CALCULUS OF COMPLEX FUNCTIONS

Recapitulation:

A number of the form $z = x+iy$ where x, y are real numbers and $i = \sqrt{-1}$ or $i^2 = -1$ is called a complex number.

$x \rightarrow$ Real part $\qquad y \rightarrow$ Imaginary part.

Complex Conjugate : $\bar{z} = x - iy$

Euler's formula : $e^{i\theta} = \cos\theta - i\sin\theta$

Polar form of a complex number:

$z = r(\cos\theta + i\sin\theta)$ or $z = re^{i\theta}$ is called polar form of z.

$r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ are called modulus and amplitude (or argument) respectively.

De-Moivre's Theorem:

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is a real number.

Function of a complex variable, limit, continuity and differentiability.

Function of a complex variable.

If it is possible to find one or more complex numbers 'w' for every value 'z' in a certain domain D, we say that 'w' is a function of z defined for the domain D.

In other words $w = f(z)$ is called a function of the complex variable z.

Since $z = x+iy$ or $z = r.e^{i\theta}$ we always write:

$w = f(z) = u(x,y) + iv(x,y) \longrightarrow$ Cartesian form

$w = f(z) = u(r,\theta) + iv(r,\theta) \longrightarrow$ Polar form.

Ex:- Consider $f(z) = z^2$

Then $u + iv = (x+iy)^2 = x^2 + 2ixy + i^2y^2$

$u + iv = (x^2 - y^2) + i(2xy)$

$\Rightarrow u = x^2 - y^2 \qquad v = 2xy$ in the cartesian form.

Neighbourhood :

A neighbourhood of a point z_0 in the complex plane is the set of all points z such that $|z-z_0| < \delta$ where δ is a small positive real number.

Geometric meaning : $(x-x_0)^2 + (y-y_0)^2 = \delta^2$; circle with centre (x_0, y_0), radius δ.

ie. neighbourhood of a point z_0 $(|z-z_0| < \delta)$ is the set of all points inside a circle having z_0 as centre and δ as radius.

Limit :

A complex valued function $f(z)$ defined in the neighbourhood of a point z_0 is said to have l as z tends to z_0, if for every $\varepsilon > 0$ however small there exists a positive real number δ such that $|f(z) - l| < \varepsilon$ when $|z-z_0| < \delta$. This is written as $\lim_{z \to z_0} f(z) = l$.

Continuity:

A complex valued function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ exists and $\lim_{z \to z_0} f(z) = f(z_0)$. This is to say $|f(z) - f(z_0)| < \varepsilon$ when $|z-z_0| < \delta$.

Differentiability :

A complex valued function $f(z)$ is said to be differentiable at $z = z_0$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and is unique. This limit when exists is called the derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$\delta z = z - z_0$
$z \to z_0$
$\delta z \to 0$

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$$

Analytic Function :

A complex valued function $w = f(z)$ is said to be analytic at a point $z = z_0$ if $\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$ exists and is unique at z_0 and in the neighbourhood of z_0.

Analytic function is also called regular function or holomorphic function.

$f(z)$ is analytic at a point z_0 if it is differentiable at z_0 and in the neighbourhood of z_0.

Theorem 1: The necessary conditions that the function $w = f(z) = u(x,y) + iv(x,y)$ may be analytic at any point $z = x+iy$ is that, there exists continuous first order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and satisfy the equations:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These are known as Cauchy-Riemann equations.

Proof: let $f(z)$ be analytic function at a point $z = x+iy$

By definition. $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exists and is unique.

In cartesian form $f(z) = u(x,y) + iv(x,y)$

δz = increment in z corresponding to increment $\delta x, \delta y$ in x, y.

$$f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \left[\frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta x + i\delta y} \right]$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \left[\frac{[u(x+\delta x, y+\delta y) - u(x,y)]}{\delta x + i\delta y} + i\frac{[v(x+\delta x, y+\delta y) - v(x,y)]}{\delta x + i\delta y} \right] \quad \text{—(1)}$$

Since $\delta z \to 0$, we have two possibilities

Case (i) when δz is wholly real, then $\delta y = 0$ and $\delta z = \delta x$

$$f'(z) = \lim_{\delta x \to 0} \left[\frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i\frac{v(x+\delta x, y) - v(x,y)}{\delta x} \right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{—(2)}$$

Case (ii) when δz is wholly imaginary, then $\delta x = 0$ and $\delta z = \delta y$

$$f'(z) = \lim_{\delta y \to 0} \left[\frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \cdot \frac{v(x, y+\delta y) - v(x,y)}{i\delta y} \right]$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{—(3)} \qquad \left[1/i = i/i^2 = i/-1 = -i \right]$$

Equating the RHS of (2) and (3) : $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

Now equating the real and imaginary parts :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Show that polar form of Cauchy Riemann equations are :

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$. Also deduce : $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

Proof: If (r,θ) be the co-ordinates of a point whose cartesian co-ordinates are (x,y) then $z = x+iy = re^{i\theta}$.

$\therefore \quad u + iv = f(z) = f(re^{i\theta})$ where u, v are now expressed in terms of r and θ.

Differentiating partially wrt r and θ,

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}).\, e^{i\theta}$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}).\, ire^{i\theta} = ir.\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Equating real and imaginary parts,

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{and} \quad r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$$

Or. $$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad -(1) \quad \text{and} \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad -(2)$$

Differentiating (2) partially wrt r :

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta\, \partial r} \quad —— (3)$$

Differentiating (1) partially with θ :

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r\, \partial \theta} \quad —— (4)$$

Using equations (4) (2) in equation (3)

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\, r.\frac{\partial u}{\partial r} - \frac{1}{r}.\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Properties of Analytic functions.

Harmonic function.

A function ϕ is said to be harmonic if it satisfies Laplace's equation $\nabla^2\phi = 0$. In the cartesian form $\phi(x,y)$ is harmonic if $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$.

