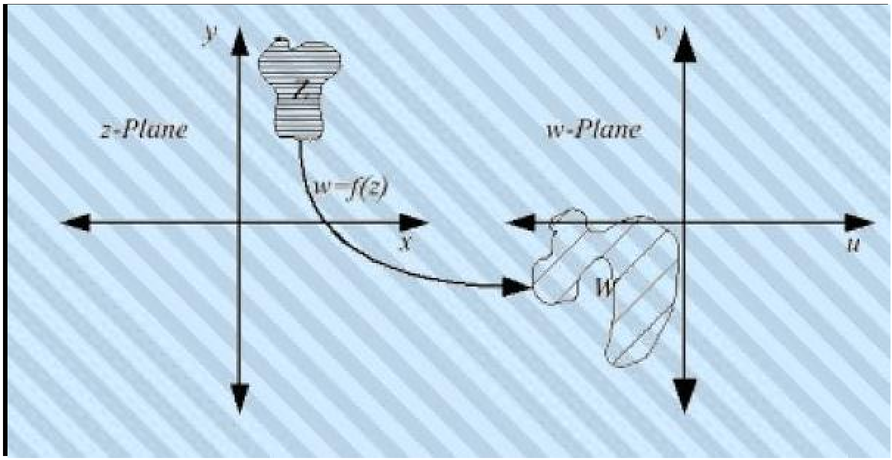
MODULE - 2

CONFORMAL TRANSFORMATIONS

2.1 Introduction

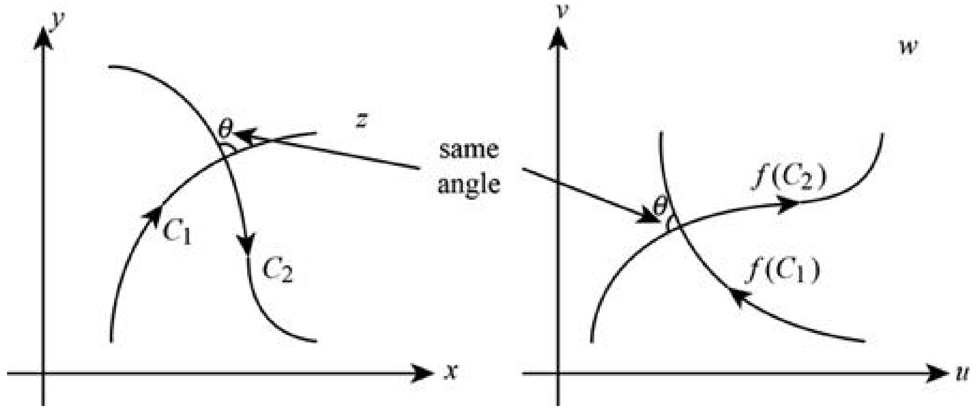
The transformation or mapping by the set of equations $u = u(x, y)$, $v = v(x, y)$ is a correspondence between the points in the z-plane and w-plane. If we solve the above equations for x and y in terms of u and v, we obtain $x = x(u, v)$, $y = y(u, v)$ which is called inverse transformation.

If the point z describes some curve in the z-plane, the point w will move along a corresponding curve in the w-plane, since to each point (x, y), there corresponds a point (u, v). We then, say that a curve in the z-plane is mapped into the corresponding curve in the w-plane by the function $w = f(z)$ which defines a mapping or transformation of the z-plane into the w-plane.



