\,x^2\right)dz=\int_{OB}+\int_{BP}$$

Along *OB*

Its equation is $x = 0$.

$\Rightarrow$ $dx = 0$ and y varies from 0 to 1.

$$dz=dx+i\,dy\Rightarrow dz=idy$$

$$\therefore \quad \int_{OB}\left(x-y+i\,x^2\right)dz=\int_0^1(-y)(-idy)$$

$$=-i\int_0^1 y\,dy=-i\left[\frac{y^2}{2}\right]_0^1=-i\left(\frac{1}{2}\right)=-\frac{i}{2}$$

Along *BP*

Its equation is $y = 1$.

$\Rightarrow$ $dy = 0$ and x varies from 0 to 1.

$$dz=dx+i\,dy\Rightarrow dz=dx$$

$$\therefore \quad \int_{BP}\left(x-y+i\,x^2\right)dz=\int_0^1\left(x-1+i\,x^2\right)dx$$

$$=\left[\frac{x^2}{2}-x+\frac{i\,x^3}{3}\right]_0^1=\frac{1}{2}-1+\frac{i}{3}=-\frac{1}{2}+\frac{i}{3}$$

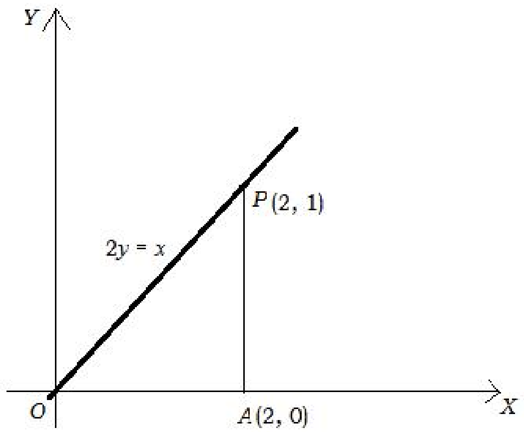
$$\therefore \int_0^{1+i}\left(x-y+i\,x^2\right)dz=-\frac{i}{2}-\frac{1}{2}+\frac{i}{3}=-\frac{1}{2}-\frac{i}{6}$$

Example 2.2.2: Evaluate $\int_0^{2+i} \left(\bar{z}\right)^2 dz$,

(i) along the real axis to 2 and then vertically to $2+i$

(ii) along the line 2y = x.

Solution: **(VTU 2001, 2009, 2014)**



Let $z = x + iy \Rightarrow \bar{z} = x - iy$

$$\left(\bar{z}\right)^2 = \left(x - iy\right)^2 = x^2 - y^2 - i\left(2xy\right)$$

(i) Along the path OAP

$$\int_0^{2+i} \left(\bar{z}\right)^2 dz = \int_{OA} + \int_{AP}$$

Along OA

Its equation is y = 0.

$\Rightarrow$ $dy = 0$ and x varies from 0 to 2.

$$dz = dx + i\,dy \Rightarrow dz = dx$$

$$\therefore \quad \int_{OA} \left(\bar{z}\right)^2 dz = \int_{OA} \left(x^2 - y^2 - 2ixy\right)^2 dz$$

$$= \int_0^2 x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

Along AP

Its equation is x = 2.

$\Rightarrow$ $dx = 0$ and y varies from 0 to 1.

$$dz = dx + i\,dy \Rightarrow dz = i\,dy$$

$$\therefore \quad \int_{AP} \left(\bar{z}\right)^2 dz = \int_{AP} \left(x^2 - y^2 - 2ixy\right)^2 dz$$

$$= \int_0^1 \left(4 - y^2 - 4iy\right)\left(i dy\right) = i\left[4y - \frac{y^3}{3} - 4i\left(\frac{y^2}{2}\right)\right]_0^1$$

$$= i\left[\left(4-\frac{1}{3}-2i\right)-(0)\right] = 2+\frac{11i}{3}$$

$$\therefore \int_0^{2+i} \left(\bar{z}\right)^2 dz = \frac{8}{3}+2+\frac{11i}{3} = \frac{14}{3}+\frac{11i}{3}$$

Example 2.2.3: Evaluate $\int_{1-i}^{2+i} (2x+iy+1)\,dz$, along the paths

(i) $x = t+1$ and $y = 2t^2-1$ (ii) straight line joining $(1-i)$ and $(2+i)$.