In the polar form $\phi(r,\theta)$ is harmonic if $\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$

Harmonic Property.

The real and imaginary parts of an analytic function are harmonic.

Proof: Let $f(z) = u(x,y) + i\,v(x,y)$ be analytic.

Since $f(z)$ is analytic, we have Cauchy Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad -(1) \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad -(2)$$

Differentiating (1) partially wrt x and (2) partially wrt y,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y\,\partial x} \quad -(3) \qquad \frac{\partial^2 v}{\partial y\,\partial x} = -\frac{\partial^2 u}{\partial y^2}. \quad -(4)$$

From (3) and (4): $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$ or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u$ is harmonic

Similarly, by differentiating (1) partially wrt y and (2) partially wrt x

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Thus both the functions u and v satisfy the Laplace's equation.

Orthogonal property.

If $f(z) = u+iv$ is analytic then the family of curves $u(x,y) = c_1$, $v(x,y) = c_2$, c_1 and c_2 being constants, intersect each other orthogonally.

Proof: Consider $u(x,y) = c_1$ -(1) and $v(x,y) = c_2$ -(2)

Differentiating (1): $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} = +\frac{\partial v/\partial y}{\partial v/\partial x} = m_1$ (CR equations) -(3)

Similarly (2) gives: $\frac{dy}{dx} = -\frac{\partial v/\partial x}{\partial v/\partial y} = m_2$

$\therefore m_1 \times m_2 = -1$ i.e. (1) and (2) form an orthogonal system.

Note:

1. $\cos(A+B) = \cos A \cos B - \sin A \sin B$
2. $\cos(A-B) = \cos A \cos B + \sin A \sin B$
3. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
4. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
5. $\cos(i\theta) = \cosh\theta$
6. $\sin(i\theta) = i\sinh\theta$
7. $f'(z) = u_x + iv_x$

   $f'(z) = e^{-i\theta}(u_r + iv_r)$

Problems:

1. Show that $w = z + e^z$ is analytic and hence find $\frac{dw}{dz}$.

Soln: $w = z + e^z$

$$u + iv = (x+iy) + e^{(x+iy)}$$
$$= x + iy + e^x \cdot e^{iy}$$
$$= x + iy + e^x(\cos y + i\sin y)$$
$$= (x + e^x \cos y) + i(y + e^x \sin y)$$

$\Rightarrow u = x + e^x \cos y \qquad v = y + e^x \sin y$

$\frac{\partial u}{\partial x} = 1 + e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = 1 + e^x \cos y$.

Cauchy Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied.

$\therefore$ $w = z + e^z$ is analytic.

$$\frac{dw}{dz} = f'(z) = u_x + iv_x = 1 + e^x \cos y + i(e^x \sin y)$$
$$= 1 + e^x(\cos y + i\sin y)$$
$$= 1 + e^x \cdot e^{iy}$$
$$= 1 + e^{x+iy} = 1 + e^z$$

$\therefore \frac{dw}{dz} = 1 + e^z$

2. Show that $f(z) = e^{x}(\cos y + i \sin y)$ is holomorphic.

Soln: $f(z) = e^{x}(\cos y + i \sin y)$

$u + iv = (e^{x}\cos y) + i(e^{x}\sin y)$

$\Rightarrow u = e^{x}\cos y \qquad v = e^{x}\sin y$

$\frac{\partial u}{\partial x} = e^{x}\cos y \qquad \frac{\partial v}{\partial x} = e^{x}\sin y$

$\frac{\partial u}{\partial y} = -e^{x}\sin y \qquad \frac{\partial v}{\partial y} = e^{x}\cos y$

Cauchy Riemann equations : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied.

Thus the given function $f(z)$ is holomorphic.

3. Show that $w = z\bar{z}$ is not analytic.

Soln: $w = z\bar{z}$. $z = x + iy \quad \bar{z} = x - iy$

$u + iv = (x+iy)(x-iy)$

$= x^2 - i^2y^2 = x^2 + y^2 \quad (i^2 = -1)$

$\Rightarrow u = x^2 + y^2 \qquad v = 0$

$u_x = 2x \qquad v_x = 0$

$u_y = 2y \qquad v_y = 0$

Cauchy Riemann equations : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are not satisfied.

$\therefore w = z\bar{z}$ is not analytic.

4. Show that $f(z) = \sin z$ is analytic and hence find $f'(z)$.

Soln: $f(z) = \sin z$

$u + iv = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy$

$= (\sin x \cosh y) + i(\cos x \sinh y) \quad [\cos iy = \cosh y \; ; \; \sin iy = i \sinh y]$

$\Rightarrow u = \sin x \cosh y \qquad v = \cos x \sinh y$

$u_x = \cos x \cosh y \qquad v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y \qquad v_y = \cos x \cosh y$

Cauchy Riemann equations : $u_x = v_y$ and $u_y = -v_x$ are satisfied.

$\therefore f(z) = \sin z$ is analytic.

Also, $f'(z) = u_x + iv_x$

$= \cos x \cosh y + i(-\sin x \sinh y)$

$= \cos x \cos iy - \sin x \sin iy$

$= \cos(x+iy)$

$= \cos z$

$\therefore f'(z) = \cos z.$

5. Show that $f(z) = \cosh z$ is analytic and hence find $f'(z)$.

Soln: $f(z) = \cosh z$

$u+iv = \cosh(x+iy)$

$= \cos i(x+iy) \qquad (\cosh\theta = \cos i\theta)$

$= \cos(ix-y)$

$= \cos ix \cos y + \sin ix \sin y$

$= (\cosh x \cos y) + i(\sinh x \sin y)$

$\Rightarrow u = \cosh x \cos y \qquad v = \sinh x \sin y$

$u_x = \sinh x \cos y \qquad v_x = \cosh x \sin y$

$u_y = -\cosh x \sin y \qquad v_y = \sinh x \cos y$

Cauchy Riemann equations: $u_x = v_y$ and $u_y = -v_x$ are satisfied.