2.2 Conformal Transformations

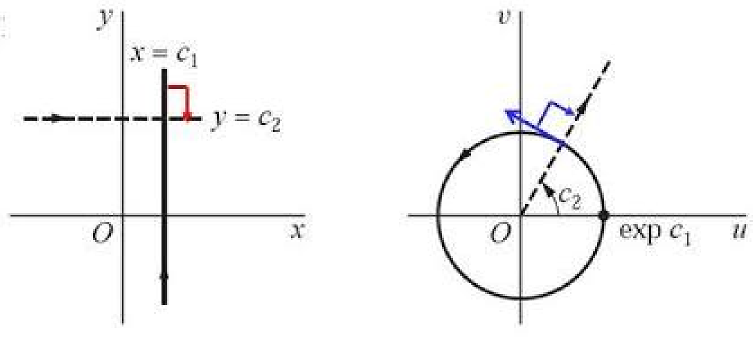
The transformation $w = f(z)$ is said to be conformal if it preserves the angle between any two curves in the z-plane and w-plane both in magnitude and sense.



z - plane   w-plane

Theorem: The condition for the transformation $w = f(z)$ to be conformal is that $f(z)$ must be analytic and $f'(z) \neq 0$ at the points of analyticity.

Example: The mapping $w = e^z$ is conformal throughout the entire z-plane since $\frac{d}{dz}\left(e^z\right) = e^z \neq 0$ for each z. Consider any two lines $x = c_1$ and $y = c_2$ in the z-plane, the first directed upward and the second directed to the right.



Note: If a mapping preserves the magnitudes of the angles but not necessarily the sense is called *isogonal.*

Example: The transformation $w = \bar{z}$, which is a reflection in the real axis, is isogonal but not conformal.

Critical Point

Suppose that f is not constant function and is analytic at a point z_0. If, in addition, $f'(z)=0$ then z_0 is called a critical point of the transformation $w=f(z)$.

Example: The point $z_0=0$ is a critical point of the transformation $w=1+z^2$.

2.3 Discussions of some transformations

1. Discussion of the transformation $w = z^2$

$$w=z^2 \qquad \text{---- (1)}$$

$$\Rightarrow \qquad \frac{dw}{dz}=2z$$

$$\Rightarrow \qquad \left(\frac{dw}{dz}\right)_{z=0}=0$$

$\therefore$ Transformation $w=z^2$ is not conformal at z = 0 and is analytic at every other point of the z-plane.

Let $\qquad z=x+iy$ and $w=u+iv$

$$(1) \Rightarrow \qquad u+iv=(x+iy)^2$$

$$\Rightarrow \qquad u+iv=x^2-y^2+i(2xy)$$

Equating *Re* and *Im* parts on both sides, we have

$$u=x^2-y^2 \text{ and } v=2xy \qquad \text{---- (2)}$$

Then we have the following cases.

Case -1:

Let $x=k_1$, where k_1 is a constant. This represents a family of straight lines parallel to y – axis in the z-plane.

Then from equation (2), we get

$$u={k_1}^2-y^2 \text{ and } v=2k_1y$$

Eliminating y between these equations, we get

$$u = k_1^2 - \left(\frac{v}{2k_1}\right)^2$$

$$\Rightarrow \qquad u = k_1^2 - \frac{v^2}{4k_1^2}$$

$$\Rightarrow \qquad 4k_1^2 u = 4k_1^4 - v^2$$

$$\Rightarrow \qquad v^2 = 4k_1^4 - 4k_1^2 u$$

$$\Rightarrow \qquad v^2 = -4k_1^2\left(u - k_1^2\right)$$

This represents a family of parabolas with axis along the negative direction of u-axis in the w-plane.

Case -2:

Let $y = k_2$, where k_2 is a constant. This represents a family of straight lines parallel to $x-$ axis in the z-plane.

Then from equation (2), we get

$$u = x^2 - k_2^2 \quad \text{and} \quad v = 2xk_2$$

Eliminating x between these equations, we get

$$u = \left(\frac{v}{2k_2}\right)^2 - k_2^2$$

$$\Rightarrow \qquad u = \frac{v^2}{4k_2^2} - k_2^2$$

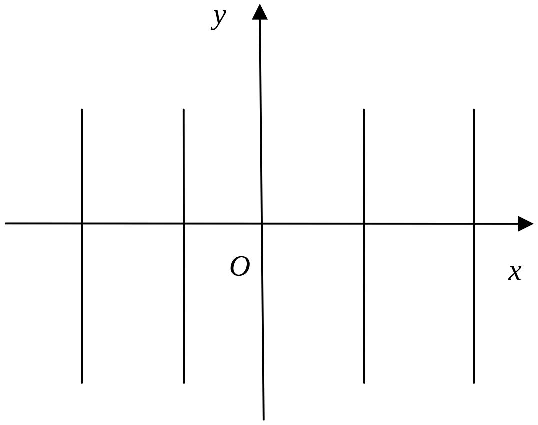
$$\Rightarrow \qquad 4k_2^2 u = v^2 - 4k_1^4$$