Solution:

(i) Given, $x = t+1$ and $y = 2t^2-1$

$\Rightarrow$ $dx = dt$ and $dy = 4t\,dt$

$$dz = dx + i\,dy \Rightarrow dz = dt + i4tdt = (1+4it)\,dt$$

Here x varies from 1 to 2,

When $x = 1 \Rightarrow t = 0$

When $x = 2 \Rightarrow t = 1$

$$\therefore \int_{1-i}^{2+i} (2x+iy+1)\,dz = \int_{t=0}^{1} \left[2(t+1)+i\left(2t^2-1\right)+1\right](1+i4t)\,dt$$

$$= \int_{t=0}^{1} \left[2t+2it^2+3-i\right](1+i4t)\,dt$$

$$= \int_{t=0}^{1} \left[-8t^3+10it^2+12it+6t+3-i\right]dt$$

$$= \left[-\frac{8t^4}{4}+\frac{10it^3}{3}+\frac{12it^2}{2}+\frac{6t^2}{2}+(3-i)t\right]_0^1$$

$$= -2+\frac{10i}{3}+\frac{12i}{2}+3+3-i = 4+\frac{25i}{3}$$

(ii) The equation of the straight line joining the points (1, -1) and (2, 1) is

$$\frac{y+1}{1+1}=\frac{x-1}{2-1}$$

$\Rightarrow$ $$y=2x-3$$

$\Rightarrow$ $$dy=2dx$$

$$dz=dx+i\,dy\Rightarrow dz=dx+i\left(2\,dx\right)=\left(1+2i\right)dx$$

and x varies from 1 to 2

$$\therefore \int_{1-i}^{2+i}\left(2x+i\,y+1\right)dz=\int_{x=1}^{2}\left[2x+i\left(2x-3\right)+1\right]\left(dx+2idx\right)$$

$$=\int_{1}^{2}\left[\left(2x+1\right)+i\left(2x-3\right)\right]\left(1+2i\right)dx$$

$$=\left(1+2i\right)\int_{1}^{2}\left[\left(2x+1\right)+i\left(2x-3\right)\right]dx$$

$$=\left(1+2i\right)\left[x^2+x+i\left(x^2-3x\right)\right]_1^2$$

$$=\left(1+2i\right)\left\{\left[4+2+i\left(4-6\right)\right]-\left[1+1+i\left(1-3\right)\right]\right\}$$

$$=\left(1+2i\right)\left\{6-2i-2+2i\right\}=4\left(1+2i\right)$$

Example 2.2.4: Evaluate $\int_{1-i}^{2+3i}\left(z^2+z\right)dz$, along the line joining the points (1, −1) and (2, 3).

Solution: **(VTU 2004, 2013)**

Let $$z=x+iy$$

$$z^2=\left(x+iy\right)^2=x^2-y^2+i\left(2xy\right)$$

$$z^2+z=\left(x^2-y^2+x\right)+i\,y\left(2x+1\right)$$

The equation of the straight line joining the points (1, −1) and (2, 3) is

$$\frac{y+1}{3+1}=\frac{x-1}{2-1}$$

$\Rightarrow$ $$y=4x-5$$

$\Rightarrow$ $$dy=4dx$$

$$dz=dx+i\,dy\Rightarrow dz=dx+i\left(4\,dx\right)=\left(1+4i\right)dx$$

and x varies from 1 to 2

$$\therefore \int_{1-i}^{2+3i} \left(z^2+z\right)dz = \int_{x=1}^{2} \left[\left(x^2-y^2+x\right)+iy\left(2x+1\right)\right]dz$$

$$= \int_{1}^{2} \left[x^2-\left(4x-5\right)^2+x+i\left(4x-5\right)\left(2x+1\right)\right]\left(1+4i\right)dx$$

$$= \left(1+4i\right)\int_{1}^{2} \left[\left(x^2-16x^2-25+40x+x\right)+i\left(8x^2-6x-5\right)\right]dx$$

$$= \left(1+4i\right)\int_{1}^{2} \left[\left(-15x^2+41x-25\right)+i\left(8x^2-6x-5\right)\right]dx$$

$$= \left(1+4i\right)\left[\left(\frac{-15x^3}{3}+\frac{41x^2}{2}-25x\right)i\left(\frac{8x^3}{3}-\frac{6x^2}{2}-5x\right)\right]_1^2$$

$$= \left(1+4i\right)\left\{\left[\left(-40+82-50\right)+i\left(\frac{64}{3}-12-10\right)\right] - \left[\left(-5+\frac{41}{2}-25\right)+i\left(\frac{8}{3}-3-5\right)\right]\right\}$$

$$= \left(1+4i\right)\left\{\left(-8\right)+i\left(-\frac{2}{3}\right)+\frac{19}{2}+i\left(\frac{16}{3}\right)\right\}$$

$$= \left(1+4i\right)\left(\frac{3}{2}+\frac{14i}{3}\right) = \frac{64i-103}{6}$$

Example 2.2.5: Prove that (i) $\oint_C \frac{dz}{z-a} = 2\pi i$**, and (ii)** $\oint_C (z-a)^n\, dz = 0,\ (n \neq -1),$

where C is the circle $|z-a| = r$**.**

Solution:

The equation of c is $z-a=re^{i\theta}$

$\Rightarrow$ $$dz = i\,r\,e^{i\theta}\,d\theta$$

Here θ varies from 0 to 2π

(i) $$\oint_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}}\left(i\,r\,e^{i\theta}\,d\theta\right)$$

$$= \int_0^{2\pi} 1\,d\theta = i[\theta]_0^{2\pi} = 2\pi i$$

$$\therefore \oint_C \frac{dz}{z-a} = 2\pi i$$

(ii)
$$\oint_C (z-a)^n\,dz = \int_0^{2\pi} r^n e^{in\theta}\left(i r e^{i\theta}\,d\theta\right)$$

$$= i\,r^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta = i\,r^{n+1}\left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i(n+1)(2\pi)} - e^0\right]$$

$$= \frac{r^{n+1}}{n+1}[1-1] = 0 \quad \left[\because e^{i(n+1)(2\pi)} = 1\right]$$

$$\therefore \quad \oint_C (z-a)^n\,dz = 0,\ (n \neq -1)$$

EXERCISE 2.2

1. Evaluate $\int_0^{3+i} z^2\,dz$, along
   (i) the line $3y = x$,
   (ii) the real axis to 3 and then vertically to $3 + i$
   (iii) the parabola $x = 3y^2$
2. Evaluate $\oint_C \left(y - x - 3x^2 i\right)dz$, where C is the straight line $z = 0$ to $z = 1 + i$
3. Evaluate $\oint_C \left(z - z^2\right)dz$, where C is the boundary of the square whose vertices are at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$
4. Evaluate $\oint_C (z+1)\,dz$, where C is the boundary of the square whose vertices are at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$
5. Evaluate $\int_C |z|\,dz$, where C is the contour
   (i) the straight line from $z = -i$ to $z = i$
   (ii) left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$
   (iii) the circle given by $|z + 1| = 1$ described in the clockwise sense.

6. Evaluate $\oint_C \left(z - z^2\right) dz$, where C is the upper half of the circle $|z| = 1$

ANSWERS

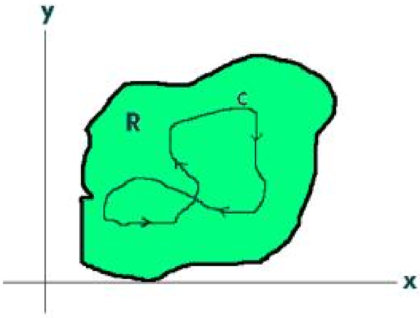
1.	(i) $6+\frac{26}{3}i$, (ii) $6+\frac{26}{3}i$, (iii) $6+\frac{26}{3}i$	2.	$1-i$
3.	0	4.	0
5.	(i) i, (ii) $2i$, (iii) 0	6.	2/3

2.7 Cauchy's Integral Theorem

Statement: If $f(z)$ is analytic at all points inside and on a simple closed curve C then

$$\oint_C f(z)\, dz = 0$$

Proof:



Let R be the region bounded by the curve C.

Let $f(z) = u(x,y) + i\,v(x,y)$ and $dz = dx + i\,dy$

Then,

$$\oint_C f(z)\, dz = \oint_C (u + i\,v)(dx + i\,dy)$$

$$\Rightarrow \quad \oint_C f(z)\, dz = \oint_C (udx + iudy + ivdx - vdy)$$

$$\Rightarrow \quad \oint_C f(z)\, dz = \oint_C (udx - vdy) + i\oint_C (vdx + udy) \quad \text{----- (1)}$$

By Green's Theorem, we have

$$\int_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy \quad \text{----- (2)}$$

Applying this theorem to the RHS of (1), we get

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \qquad \text{----- (3)}$$

Since $f(z)$ is analytic, by C-R equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

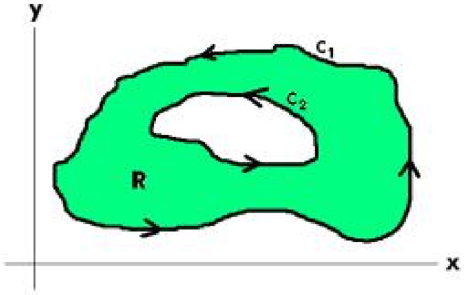
Using this in (3), we get

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$\Rightarrow$ $$\oint_C f(z)\,dz = 0 + i(0)$$

$\Rightarrow$ $$\oint_C f(z)\,dz = 0$$

Corollary -1: If $f(z)$ is analytic in a region R and $z = a$ and $z = b$ are two points in R then $\int_a^b f(z)\,dz$ is always independent of the path joining the points a and b.



Corollary -2: If $f(z)$ is analytic in the region R bounded by two simple closed curves C_1 and C_2 then $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$.