$\therefore f(z) = \cosh z$ is analytic.

Also, $f'(z) = u_x + iv_x$

$= \sinh x \cos y + i \cosh x \sin y$

$= \frac{1}{i}[i \sinh x \cos y - \cosh x \sin y]$

$= \frac{1}{i}[\sin ix \cos y - \cos ix \sin y]$

$= \frac{1}{i}\sin(ix-y) = \frac{1}{i}\sin i(x+iy)$

$= \frac{1}{\cancel{i}}\cancel{i}\sinh(x+iy)$

$\therefore f'(z) = \sinh z.$

6. Show that $w = \log z$, $z \neq 0$ is analytic and hence find $\frac{dw}{dz}$.

Soln: $w = \log z = \log(re^{i\theta})$

$u + iv = \log r + i\theta$

$\Rightarrow u = \log r \qquad v = \theta$

$u_r = \frac{1}{r} \qquad v_r = 0$

$u_\theta = 0 \qquad v_\theta = 1$

Cauchy Riemann equations in Polar form : $r.u_r = v_\theta$ and $r.v_r = -u_\theta$ are satisfied.

Thus $w = \log z$ is analytic.

$\frac{dw}{dz} = f'(z) = e^{-i\theta}(u_r + iv_r)$

$= e^{-i\theta}\left(\frac{1}{r} + 0\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}$

$\therefore f'(z) = \frac{1}{z}$.

7. Show that $f(z) = z^n$, where n is a positive integer is analytic ; find $f'(z)$.

Soln: $f(z) = z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

$u + iv = (r^n \cos n\theta) + i(r^n \sin n\theta)$

$\Rightarrow u = r^n \cos n\theta \qquad v = r^n \sin n\theta$

$u_r = n.r^{n-1}\cos n\theta \qquad v_r = n.r^{n-1}\sin n\theta$

$u_\theta = -r^n n. \sin n\theta \qquad v_\theta = r^n. n \cos n\theta$

Cauchy Riemann equations in polar form : $r.u_r = v_\theta$ and $r.v_r = -u_\theta$ are satisfied.

$f'(z) = e^{-i\theta}(u_r + iv_r)$

$= e^{-i\theta}(n.r^{n-1}\cos n\theta + i\, n.r^{n-1}\sin n\theta)$

$= e^{-i\theta}.\, n.r^{n-1}(\cos n\theta + i\sin n\theta)$

$= n.r^{n-1}.e^{-i\theta}.e^{in\theta}$

$= n.r^{n-1}.\, e^{i\theta(n-1)}$

$= n.(r.e^{i\theta})^{n-1}$

$= n.z^{n-1}$

$\therefore f'(z) = n.z^{n-1}$.

8. Show that $f(z) = \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta$, $r \neq 0$ is a regular function of $z = r.e^{i\theta}$. Also find $f'(z)$.

Soln: By data $u = \left(r + \frac{k^2}{r}\right)\cos\theta$ and $v = \left(r - \frac{k^2}{r}\right)\sin\theta$

$$u_r = \left(1 - \frac{k^2}{r^2}\right)\cos\theta \qquad v_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta$$

$$u_\theta = \left(r + \frac{k^2}{r}\right)(-\sin\theta) \qquad v_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$$

Cauchy Riemann equations in Polar form : $r u_r = v_\theta$ and $r v_r = -u_\theta$ are satisfied.

$\therefore$ $f(z)$ is analytic.

Also, $f'(z) = e^{-i\theta}(u_r + i v_r)$

$$= e^{-i\theta}\left[\left(1 - \frac{k^2}{r^2}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right]$$

$$= e^{-i\theta}\left[(\cos\theta + i\sin\theta) - \frac{k^2}{r^2}(\cos\theta - i\sin\theta)\right]$$

$$= e^{-i\theta}\left[e^{i\theta} - \frac{k^2}{r^2}e^{-i\theta}\right]$$

$$= 1 - \frac{k^2}{r^2(e^{i\theta})^2} = 1 - \frac{k^2}{z^2}$$

$\therefore$ $f'(z) = 1 - \frac{k^2}{z^2}$

Construction of Analytic function $f(z)$ given its real or imaginary part.

Milne – Thompson Method.

Working Procedure:

Cartesian form –

- Given u or v as functions of x, y we find u_x, u_y or v_x, v_y and consider $f'(z) = u_x + i v_x$.
  - → Given u, we use CR equation $v_x = -u_y$
  - → Given v, we use CR equation $u_x = -v_y$

  So that $f'(z) = u_x - i u_y$ or $f'(z) = v_y + i v_x$
- We substitute the expression for the partial derivatives in the RHS and then put $x = z$, $y = 0$ to obtain $f'(z)$ as function of z.
- Integrating wrt z, we get $f(z)$.

Polar form –

- Given u or v as functions of r, θ we find u_r, u_θ or v_r, v_θ and consider $f'(z) = e^{-i\theta}(u_r + i v_r)$
  - → Given u, we use CR equation $v_r = -\frac{1}{r} u_\theta$
  - → Given v, we use CR equation, $u_r = \frac{1}{r} v_\theta$.

  So that $f'(z) = e^{-i\theta}\left(u_r - \frac{i}{r} u_\theta\right)$ or $f'(z) = e^{-i\theta}\left(\frac{1}{r} v_\theta + i v_r\right)$
- We use the substitution $r = z$, $\theta = 0$ to obtain $f'(z)$ as function of z.
- Integrating wrt z, we get $f(z)$.