$$\Rightarrow \qquad v^2 = 4k_2^2 u + 4k_2^4$$

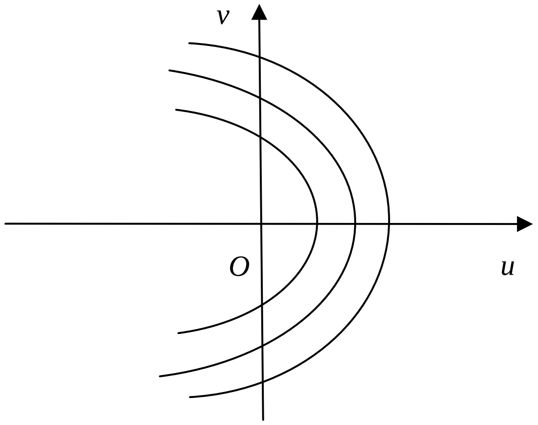
$$\Rightarrow \qquad v^2 = 4k_2^2\left(u + k_2^2\right)$$

This represents a family of parabolas with axis along the positive direction of u-axis in the w-plane.

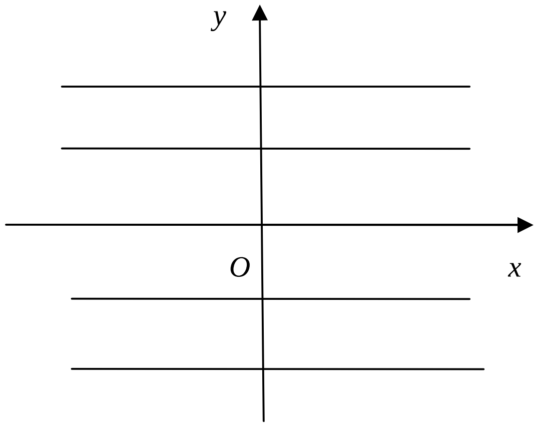
Conclusion: Thus, the transformation $w = z^2$ transforms straight lines parallel to y-axis to parabolas having negative u-axis as its axis and the straight lines parallel to the x-axis to parabolas having positive u-axis as its axis.



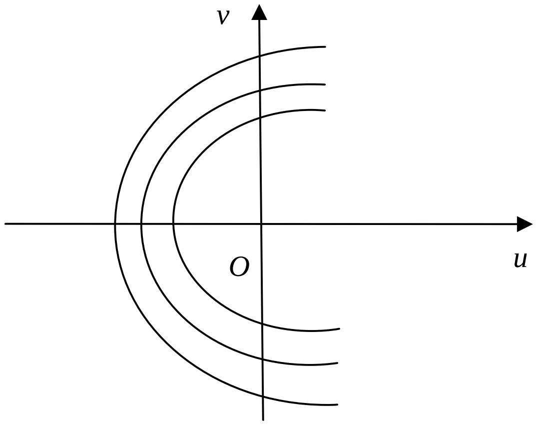
z - Plane



w - Plane



z - Plane



w - Plane

2. Discussion of the transformation $w = e^z$

$$w = e^z \qquad \text{---- (1)}$$

$\Rightarrow$ $$\frac{dw}{dz} = e^z$$

$\Rightarrow$ $$\left(\frac{dw}{dz}\right) \neq 0 \quad \text{for any } z$$

$\therefore$ Transformation $w = e^z$ is conformal at all points

Let $$z = x + iy \text{ and } w = u + iv$$

(1) $\Rightarrow$ $$u + iv = e^{x+iy}$$

$\Rightarrow$ $$u + iv = e^{x} e^{iy}$$

$\Rightarrow$ $$u + iv = e^{x}\left(\cos^{-} + i\sin^{-}\right)$$

Equating *Re* and *Im* parts onboth sides, we have

$$u = e^{x}\cos y \text{ and } v = e^{x}\sin y \qquad \text{---- (2)}$$

Then we have the following cases.