2.8 Cauchy's Integral Formula

Statement: If $f(z)$ is analytic within and on a closed curve C and $z = a$ is an interior point of C, then

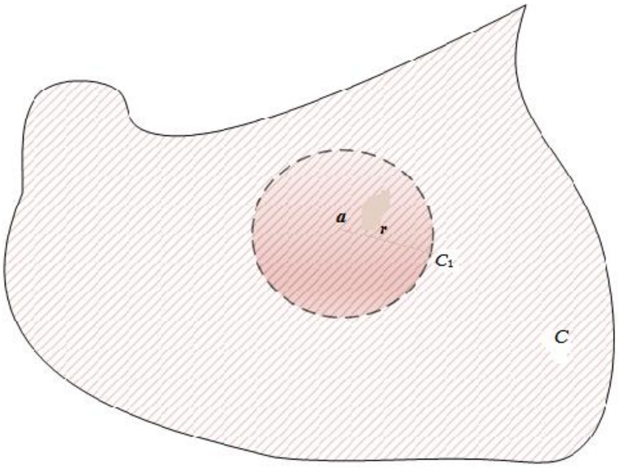
$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

Proof:

Let $$\phi(z)=\frac{f(z)}{z-a}$$

Since $f(z)$ and $z-a$ are analytic within and on the closed curve C.

$\therefore$ $\phi(z)$ is analytic except at $z = a$.



Draw a circle C_1 with centre at 'a' and radius equal to 'r', so that equation of C_1 is

$$|z-a| = re^{i\theta}$$

Any point on C_1 is $z = a + re^{i\theta}$

$\Rightarrow$ $$dz = i\,r\,e^{i\theta}\,d\theta$$

Now $\phi(z)$ is analytic between C and C_1

$\therefore$ By Corollary -2 of Cauchy's theorem, we have

$$\oint_C \phi(z)\,dz = \oint_{C_1} \phi(z)\,dz$$

$\Rightarrow$ $$\oint_C \frac{f(z)}{z-a}dz = \oint_{C_1} \frac{f(z)}{z-a}dz$$

$\Rightarrow$ $$\oint_C \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f\left(a+re^{i\theta}\right)}{re^{i\theta}}\left(i\,r\,e^{i\theta}\,d\theta\right)$$

$$\Rightarrow \qquad \oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f\left(a+re^{i\theta}\right)d\theta$$

Taking limit as $r \to 0$, we get

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta$$

$$\Rightarrow \qquad \oint_C \frac{f(z)}{z-a}\,dz = i\,f(a)\int_0^{2\pi} 1\,d\theta$$

$$\Rightarrow \qquad \oint_C \frac{f(z)}{z-a}\,dz = i\,f(a)\left[\theta\right]_0^{2\pi}$$

$$\Rightarrow \qquad \oint_C \frac{f(z)}{z-a}\,dz = i\,f(a)\left[2\pi - 0\right] = 2\pi i\,f(a)$$

$$\therefore \qquad \oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\,f(a)$$

Corollary: By Cauchy's Integral formula, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\,f(a)$$

$$\Rightarrow \qquad f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Differentiating this successively w.r.t. 'a', we get

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$$

$$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz$$

$$f'''(a) = \frac{3!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^4}\,dz$$

and so on.

In general,

$$f^{n}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Working Procedure to evaluate $\oint_C \frac{f(z)}{z-a}dz$ and $\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$

by using Cauchy's Integral theorem and Cauchy's Integral formula

- First check whether the point $z = a$ lies inside or outside the given curve C.
- If $z = a$ lies outside C then by Cauchy's Integral theorem

$$\oint_C \frac{f(z)}{z-a}dz = 0 \text{ and } \oint_C \frac{f(z)}{(z-a)^{n+1}}dz = 0$$

- If $z = a$ lies inside C then by Cauchy's Integral formula

$$\oint_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a) \text{ and } \oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^n(a)$$

WORKED EXAMPLES

Example 2.3.1: Evaluate $\oint_C \frac{e^{-z}}{z+1}dz$, where C is the circle $|z| = 2$.

Solution:

Given C: $|z| = 2$, which is a circle with centre (0, 0) and radius 2.

Let $f(z) = e^{-z}$

$$\oint_C \frac{e^{-z}}{z+1}dz = \oint_C \frac{f(z)}{z-(-1)}dz$$

Here the point z = –1 = (–1, 0) lies inside C.