Problems:

1. Construct the analytic function whose real part is $u = \log\sqrt{x^2+y^2}$

Soln: $u = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2)$

$$u_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$$

$$u_y = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2}$$

Consider $f'(z) = u_x + iv_x$ But $v_x = -u_y$ (CR equation)

$$= u_x - iu_y$$

$$f'(z) = \frac{x-iy}{x^2+y^2} \quad \text{——} \quad (1)$$

Putting $x = z$ and $y = 0$ we have

$$f'(z) = \frac{z-0}{z^2+0} = \frac{1}{z}$$

$$\therefore f(z) = \int\frac{1}{z}dz + c$$

$$f(z) = \log z + c.$$

2. Find the analytic function $f(z)$ whose imaginary part is $e^x(x\sin y + y\cos y)$.

Soln: $v = e^x(x\sin y + y\cos y)$

$$v_x = e^x(x\sin y + y\cos y) + e^x\sin y = e^x(x\sin y + y\cos y + \sin y)$$

$$v_y = e^x(x\cos y + \cos y - y\sin y)$$

Consider $f'(z) = u_x + iv_x$ But $u_x = v_y$ (CR equation)

$$= v_y + iv_x$$

$$f'(z) = e^x(x\cos y + \cos y - y\sin y) + ie^x(x\sin y + y\cos y + \sin y)$$

$$= e^x\left[x(\cos y + i\sin y) + (\cos y + i\sin y) + (-y + xi)\sin y\right]$$

Putting $x = z$ and $y = 0$ we have

$$f'(z) = e^z(z+1) \quad (\cos 0 = 1 ;\ \sin 0 = 0)$$

$$\therefore f(z) = \int(z+1)e^z dz + c$$

$$= (z+1)e^z - e^z + c$$

$$f(z) = ze^z + c.$$

3. Find the analytic function whose real part is $\dfrac{x^4-y^4-2x}{x^2+y^2}$. Hence determine v.

Soln:

$$u = \frac{x^4-y^4-2x}{x^2+y^2}$$

$$u_x = \frac{(x^2+y^2)(4x^3-2)-(x^4-y^4-2x)(2x)}{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2)(-4y^3)-(x^4-y^4-2x)(2y)}{(x^2+y^2)^2}$$

Consider $f'(z) = u_x + iv_x$ But $v_x = -u_y$

$$= u_x - iu_y$$

Putting $x=z$ and $y=0$ we have,

$$f'(z) = \frac{2z^5+2z^2}{z^4} = 2z + \frac{2}{z^2}$$

$$\therefore f(z) = \int\left(2z+\frac{2}{z^2}\right)dz + c$$

$$f(z) = z^2 - \frac{2}{z} + c$$

To find v:

$$u+iv = (x+iy)^2 - \frac{2}{x+iy} + c$$

$$= x^2+i^2y^2+2ixy - \frac{2(x-iy)}{x^2-i^2y^2} + c$$

$$= x^2-y^2+2ixy - \frac{2(x-iy)}{x^2+y^2} + c$$

$$= \left(x^2-y^2-\frac{2x}{x^2+y^2}\right) + i\left(2xy+\frac{2y}{x^2+y^2}\right) + c$$

$$= \left(\frac{x^4-y^4-2x}{x^2+y^2}\right) + i\left(\frac{2x^3y+2xy^3+2y}{x^2+y^2}\right) + c$$

Equating real and imaginary parts we observe that real part u is same as in the given problem and imaginary part is:

$$v = \frac{2x^3y+2xy^3+2y}{x^2+y^2}$$

4. Determine the analytic function $f(z) = u + iv$ given that the real part $u = e^{2x}(x\cos 2y - y\sin 2y)$

Soln: $u = e^{2x}(x\cos 2y - y\sin 2y)$

$$u_x = e^{2x}(\cos 2y) + 2e^{2x}(x\cos 2y - y\sin 2y)$$

$$= e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y)$$

$$u_y = e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)$$

Consider $f'(z) = u_x + iv_x$ But $v_x = -u_y$

$$= u_x - iu_y$$

$$= e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y) + i\,e^{2x}(2x\sin 2y + \sin 2y + 2y\cos 2y)$$

Putting $x = z$ and $y = 0$ we get

$$f'(z) = e^{2z}(1 + 2z)$$

$$\therefore f(z) = \int (1+2z)e^{2z}\,dz + c$$

$$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + c$$

$$= ze^{2x} + c$$

5. Find the analytic function $f(z)$ whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ and hence find the imaginary part.

Soln: $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(-(-2\sin 2x))}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = -\frac{\sin 2x}{(\cosh 2y - \cos 2x)^2} \times 2\sinh 2y$$

Consider $f'(z) = u_x + iv_x$ But $v_x = -u_y$

$$= u_x - iu_y$$

$$f'(x) = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2} - i\,\frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

Putting $x = z$ and $y = 0$

$$f'(z) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2} - i(0)$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z} = \frac{-2}{2\sin^2 z}$$

$$= -\operatorname{cosec}^2 z$$

$$\therefore f(z) = \int (-\operatorname{cosec}^2 z)\, dz + c$$

$$= \cot z + c$$

To find v:

$$u + iv = \frac{\cos(x+iy)}{\sin(x+iy)} \times \frac{\sin(x-iy)}{\sin(x-iy)} + c$$

$$= \frac{\frac{1}{2}[\sin(x+iy+x-iy) + \sin(x-iy-x-iy)]}{\frac{1}{2}[\cos(x+iy-x+iy) - \cos(x+iy+x-iy)]}$$

$$= \frac{\sin 2x + \sin(-2iy)}{\cos(2iy) - \cos 2x} = \frac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x}$$

$$= \left[\frac{\sin 2x}{\cosh 2y - \cos 2x}\right] + i\left[\frac{-\sinh 2y}{\cosh 2y - \cos 2x}\right]$$

$$\therefore v = -\frac{\sinh 2y}{\cosh 2y - \cos 2x}$$

6. If $\phi + i\psi$ represents the complex Potential of an electrostatic field where $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$ find the complex potential as a function of the complex variable z and hence determine ϕ.