Case -1:

Let $x = k_1$, where k_1 is a constant. This represents a family of straight lines parallel to y – axis in the z-plane.

Then from equation (2), we get

$$u = e^{k_1}\cos y \text{ and } v = e^{k_1}\sin y$$

Eliminating y between these equations, we get

$$u^2 + v^2 = \left(e^{k_1}\right)^2 \quad \text{(by squaring and adding)}$$

This represents a family of circles with centre origin and radius e^{k_1} in the w-plane.

Case -2:

Let $y = k_2$, where k_2 is a constant. This represents a family of straight lines parallel to x – axis in the z-plane.

Then from equation (2), we get

$$u = e^{x}\cos k_2 \text{ and } v = e^{x}\sin k_2$$

Eliminating x between these equations, we get

$$\frac{u}{v} = \cot k_2$$

$\Rightarrow$ $$\tan k_2 = \frac{v}{u}$$

$$\Rightarrow \qquad v = u \tan k_2$$

This represents a family of straight lines passing through the origin in the w-plane with slope $m = \tan k_2$.

Case – 3:

Let $x = 0$. This represents y – axis (imaginary axis) in the z-plane.

Then from equation (2), we get

$$u = \cos y \text{ and } v = \sin y$$

Eliminating y between these equations, we get

$$u^2 + v^2 = 1 \quad \text{(by squaring and adding)}$$

This represents a unit circle with centre origin in the w-plane.

Case – 4:

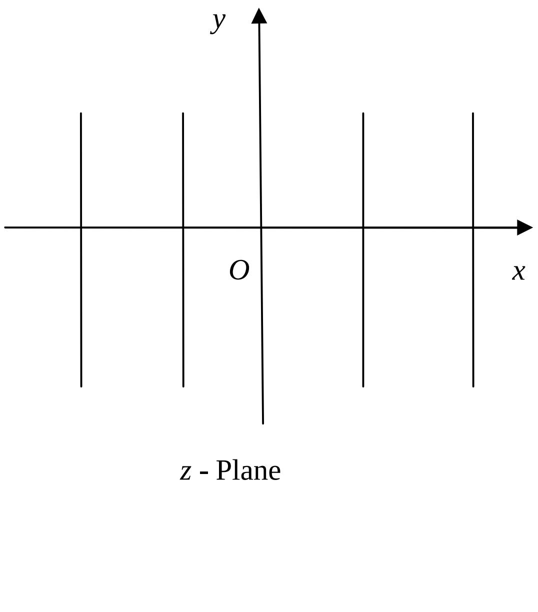
Let $y = 0$. This represents x – axis (real axis) in the z-plane.

Then from equation (2), we get

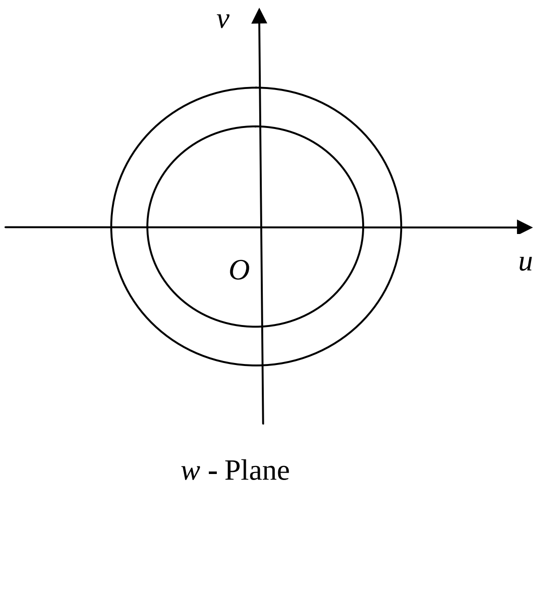
$$u = e^x \text{ and } v = 0$$

Here $v = 0$ represents a real axis in the w-plane.