$\therefore$ By Cauchy's Integral formula, we have $\oint_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$

$$\therefore \quad \oint_C \frac{f(z)}{z-(-1)}dz = 2\pi i\, f(-1)$$

$$\Rightarrow \quad \oint_C \frac{f(z)}{z-(-1)}dz = 2\pi i(e) \qquad \left[\because f(z) = e^{-z}\right]$$

$$\therefore \oint_C \frac{e^{-z}}{z+1}dz = 2\pi e i$$

Example 2.3.2: Evaluate $\oint_C \frac{e^{Jz}}{z\,H\,1}dz,$ **where C is the circle** $|z|\,Z\,\frac{1}{2}.$

Solution:

Given $C: |z| = 2,$ which is a circle with centre (0, 0) and radius $\frac{1}{2}$

Let $f(z) = e^{-z}$

$$\oint_C \frac{e^{-z}}{z+1}dz = \oint_C \frac{f(z)}{z-(-1)}dz$$

Here the point z = –1 = (–1, 0) lies outside *C*.

By Cauchy's Integral theorem, we have $\oint_C \frac{f(z)}{z-a}dz = 0$

$\therefore$ $$\oint_C \frac{f(z)}{z-(-1)}dz = 0$$

$$\therefore \oint_C \frac{e^{-z}}{z+1}dz = 0$$

Example 2.3.3: Evaluate $\oint_C \frac{\cos ® z^2}{E z\,J\,1 F E z\,J\,2 F}dz,$ **where C :** $|z|\,Z\,3.$

Solution:

Given *C*: $|z| = 3,$ which is a circle with centre (0, 0) and radius 3.

Let $f(z) = \cos ® z^2$

$$\oint_C \frac{\cos ® z^2}{(z-1)(z-2)}dz = \oint_C \frac{f(z)}{(z-1)(z-2)}dz$$

Consider, $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

We have, $1 = A(z-2) + B(z-1)$

$z = 1 \Rightarrow A = -1$ and $z = 2 \Rightarrow B = 1$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$\Rightarrow \qquad \oint_C \frac{f(z)}{(z-1)(z-2)}dz = \oint_C \left(\frac{-1}{z-1} + \frac{1}{z-2} \right) f(z)\,dz$

$\Rightarrow \qquad \oint_C \frac{f(z)}{(z-1)(z-2)}dz = -\oint_C \frac{f(z)}{z-1}dz + \oint_C \frac{f(z)}{z-2}dz$

Here the point z = 1 = (1, 0) lies inside C and the point z = 2 = (2, 0) also lies inside C.

By Cauchy's Integral formula, we have $\oint_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$

$\therefore \qquad \oint_C \frac{f(z)}{z-1}dz = 2\pi i\, f(1) \quad \text{and} \quad \oint_C \frac{f(z)}{z-2}dz = 2\pi i\, f(2)$

$\because\ f(z) = \cos \pi z^2$

$\Rightarrow \qquad f(1) = \cos \pi = -1 \quad \text{and} \quad f(2) = \cos 4\pi = 1$

$\therefore \qquad \oint_C \frac{f(z)}{z-1}dz = 2\pi i(-1) = -2\pi i \quad \text{and} \quad \oint_C \frac{f(z)}{z-2}dz = 2\pi i(1) = 2\pi i$

$\therefore\ \oint_C \frac{f(z)}{(z-1)(z-2)}dz = -(-2\pi i\ ) + 2\pi i = 4\pi i$

$\therefore \qquad \oint_C \frac{\cos \pi z^2}{(z-1)(z-2)}dz = 4\pi i$

Example 2.3.4: Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$**, where C :** $|z| = 3$**.**

Solution:

Given *C*: $|z| = 3$, which is a circle with centre (0, 0) and radius 3.

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz = \oint_C \frac{f(z)}{(z-1)(z-2)}dz$$

Consider, $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

We have, $1 = A(z-2) + B(z-1)$

$$z = 1 \Rightarrow A = -1 \text{ and } z = 2 \Rightarrow B = 1$$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\Rightarrow \oint_C \frac{f(z)}{(z-1)(z-2)} dz = \oint_C \left(\frac{-1}{z-1} + \frac{1}{z-2}\right) f(z) dz$$

$$\Rightarrow \oint_C \frac{f(z)}{(z-1)(z-2)} dz = -\oint_C \frac{f(z)}{z-1} dz + \oint_C \frac{f(z)}{z-2} dz$$

Here the point z = 1 = (1, 0) lies inside C and the point z = 2 = (2, 0) also lies inside C.

By Cauchy's Integral formula, we have $\oint_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$

$$\therefore \oint_C \frac{f(z)}{z-1} dz = 2\pi i\, f(1) \quad \text{and} \quad \oint_C \frac{f(z)}{z-2} dz = 2\pi i\, f(2)$$

$$\because f(z) = \sin \pi z^2 + \cos \pi z^2$$

$$\Rightarrow f(1) = \sin \pi + \cos \pi = 0 + (-1) = -1 \quad \text{and} \quad f(2) = \sin 4\pi + \cos 4\pi = 0 + 1 = 1$$

$$\therefore \oint_C \frac{f(z)}{z-1} dz = 2\pi i(-1) = -2\pi i \quad \text{and} \quad \oint_C \frac{f(z)}{z-2} dz = 2\pi i(1) = 2\pi i$$

$$\therefore \oint_C \frac{f(z)}{(z-1)(z-2)} dz = -(-2\pi i) + 2\pi i = 4\pi i$$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 4\pi i$$

Example 2.3.5: Evaluate $\oint_C \frac{z^2+5}{(z-2)(z-3)} dz$**, where** $C: |z| = \frac{5}{2}$**.**

Solution:

Given C: $|z| = \frac{5}{2}$, which is a circle with centre (0, 0) and radius $\frac{5}{2}$.