Soln: $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$

$$\psi_x = 2x + \frac{(x^2+y^2)1 - x.2x}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\psi_y = -2y - \frac{2yx}{(x^2+y^2)^2}$$

Consider $f'(z) = \psi_y + i\psi_x$ $\quad\because \phi_x = \psi_y$ (CR equation)

$$= -2y - \frac{2xy}{(x^2+y^2)^2} + i\left[2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right]$$

Putting $x = z$ and $y = 0$

$$= 0 + i\left[2z - \frac{z^2}{z^4}\right] = i\left[2z - \frac{1}{z^2}\right]$$

$$\therefore f(z) = i\int\left[2z - \frac{1}{z^2}\right]dz + c = i\left(z^2 + \frac{1}{z}\right) + c$$

To find ϕ :

$$\phi + i\psi = i\left[(x+iy)^2 + \frac{1}{x+iy}\right] + c$$

$$= i\left[x^2 - y^2 + 2ixy\right] + \frac{i(x-iy)}{(x+iy)(x-iy)} + c$$

$$= i(x^2-y^2) - 2xy + \frac{ix+y}{x^2+y^2} + c$$

$$= \left[-2xy + \frac{y}{x^2+y^2}\right] + i\left[x^2 - y^2 + \frac{x}{x^2+y^2}\right] + c$$

$$\therefore \phi = -2xy + \frac{y}{x^2+y^2}.$$

7. Construct the analytic function whose real part is $r^2 \cos 2\theta$.

Soln: $u = r^2 \cos 2\theta$

$u_r = 2r \cos 2\theta$

$u_\theta = -2r^2 \sin 2\theta$

Consider $f'(z) = e^{-i\theta}(u_r + iv_r)$ But $v_r = -\frac{1}{r} u_\theta$ (CR equation)

$= e^{-i\theta}\left(2r\cos 2\theta + i\left(-\frac{1}{r}\right)(-2r^2 \sin 2\theta)\right)$

$= e^{-i\theta}(2r\cos 2\theta + i\,2r\sin 2\theta)$

Putting $r = z$ and $\theta = 0$

$f'(z) = 2z$

$\therefore f(z) = z^2 + c$

8. Determine the analytic function $f(z)$ whose imaginary part is $\left(r - \frac{k^2}{r}\right)\sin\theta$, $r \neq 0$. Hence find the real part of $f(z)$ and prove that it is harmonic.

Soln: $v = \left(r - \frac{k^2}{r}\right)\sin\theta$

$v_r = \left(1 + \frac{k^2}{r^2}\right)\sin\theta$

$v_\theta = \left(r - \frac{k^2}{r}\right)\cos\theta$

Consider $f'(z) = e^{-i\theta}(u_r + iv_r)$ But $u_r = \frac{1}{r} v_\theta$ (CR equation)

$= e^{-i\theta}\left(\frac{1}{r} v_\theta + i v_r\right)$

$= e^{-i\theta}\left(\frac{1}{r}\left(r - \frac{k^2}{r}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right)$

$= e^{-i\theta}\left(\left(1 - \frac{k^2}{r^2}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right)$

Putting $r = z$ and $\theta = 0$

$f'(z) = 1 - \frac{k^2}{z^2}$

$\therefore f(z) = z + \frac{k^2}{z} + c$

To find u:

$$u + iv = (re^{i\theta}) + \frac{k^2}{re^{i\theta}}$$

$$= r(\cos\theta + i\sin\theta) + \frac{k^2}{r}(\cos\theta - i\sin\theta)$$

$$= \left(r + \frac{k^2}{r}\right)\cos\theta + i\left(r - \frac{k^2}{r}\right)\sin\theta$$

$$\therefore \quad u = \left(r + \frac{k^2}{r}\right)\cos\theta.$$

To prove u is harmonic,

we need to prove $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ —— (1)

- $u_r = \left(1 - \frac{k^2}{r^2}\right)\cos\theta$

$u_{rr} = \frac{2k^2}{r^3}\cos\theta$

- $u_\theta = -\left(r + \frac{k^2}{r}\right)\sin\theta$

$u_{\theta\theta} = -\left(r + \frac{k^2}{r}\right)\cos\theta$

Substituting in the LHS of (1):

$$\frac{2k^2}{r^3}\cos\theta + \frac{1}{r}\left(1 - \frac{k^2}{r^2}\right)\cos\theta + \frac{1}{r^2}(-)\left(r + \frac{k^2}{r}\right)\cos\theta$$

$$\frac{2k^2}{r^3}\cos\theta + \frac{\cos\theta}{r} - \frac{k^2}{r^3}\cos\theta - \frac{\cos\theta}{r} - \frac{k^2}{r^3}\cos\theta$$

$$= 0.$$

$\therefore \quad u = \left(r + \frac{k^2}{r}\right)\cos\theta$ is harmonic.

Finding the conjugate harmonic function and the analytic function.

The real and imaginary parts of an analytic function $f(z) = u + iv$ are harmonic. u and v are called conjugate harmonic functions. (Harmonic conjugates). Given u we can find v and vice-versa.

Problems :

1. Show that $u = e^x(x\cos y - y\sin y)$ is harmonic and find its harmonic conjugate. Also determine the corresponding analytic function.

<u>Soln:</u> $\quad u = e^x(x\cos y - y\sin y)$

$$u_x = e^x(x\cos y - y\sin y) + e^x\cos y$$

$$= e^x(x\cos y + \cos y - y\sin y) \quad \text{—} \; (1)$$

$$u_{xx} = e^x(x\cos y + \cos y - y\sin y) + e^x\cos y$$

$$= e^x(x\cos y + 2\cos y - y\sin y)$$

$$u_y = e^x(x(-\sin y) - \sin y - y\cos y)$$

$$= -e^x(x\sin y + \sin y + y\cos y) \quad \text{—} \; (2)$$

$$u_{yy} = -e^x(x\cos y + \cos y + \cos y - y\sin y)$$

$$= -e^x(x\cos y + 2\cos y - y\sin y)$$

Consider $u_{xx} + u_{yy} = e^x(x\cos y + 2\cos y - y\sin y) - e^x(x\cos y + 2\cos y - y\sin y)$

$$= 0$$

$\therefore$ u is harmonic.