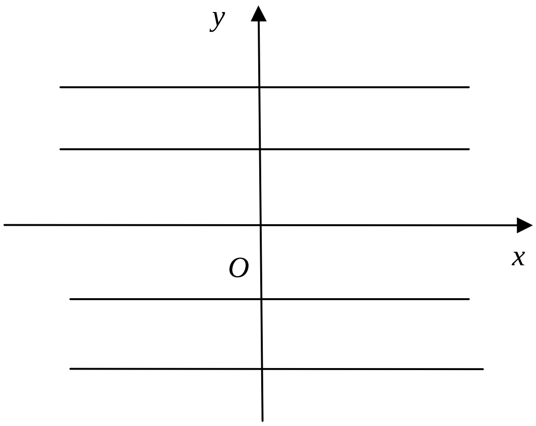
Conclusion: Thus, the transformation $w = e^z$ transforms the straight lines parallel to y-axis to circles with centre origin and radius e^{k_1}, the straight lines parallel to x-axis to the straight lines passing through the origin, imaginary axis to unit circle and real axis to real axis.



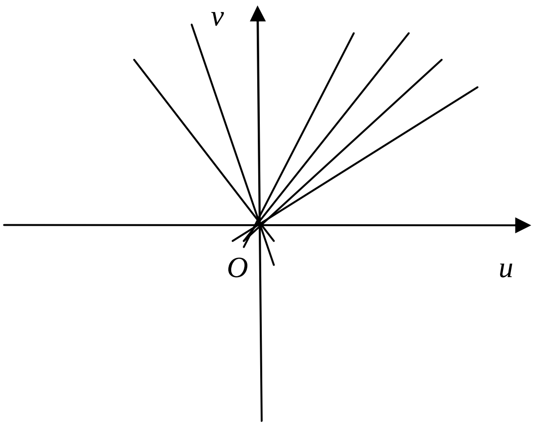
z - Plane



w - Plane



z - Plane



w - Plane

3. Discussion of the transformation $w = z + \frac{1}{z}, \left(z \neq 0\right)$

$$w = z + \frac{1}{z}, \left(z \neq 0\right) \qquad \text{---- (1)}$$

$\Rightarrow$
$$\frac{dw}{dz} = 1 - \frac{1}{z^2}$$

$\Rightarrow$
$$\left(\frac{dw}{dz}\right) = 0 \text{ at } z = \pm 1$$

$\therefore$ Transformation $w = z + \frac{1}{z}$ is not conformal at $z = \pm 1$ and is analytic at every other point of the z-plane.

Let $\qquad z = re^{i\theta}$ and $w = u + iv$

(1) $\Rightarrow$
$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$\Rightarrow$
$$u + iv = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$\Rightarrow$
$$u + iv = r\left(\cos\theta + i\sin\theta\right) + \frac{1}{r}\left(\cos\theta - i\sin\theta\right)$$

$$\Rightarrow \qquad u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Equating *Re* and *Im* both sides, we have

$$u = \left(r + \frac{1}{r}\right)\cos\theta \quad \text{and} \quad v = \left(r - \frac{1}{r}\right)\sin\theta \qquad \text{---- (2)}$$

Then we have the following cases.

Case -1:

Eliminating θ between these equations, we get

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = \cos^2\theta + \sin^2\theta$$

(by squaring and adding)

$$\Rightarrow \qquad \frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1 \qquad \text{---- (3)}$$

Let $r = k_1$, where k_1 is a constant. This represents a circle centered at origin in the *z*-plane.