Let $f(z) = z^2 + 5$

$$\oint_C \frac{z^2+5}{(z-2)(z-3)} dz = \oint_C \frac{f(z)}{(z-2)(z-3)} dz$$

Consider, $\dfrac{1}{(z-2)(z-3)}=\dfrac{A}{z-2}+\dfrac{B}{z-3}$

We have, $1=A(z-3)+B(z-2)$

$z=2\Rightarrow A=-1$ and $z=3\Rightarrow B=1$

$$\therefore \frac{1}{(z-2)(z-3)}=\frac{-1}{z-2}+\frac{1}{z-3}$$

$$\Rightarrow \oint_C \frac{f(z)}{(z-2)(z-3)}dz=\oint_C\left(\frac{-1}{z-2}+\frac{1}{z-3}\right)f(z)dz$$

$$\Rightarrow \oint_C \frac{f(z)}{(z-2)(z-3)}dz=-\oint_C \frac{f(z)}{z-2}dz+\oint_C \frac{f(z)}{z-3}dz$$

Here the point $z = 2 = (2, 0)$ lies inside *C*. and the point $z = 3 = (3, 0)$ lies outside *C*.

By Cauchy's Integral formula, we have $\oint_C \dfrac{f(z)}{z-a}dz=2\pi i\, f(a)$

$$\therefore \oint_C \frac{f(z)}{z-2}dz=2\pi i\, f(2)=2\pi i(9)=18\pi i \qquad \left[\because f(z)=z^2+5\right]$$

And the point $z = 3 = (3, 0)$ lies outside *C*.

By Cauchy's Integral theorem, we have $\oint_C \dfrac{f(z)}{z-a}dz=0$

$$\therefore \oint_C \frac{f(z)}{z-3}dz=0$$

$$\Rightarrow \oint_C \frac{f(z)}{(z-2)(z-3)}dz=-18\pi i+0=-18\pi i$$

$$\therefore \oint_C \frac{z^2+5}{(z-2)(z-3)}dz=-18\pi i$$

Example 2.3.6: Evaluate $\oint_C \dfrac{e^{2z}}{(z+1)^4}dz$, where $C: |z|=2$.

Solution:

Given *C*: $|z|=2$, which is a circle with centre (0, 0) and radius 2.

Let $f(z)=e^{2z}$

$$\oint_C \frac{e^{2z}}{(z+1)^4}dz=\oint_C \frac{f(z)}{(z+1)^4}dz$$

Here the point z = –1 = (–1, 0) lies inside C.

$\therefore$ By Cauchy's Integral formula, we have $\oint_C \frac{f(z)}{(z-a)^{n+1}}dz=\frac{2®i}{n!}f^{n}(a)$

$$\therefore \qquad \oint_C \frac{f(z)}{(z+1)^4}dz=\frac{2®i}{3!}f'''(-1)$$

$\because\ f(z)=e^{2z}$

$f'''(z)=8e^{2z}$

$\Rightarrow\ f'''(-1)=8e^{-2}$

$$\therefore \qquad \oint_C \frac{f(z)}{(z+1)^4}dz=\frac{2®i}{3!}\left(8e^{-2}\right)=\frac{8®i}{3e^2}$$

$$\therefore \quad \oint_C \frac{e^{2z}}{(z+1)^4}dz=\frac{8®i}{3e^2}$$

Example 2.3.7: Evaluate $\oint_C \frac{z\ \mathrm{J}\ 1}{\mathrm{E}z\ \mathrm{H}\ 1\mathrm{F}^2\ \mathrm{E}z\ \mathrm{J}\ 2\mathrm{F}}dz$**, where C :** $|z\ \mathrm{J}\ i|\ \mathrm{Z}\ 2.$

Solution:

Given C: $|z-i|=2,$ which is a circle with centre $z=i=(0,1)$ and radius 2.