Consider Cauchy Riemann equations : $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Then $\dfrac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y)$

$$v = e^x\left[x\int\cos y\,dy + \int\cos y\,dy - \int y\sin y\,dy\right] + f(x)$$

$$= e^x(x\sin y + \cancel{\sin y} + y\cos y - \cancel{\sin y}) + f(x) \quad \text{—} \; (3)$$

Then $\frac{\partial v}{\partial x} = e^x(x\sin y + \sin y + y\cos y)$

$v = \sin y \int x e^x dx + \sin y \int e^x dx + y\cos y \int e^x dx + g(y)$

$= \sin y\,(xe^x - e^x) + \sin y\, e^x + y\cos y\, e^x + g(y).$

$= x e^x \sin y + y e^x \cos y + g(y)$ —— (4)

Comparing (3) and (4) we choose $f(x) = 0$ and $g(y) = 0$

$\therefore v = e^x(x\sin y + y\cos y)$

Now $f(z) = u + iv$

$= e^x(x\cos y - y\sin y) + i\, e^x(x\sin y + y\cos y)$

Putting $x = z$ and $y = 0$

$f(z) = z e^z$

2. Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function $f(z)$.

Soln: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$ — (1) $\qquad u_y = -6xy - 6y$ — (2)

$u_{xx} = 6x + 6 \qquad u_{yy} = -6x - 6$

Consider : $u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0 \Rightarrow u$ is harmonic.

Consider Cauchy Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Then $\frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 6x \qquad \frac{\partial v}{\partial x} = 6xy + 6y$

$v = 3x^2 \int dy - \int 3y^2 dy + 6x \int dy + f(x) \qquad v = 6y \int x\,dx + 6y \int dx + g(y)$

$= 3x^2 y - y^3 + 6xy + f(x)$ — (3) $\qquad = 3x^2 y + 6xy + g(y)$ — (4)

Comparing (3) and (4) we choose $f(x) = 0 \quad g(y) = -y^3$

$\therefore v = 3x^2 y + 6xy - y^3$

Now $f(z) = u + iv$

$= (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2 y + 6xy - y^3)$

Putting $x = z$ and $y = 0$

$f(z) = z^3 + 3z^2 + 1$

3. Determine which of the following function is harmonic. Find the conjugate harmonic function and express $u+iv$ as an analytic function of z.

(a) $v = \log\sqrt{x+y}$ (b) $v = \cos x \cdot \sinh y$

Soln: (a) $v = \log\sqrt{x+y} = \frac{1}{2}\log(x+y)$

$v_x = \frac{1}{2}\cdot\frac{1}{x+y}$ $\quad v_y = \frac{1}{2}\frac{1}{x+y}$

$v_{xx} = \frac{1}{2}\frac{-1}{(x+y)^2}$ $\quad v_{yy} = \frac{1}{2}\frac{-1}{(x+y)^2}$

Consider $v_{xx} + v_{yy} = \frac{-1}{(x+y)^2} \neq 0. \Rightarrow$ v is not harmonic.

(b) $v = \cos x \cdot \sinh y$

$v_x = -\sin x \sinh y$ —① $\quad v_y = \cos x \cosh y$ —②

$v_{xx} = -\cos x \sinh y$ $\quad v_{yy} = \cos x \sinh y$

Consider $v_{xx} + v_{yy} = -\cos x \sinh y + \cos x \sinh y = 0 \Rightarrow$ v is harmonic.

Consider Cauchy Riemann equations: $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial x} = \cos x \cosh y$ $\quad \frac{\partial u}{\partial y} = \sin x \sinh y$

$u = \cosh y\int \cos x\, dx + f(y)$ $\quad u = \sin x \cdot \int \sinh y\, dy + g(x)$

$= \sin x \cosh y + f(y)$ —③ $\quad = \sin x \cosh y + g(x)$ —④

Comparing ③ and ④ we choose $f(y) = 0$ $g(x) = 0$

$\therefore u = \sin x \cosh y$

Now $f(z) = u + iv$

$= (\sin x \cosh y) + i(\cos x \sinh y)$

Putting $x = z$ and $y = 0$ we get

$f(z) = \sin z$ $\quad \because \cosh(0) = 1 \quad \sinh(0) = 0.$

4. Show that $u = \left(r + \frac{1}{r}\right)\cos\theta$ is harmonic. Find its harmonic conjugate and also the corresponding analytic function.

Soln: $u = \left(r + \frac{1}{r}\right)\cos\theta$

We need to prove : $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ —①

$u_r = \left(1 - \frac{1}{r^2}\right)\cos\theta$ $\qquad$ $u_\theta = \left(r + \frac{1}{r}\right)(-\sin\theta)$

$u_{rr} = \frac{2}{r^3}\cos\theta$ $\qquad$ $u_{\theta\theta} = \left(r + \frac{1}{r}\right)(-\cos\theta)$

Substituting these in LHS of ①:

$$\frac{2}{r^3}\cos\theta + \frac{1}{r}\left(1 - \frac{1}{r^2}\right)\cos\theta + \frac{1}{r^2}\left(r + \frac{1}{r}\right)(-\cos\theta)$$

$$= \frac{2}{r^3}\cos\theta + \frac{1}{r}\cos\theta - \frac{1}{r^3}\cos\theta \mp \frac{1}{r}\cos\theta - \frac{1}{r^3}\cos\theta = 0.$$

$\Rightarrow$ u is harmonic.