Then from equation (3), we get

$$\frac{u^2}{\left(k_1 + \frac{1}{k_1}\right)^2} + \frac{v^2}{\left(k_1 - \frac{1}{k_1}\right)^2} = 1$$

This represents a family of ellipses having centre at origin in the *w*-plane.

Case -2:

Eliminating *r* from relations (2), we get

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2$$

(by squaring and subtracting)

$$\Rightarrow \qquad \frac{u^2}{\cos^2 -} - \frac{v^2}{\sin^2 -} = 4 \qquad \text{---- (4)}$$

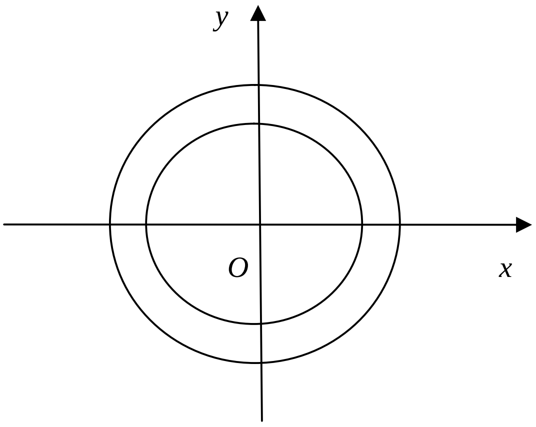
Let $^{-} = k_2$, where k_2 is a constant. This represents radial lines in the *z*-plane.

Then from equation (4), we get

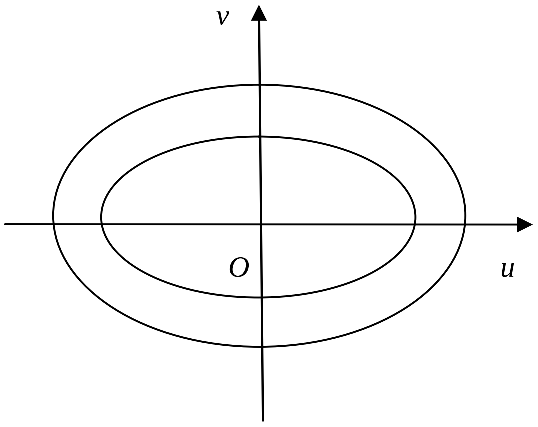
$$\frac{u^2}{\cos^2 k_2} - \frac{v^2}{\sin^2 k_2} = 4$$

This represents a family of hyperbolas having centre at origin in the *w*-plane.

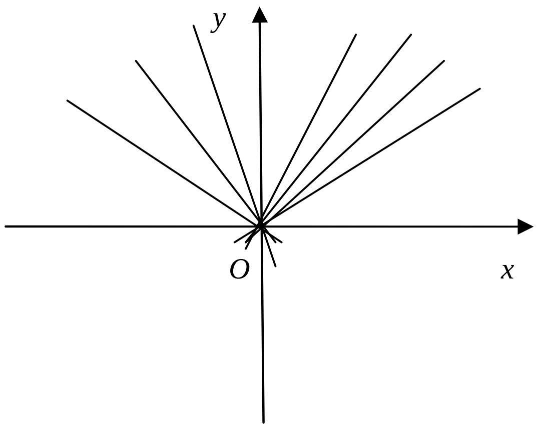
Conclusion: Thus, the transformation $w = z + \frac{1}{z}, (z \neq 0)$ transforms circles with centre origin to ellipses having centre at origin and the radial lines to hyperbolas having centre at origin.



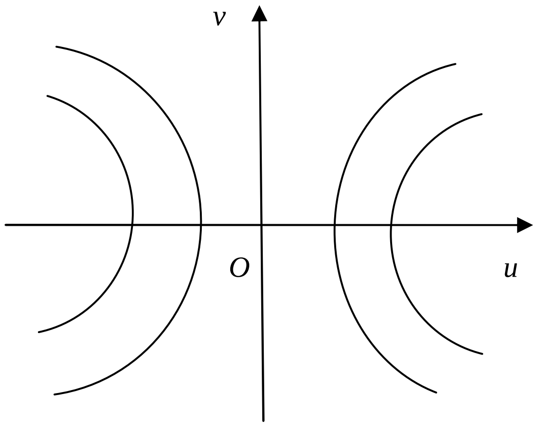
z - Plane



w - Plane



z - Plane



w - Plane

The Linear Transformation

The transformation $w = az + b$ where a and b are given complex constants is called a linear transformation.