Let $f(z)=z-1$

$$\oint_C \frac{z-1}{(z+1)^2(z-2)}dz=\oint_C \frac{f(z)}{(z+1)^2(z-2)}dz$$

Consider, $\frac{1}{(z+1)^2(z-2)}=\frac{A}{z+1}+\frac{B}{(z+1)^2}+\frac{C}{z-2}$

We have, $1=A(z+1)(z-2)+B(z-2)+c(z+1)^2$

$z=-1\Rightarrow B=-\frac{1}{3},\ z=2\Rightarrow C=\frac{1}{9}$ and

$$z=0 \Rightarrow -2A-2B+C=1 \Rightarrow 2A=-2\left(-\frac{1}{3}\right)+\frac{1}{9}-1$$

$$\Rightarrow A=-\frac{1}{9}$$

$$\therefore \frac{1}{(z+1)^2(z-2)}=\frac{-1/9}{z+1}+\frac{-1/3}{(z+1)^2}+\frac{1/9}{z-2}$$

$$\Rightarrow \qquad \oint_C \frac{z-1}{(z+1)^2(z-2)}dz=\oint_C\left(\frac{-1/9}{z+1}+\frac{-1/3}{(z+1)^2}+\frac{1/9}{z-2}\right)f(z)dz$$

$$\Rightarrow \qquad \oint_C \frac{z-1}{(z+1)^2(z-2)}dz=-\frac{1}{9}\oint_C\frac{f(z)}{z+1}dz-\frac{1}{3}\oint_C\frac{f(z)}{(z+1)^2}dz+\frac{1}{9}\oint_C\frac{f(z)}{z-2}dz \qquad \text{---- (1)}$$

Here the point z = –1 = (–1, 0) lies inside *C*

By Cauchy's Integral formula, we have

$$\oint_C \frac{f(z)}{z-a}dz=2\pi i\, f(a) \text{ and } \oint_C \frac{f(z)}{(z-a)^{n+1}}dz=\frac{2\pi i}{n!}f^n(a)$$

$$\therefore \qquad \oint_C \frac{f(z)}{z+1}dz=2\pi i\, f(-1) \text{ and } \oint_C \frac{f(z)}{(z+1)^2}dz=\frac{2\pi i}{1!}f'(-1)$$

$$\Rightarrow \qquad \oint_C \frac{f(z)}{z+1}dz=2\pi i(-2) \text{ and } \oint_C \frac{f(z)}{(z+1)^2}dz=\frac{2\pi i}{1!}(1) \qquad [\because f(z)=z-1]$$

$$\Rightarrow \qquad \oint_C \frac{f(z)}{z+1}dz=-4\pi i \text{ and } \oint_C \frac{f(z)}{(z+1)^2}dz=2\pi i$$

And the point z = 2 = (2, 0) lies outside *C*.

By Cauchy's Integral theorem, we have $\oint_C \frac{f(z)}{z-a}dz=0$

$$\therefore \oint_C \frac{f(z)}{z-2}dz=0$$

Equation (1) becomes,

$$\oint_C \frac{z-1}{(z+1)^2(z-2)}dz = -\frac{1}{9}(-4\pi i)-\frac{1}{3}(2\pi i)+\frac{1}{9}(0)$$

$$= \frac{4\pi i}{9}-\frac{2\pi i}{3} = -\frac{2\pi i}{9}$$

$$\therefore \quad \oint_C \frac{z-1}{(z+1)^2(z-2)}dz = -\frac{2\pi i}{9}$$

Example 2.3.8: Evaluate $\oint_C \frac{z}{(z^2+1)(z^2-9)}dz$, **where** $C: |z-2| = 2$.

Solution:

Given C: $|z-2|=2$, which is a circle with centre $z=2=(2, 0)$ and radius 2.

Here $(z^2+1)(z^2-9)=0 \Rightarrow z=\pm i, \pm 3$

The points $z=\pm i$ and -3 lies outside the circle C and the point $z=3$ lies inside the circle C.

Let $f(z)=\frac{z}{(z^2+1)(z+3)}$

$$\oint_C \frac{z}{(z^2+1)(z+3)(z-3)}dz = \oint_C \frac{f(z)}{z-3}dz$$

By Cauchy's Integral formula, we have

$$\oint_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$$

$$\therefore \quad \oint_C \frac{f(z)}{z-3}dz = 2\pi i\, f(3)$$

$$\because \quad f(z)=\frac{z}{(z^2+1)(z+3)}$$

$$\Rightarrow f(3)=\frac{3}{(3^2+1)(3+3)}=\frac{1}{20}$$

$$\therefore \quad \oint_C \frac{f(z)}{z-3}dz = 2\pi i\left(\frac{1}{20}\right)$$

$$\Rightarrow \quad \oint_C \frac{f(z)}{z-3}dz = \frac{\pi i}{10}$$

$$\therefore \quad \oint_C \frac{z}{(z^2+1)(z^2-9)}dz = \frac{\pi i}{10}$$

Example 2.3.9: Evaluate $\oint_C \frac{z}{(z^2-3z+2)}dz$**, where** $C: |z-2| = \frac{1}{2}$**.**

Solution:

Given C: $|z-2| = \frac{1}{2}$, which is a circle with centre $z=2=(2,0)$ and radius $\frac{1}{2}$.