Consider Cauchy Riemann equations : $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

$\frac{\partial v}{\partial \theta} = r\left(1 - \frac{1}{r^2}\right)\cos\theta$ $\qquad$ $\frac{\partial v}{\partial r} = -\frac{1}{r}\left(r + \frac{1}{r}\right)(-\sin\theta)$

$\frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta$ $\qquad$ $\frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right)\sin\theta$

$v = \left(r - \frac{1}{r}\right)\int \cos\theta \, d\theta + f(r)$ $\qquad$ $v = \sin\theta \int \left(1 + \frac{1}{r^2}\right)dr + g(\theta)$

$= \left(r - \frac{1}{r}\right)\sin\theta + f(r)$ —② $\qquad$ $= \sin\theta\left(r - \frac{1}{r}\right) + g(\theta)$ —③

Comparing ② and ③ we choose $f(r) = 0$ and $g(\theta) = 0$.

$\therefore$ $v = \left(r - \frac{1}{r}\right)\sin\theta$

Now $f(z) = u + iv$

$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$

Putting $r = z$ and $\theta = 0$ we get

$f(z) = z + \frac{1}{z}$

Miscellaneous Problems.

1. Find the analytic function $f(z) = u + iv$ given $u - v = e^x(\cos y - \sin y)$.

Soln: $u - v = e^x(\cos y - \sin y)$

Differentiate partially w.r.t x and y:

$$u_x - v_x = e^x(\cos y - \sin y) \quad \text{——} ①$$

$$u_y - v_y = e^x(-\sin y - \cos y) \quad \text{——} ②$$

Using Cauchy Riemann equations: $u_y = -v_x$ and $v_y = u_x$.

② : $-v_x - u_x = e^x(-\sin y - \cos y)$

Or $u_x + v_x = e^x(\sin y + \cos y)$ —— ③

① + ③ : $2u_x = 2e^x \cos y \Rightarrow u_x = e^x \cos y$

① − ③ : $-2v_x = -2e^x \sin y \Rightarrow v_x = e^x \sin y$

We have $f'(z) = u_x + iv_x$

$= e^x \cos y + ie^x \sin y$

$= e^x(\cos y + i \sin y) = e^{x+iy} = e^z$

$f(z) = \int e^z \, dz + C = e^z + C$.

2. Find the analytic function $f(z)$ as a function of z given that sum of its real and imaginary parts is $x^3 - y^3 + 3xy(x-y)$.

Soln: Given: $u + v = x^3 - y^3 + 3xy(x-y) = x^3 - y^3 + 3x^2y - 3xy^2$

Differentiate partially w.r.t x and y:

$$u_x + v_x = 3x^2 + 6xy - 3y^2 \quad \text{——} ①$$

$$u_y + v_y = -3y^2 + 3x^2 - 6xy \quad \text{——} ②$$

Using Cauchy Riemann equations: $u_y = -v_x$ and $v_y = u_x$

② : $-v_x + u_x = -3y^2 + 3x^2 - 6xy$ —— ③

① + ③ : $2u_x = 2(3x^2 - 3y^2) \Rightarrow u_x = 3x^2 - 3y^2$

① − ③ : $2v_x = 2(6xy) \Rightarrow v_x = 6xy$

We have : $f'(z) = u_x + iv_x = 3x^2 - 3y^2 + i6xy$

Putting $x = z$ and $y = 0$: $f'(z) = 3z^2$

$\therefore f(z) = z^3 + C$

3. If $f(z) = u + iv$ is analytic find $f(z)$ if $u - v = (x-y)(x^2+4xy+y^2)$.

Soln: $u - v = (x-y)(x^2+4xy+y^2)$

$= x^3 + 3x^2y - 3xy^2 - y^3$

Differentiate partially w.r.t x and y:

$u_x - v_x = 3x^2 + 6xy - 3y^2$ —— ①

$u_y - v_y = 3x^2 - 6xy - 3y^2$ —— ②

Using Cauchy Riemann equations : $u_y = -v_x$ and $v_y = u_x$

② : $-v_x - u_x = 3x^2 - 6xy - 3y^2$

Or $u_x + v_x = -3x^2 + 6xy + 3y^2$ —— ③

① + ③ : $2u_x = 2(6xy) \Rightarrow u_x = 6xy$

① − ③ : $-2v_x = 2(3x^2 - 3y^2) \Rightarrow v_x = 3x^2 - 3y^2$

We have $f'(z) = u_x + iv_x$

$= 6xy + i(3x^2 - 3y^2)$

Putting $x = z$ and $y = 0$

$f'(z) = i3z^2$

$f(z) = iz^3 + c$.

4. If $f(z) = u(r,\theta) + i\, v(r,\theta)$ is analytic and given that $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$, $r \neq 0$ determine the analytic function $f(z)$.

Soln: $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$

Differentiate partially w.r.t r and θ:

$u_r + v_r = -\frac{2}{r^3}(\cos 2\theta - \sin 2\theta)$ —— ①

$u_\theta + v_\theta = \frac{1}{r^2}(-2\sin 2\theta - 2\cos 2\theta)$ —— ②

Using Cauchy Riemann equations : $v_\theta = r u_r$ and $-u_\theta = r v_r$

② : $-r v_r + r u_r = \frac{1}{r^2}(-2\sin 2\theta - 2\cos 2\theta)$

$u_r - v_r = \frac{-2}{r^3}(\cos 2\theta + \sin 2\theta)$ —— ③

① + ③ : $2u_r = 2\left(\frac{-2}{r^3}\right)\cos 2\theta \Rightarrow u_r = \frac{-2}{r^3}\cos 2\theta$

① - ③ : $2V_r = 2\left(\frac{2}{r^3}\right)\sin 2\theta \Rightarrow V_r = \frac{2}{r^3}\sin 2\theta$

We have $f'(z) = e^{-i\theta}(u_r + iV_r)$

$= e^{-i\theta}\left(-\frac{2}{r^3}\cos 2\theta + \frac{2i}{r^3}\sin 2\theta\right)$

Putting $r = z$ and $\theta = 0$

$f'(z) = -\frac{2}{z^3}$

$\therefore f(z) = -2\int \frac{1}{z^3}dz + c = \frac{1}{z^2} + c.$

5. If $f(z)$ is analytic show that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$.

Soln: Let $f(z) = u + iv$ be analytic.