2.4 Bilinear Transformation (Mobius Transformation)

The transformation $w = \frac{az+b}{cz+d}$ where a, b, c and d are constants and $ad - bc \neq 0$ is known as the bilinear transformation. (First studied by *Mobius*, a German Mathematician). Hence, sometimes called *Mobius transformation.*

The condition $ad - bc \neq 0$ ensures that $\frac{dw}{dz} \neq 0$, i.e., the transformation is conformal. If $ad - bc = 0$ every point of the z-plane is a critical point.

The inverse transformation of $w = \frac{az+b}{cz+d}$ is $z = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation.

This follows that there is one to one correspondence between all points in the two planes except at $z = \frac{-d}{c}$ in the z-plane and at $w = \frac{a}{c}$ in the w-plane.

2.4.1 Invariant or fixed points of bilinear transformation

If a point z maps onto itself under the bilinear transformation then the point is called an invariant or a fixed point of bilinear transformation. A fixed point of a transformation $w = f(z) = \frac{az+b}{cz+d}$ is a point z_0 such that $z_0 = f(z_0)$.

If z maps into itself in the w-plane (i.e., w = z), then $w = \frac{az+b}{cz+d}$ gives $z = \frac{az+b}{cz+d}$ or $cz^2 + (d-a)z - b = 0$. The roots of this equation are defined as the invariant or fixed points of the bilinear transformation. If however, the two roots are equal, the bilinear transformation is said to be parabolic.

Remark: Dividing the numerator and denominator of the right side of $w = \frac{az+b}{cz+d}$ by one of the four constants, it is clear that w has only three arbitrary constants. Hence three conditions are required to determine a bilinear transformation. For instance, three distinct points z_1, z_2, z_3 can be mapped into any three specified points w_1, w_2, w_3

2.4.2 Properties of Bilinear transformation

1. A bilinear transformation maps circles into circles.
2. A bilinear transformation preserves cross-ratio of four points

Cross ratio of four points

If z_1, z_2, z_3 and z_4 are four distinct points then the ratio $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$ is called the cross ratio of the points and is denoted by $(z_1,\ z_2, z_3,\ z_4)$.

$$\therefore \quad (z_1,\ z_2, z_3,\ z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

Working procedure to find Bilinear transformation

- ☞ Given z_1, z_2, z_3 onto w_1, w_2, w_3
- ☞ Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.
- ☞ Substitute the given set of points in $\dfrac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \dfrac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$
- ☞ Solving this equation for w in terms of z, we obtain the bilinear transformation that transforms from z_1, z_2, z_3 onto w_1, w_2, w_3 .

WORKED EXAMPLES

Example 2.1.1: Find the bilinear transformation that maps the points 1, 0, J1 onto the points *i*, 0, J *i*.

Solution:

Let $z_1 = 1,\ z_2 = 0,\ z_3 = -1$ and $w_1 = i,\ w_2 = 0,\ w_3 = -i$

Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

Substitute the given set of points in $\dfrac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \dfrac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$$\Rightarrow \qquad \frac{(w-i)(0+i)}{(i-0)(-i-w)} = \frac{(z-1)(0+1)}{(1-0)(-1-z)}$$

$$\Rightarrow \qquad \frac{wi+1}{1-wi} = \frac{z-1}{-1-z}$$

$$\Rightarrow \qquad (wi+1)(1+z) = (1-wi)(1-z)$$

$$\Rightarrow \qquad wi + wzi + 1 + z = 1 - z - wi + wzi$$

$$\Rightarrow \qquad 2wi = -2z$$

$$\Rightarrow \qquad w = \frac{-z}{i}$$

$$\Rightarrow \qquad w = zi \qquad \left[\because \frac{1}{i} = -i\right]$$

This is the required bilinear transformation.