Here $z^2-3z+2=(z-1)(z-2) \Rightarrow z=1,\ 2$

The points $z=1$ lies outside the circle C and the point $z=2$ lies inside the circle C.

Let $f(z)=\frac{z}{(z-1)}$

$$\oint_C \frac{z}{(z-1)(z-2)}dz = \oint_C \frac{f(z)}{z-2}dz$$

By Cauchy's Integral formula, we have

$$\oint_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$$

$$\therefore \quad \oint_C \frac{f(z)}{z-2}dz = 2\pi i\, f(2)$$

$\because \quad f(z)=\frac{z}{(z-1)}$

$\Rightarrow f(2)=\frac{2}{(2-1)}=2$

$$\therefore \quad \oint_C \frac{f(z)}{z-2}dz = 2\pi i(2)$$

$$\Rightarrow \quad \oint_C \frac{f(z)}{z-2}dz = 4\pi i$$

$$\therefore \quad \oint_C \frac{z}{(z^2-3z+2)}dz = 4\pi i$$

Example 2.3.10: Evaluate $\oint_C \frac{z^2-4}{z(z^2+9)}dz$**, where** $C: |z| = 1$**.**

Solution:

Given C: $|z|=1$, which is a circle with centre $z=0=(0,0)$ and radius 1.

Here $z(z^2+9)=0 \Rightarrow z=0, \pm 3i$

The points $z = \pm 3i$ lies outside the circle C and the point $z = 0$ lies inside the circle C.

$$\text{Let } f(z) = \frac{z^2 - 4}{\left(z^2 + 9\right)}$$

$$\oint_C \frac{z^2 - 4}{z\left(z^2 + 9\right)} dz = \oint_C \frac{f(z)}{z} dz$$

By Cauchy's Integral formula, we have

$$\oint_C \frac{f(z)}{z - a} dz = 2\text{®}i\, f(a)$$

$$\therefore \qquad \oint_C \frac{f(z)}{z - 0} dz = 2\text{®}i\, f(0)$$

$$\because \quad f(z) = \frac{z^2 - 4}{\left(z^2 + 9\right)}$$

$$\Rightarrow f(0) = \frac{z^2 - 4}{\left(z^2 + 9\right)} = -\frac{4}{9}$$

$$\therefore \qquad \oint_C \frac{f(z)}{z - 0} dz = 2\text{®}i\left(-\frac{4}{9}\right)$$

$$\Rightarrow \qquad \oint_C \frac{f(z)}{z} dz = -\frac{8\text{®}i}{9}$$

$$\therefore \quad \oint_C \frac{z^2 - 4}{z\left(z^2 + 9\right)} dz = -\frac{8\text{®}i}{9}$$

EXERCISE 2.3

1. State and prove the Cauchy's theorem.
(VTU 2006, 2008, 2012, 2017)
2. State and prove the Cauchy's integral formula.
(VTU 2005, 2007, 2012, 2015, 2017, 2019)
3. Verify Cauchy's theorem for the function $f(z) = ze^{-z}$ over the unit circle with origins as the centre. **(VTU 2010, 2012, 2014)**

4. Evaluate $\oint_C \frac{e^z}{(z+1)(z-2)}\,dz$, where $C: |z| = 3$ **(VTU 2013, 2017)**

5. Evaluate $\oint_C \frac{z^2-z+1}{z-1}\,dz$, where C is (i) $|z| = 1$ and (ii) $|z| = 1/2$

6. Evaluate $\oint_C \frac{e^{-2z}}{(z+1)^3}\,dz$, where $C: |z| = 2$

7. Evaluate $\oint_C \frac{e^{\text{®} z}}{(2z-i)^3}\,dz$, where $C: |z| = 1$ **(VTU 2010, 2012, 2014)**

8. Evaluate $\oint_C \frac{\cos \text{®} z}{z^2-1}\,dz$, where C is a rectangle with vertices $2 \pm i, -2 \pm i.$

9. Evaluate $\oint_C \frac{e^{tz}}{z^2+1}\,dz$, where $C: |z| = 3$

10. Evaluate $\oint_C \frac{e^{2z}}{(z+i)^4}\,dz$, where $C: |z| = 3$

11. Evaluate $\oint_C \frac{\sin^2 z}{\left(z-\frac{\text{®}}{6}\right)^3}\,dz$, where $C: |z| = 1$ **(VTU 2012, 2014)**

12. Evaluate $\oint_C \frac{e^z}{(z^2+\text{®}^2)^2}\,dz$, where $C: |z| = 4$

13. Evaluate $\oint_C \frac{\sin \text{®} z^2 + \cos \text{®} z^2}{(z-1)^2(z-2)}\,dz$, where $C: |z| = 3$ **(VTU 2013, 2016)**

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