Then $|f(z)| = \sqrt{u^2+v^2}$ or $|f(z)|^2 = u^2+v^2 = \phi$ (say)

To prove that : $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\phi = 4|f'(z)|^2$

or $\phi_{xx} + \phi_{yy} = 4|f'(z)|^2$

Consider $\phi = u^2 + v^2$

Then $\phi_x = 2u.u_x + 2v.v_x$

$\phi_{xx} = 2u.u_{xx} + 2u_x^2 + 2vv_{xx} + 2v_x^2$ —— ①

Similarly $\phi_{yy} = 2u.u_{yy} + 2u_y^2 + 2vv_{yy} + 2v_y^2$ —— ②

Adding ① and ② :

$\phi_{xx} + \phi_{yy} = 2[u(u_{xx} + v_{xx}) + v(v_{xx} + v_{yy}) + u_x^2 + v_y^2 + u_y^2 + v_x^2]$ —③

Since $f(z)$ is analytic, u and v are harmonic.

i.e. $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

Using Cauchy Riemann equations: $v_y = u_x$ and $v_x = -u_y$

$\therefore$ ③ $\Rightarrow \phi_{xx} + \phi_{yy} = 2[u(0) + v(0) + u_x^2 + u_x^2 + (-v_x)^2 + v_x^2]$

or $\phi_{xx} + \phi_{yy} = 4(u_x^2 + v_x^2)$ —— ④

We have $f'(z) = u_x + iv_x \Rightarrow |f'(z)|^2 = u_x^2 + v_x^2$

$\therefore$ ④ $\Rightarrow \phi_{xx} + \phi_{yy} = 4|f'(z)|^2$

6. If $f(z)$ is a regular function of z show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

Soln: Let $f(z) = u + iv$ be analytic function. or regular function.

Then $|f(z)| = \sqrt{u^2+v^2} = \phi$

or $|f(z)|^2 = u^2 + v^2 = \phi^2$

To prove that : $\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = |f'(z)|^2$

Consider $\phi^2 = u^2 + v^2$

Differentiate partially wrt x and y :

$$\cancel{2}\phi\,\phi_x = \cancel{2}u\,u_x + \cancel{2}v\,v_x \quad \text{——} (1)$$

$$\cancel{2}\phi\,\phi_y = \cancel{2}u \cdot u_y + \cancel{2}v \cdot v_y \quad \text{——} (2)$$

Squaring and adding (1) and (2) :

$$\phi^2(\phi_x^2 + \phi_y^2) = (u u_x + v v_x)^2 + (u u_y + v v_y)^2$$

Using Cauchy Riemann equations : $u_y = -v_x$ and $v_y = u_x$

$$\phi^2(\phi_x^2 + \phi_y^2) = (u u_x + v v_x)^2 + (-u v_x + v u_x)^2$$

$$= u^2 u_x^2 + v^2 v_x^2 + \cancel{2uv u_x v_x} + u^2 v_x^2 + v^2 u_x^2 - \cancel{2uv u_x v_x}$$

$$= u^2(u_x^2 + v_x^2) + v^2(u_x^2 + v_x^2)$$

$$= (u^2+v^2)(u_x^2+v_x^2)$$

Since $\phi^2 = u^2 + v^2$,

$$\phi_x^2 + \phi_y^2 = u_x^2 + v_x^2 \quad \text{——} (3)$$

We have $f'(z) = u_x + iv_x$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}$$

$$|f'(z)|^2 = u_x^2 + v_x^2$$

$$\therefore \quad \phi_x^2 + \phi_y^2 = |f'(z)|^2.$$

7. If $\phi(x,y)$ is a differentiable function and $f(z) = u(x,y) + i\,v(x,y)$ is a regular function, show that :

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = \left\{\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right\} |f'(z)|^2$$

Soln: Treating ϕ as composite function of u and v which are functions of x and y.

By chain rule :

- $$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\phi_x = \phi_u u_x + \phi_v \cdot v_x \quad \text{——} (1)$$

- $$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\phi_y = \phi_u u_y + \phi_v \cdot v_y \quad \text{——} (2)$$

Squaring and adding (1) and (2) :

$$\phi_x^2 + \phi_y^2 = (\phi_u u_x + \phi_v v_x)^2 + (\phi_u u_y + \phi_v \cdot v_y)^2$$

using Cauchy Riemann equations : $u_y = -v_x$ and $v_y = u_x$

$$\phi_x^2 + \phi_y^2 = (\phi_u u_x + \phi_v v_x)^2 + (\phi_u(-v_x) + \phi_v u_x)^2$$

$$= \phi_u^2 u_x^2 + \phi_v^2 v_x^2 + \cancel{2\phi_u\phi_v u_x v_x} + \phi_u^2 v_x^2 + \phi_v^2 u_x^2 - \cancel{2\phi_u\phi_v u_x v_x}$$

$$= \phi_u^2(u_x^2 + v_x^2) + \phi_v^2(u_x^2 + v_x^2)$$

$$= (\phi_u^2 + \phi_v^2)(u_x^2 + v_x^2)$$

But $f'(z) = u_x + i v_x$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2} \quad \text{or} \quad |f'(z)|^2 = u_x^2 + v_x^2$$

$$\therefore \quad \phi_x^2 + \phi_y^2 = (\phi_u^2 + \phi_v^2)\,|f'(z)|^2 .$$