Example 2.1.2: Find the bilinear transformation that maps the points J1, *i*, 1 onto the points 1, *i*, J 1. And also find its fixed points.

Solution: **(VTU 2005, 2009, 2013)**

Let $z_1 = -1,\ z_2 = i,\ z_3 = 1$ and $w_1 = 1,\ w_2 = i,\ w_3 = -1$

Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

Substitute the given set of points in $\dfrac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \dfrac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$$\Rightarrow \qquad \frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z+1)(i-1)}{(-1-i)(1-z)}$$

$$\Rightarrow \qquad \frac{wi+w-i-1}{-1-w+i+wi} = \frac{zi-z+i-1}{-1+z-i+zi}$$

$$\Rightarrow \qquad \frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z+1)(i-1)}{(z-1)(i+1)}$$

$$\Rightarrow \qquad \frac{(w-1)}{(w+1)}(i) = \frac{(z+1)}{(z-1)}(-i) \qquad \left[\because \frac{1+i}{1-i} = i \text{ and } \frac{1-i}{1+i} = -i\right]$$

$$\Rightarrow \qquad \frac{(w-1)}{(w+1)} = -\frac{(z+1)}{(z-1)}$$

$$\Rightarrow \qquad (w-1)(z-1) = -(z+1)(w+1)$$

$$\Rightarrow \qquad wz - w - z + 1 = -wz - z - w - 1$$

$$\Rightarrow \qquad 2wz = -2$$

$$\Rightarrow \qquad w = -\frac{1}{z}$$

This is the required bilinear transformation.

Let z_0 be the fixed point of the transformation such that $z_0 = f(z_0)$.

We have, $w = f(z) = -\frac{1}{z}$

$$\therefore \qquad z_0 = -\frac{1}{z_0}$$

$$\Rightarrow \qquad z_0^{\,2} = -1$$

$$\Rightarrow \qquad z_0 = \pm\sqrt{-1}$$

$$\Rightarrow \qquad z_0 = \pm i$$

Example 2.1.3: Find the bilinear transformation that maps the points 0, $-i$, -1 onto the points i, 1, 0.

Solution: **(VTU 2005, 2008, 2011, 2015)**

Let $z_1 = 0,\ z_2 = -i,\ z_3 = -1$ and $w_1 = i,\ w_2 = 1,\ w_3 = 0$

Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

Substitute the given set of points in $\dfrac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \dfrac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$$\Rightarrow \qquad \frac{(w-i)(1-0)}{(i-1)(0-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

$$\Rightarrow \qquad \frac{(w-i)}{(1-i)(w)} = -\frac{(z)(1-i)}{i(1+z)}$$

$$\Rightarrow \qquad \frac{(w-i)}{w} = -\left(\frac{z}{1+z}\right)\frac{(1-i)^2}{i}$$

$$\Rightarrow \qquad \frac{(w-i)}{w} = \frac{2z}{1+z}$$

$$\Rightarrow \qquad (w-i)(1+z) = 2wz$$

$$\Rightarrow \qquad w + wz - i - iz = 2wz$$

$$\Rightarrow \qquad w - wz = i + iz$$

$$\Rightarrow \qquad w(1-z) = i(1+z)$$

$$\Rightarrow \qquad w = i\left(\frac{1+z}{1-z}\right)$$

This is the required bilinear transformation.

Example 2.1.4: Find the bilinear transformation that maps the points 1, *i*, J1 onto the points 0, 1, ã.

Solution: **(VTU 2006, 2009, 2012, 2016)**

Let $z_1 = 1,\ z_2 = i,\ z_3 = -1$ and $w_1 = 0,\ w_2 = 1,\ w_3 = \infty$

Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

Substitute the given set of points in $\dfrac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \dfrac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$$\Rightarrow \qquad \frac{(w-w_1)w_3\left(\dfrac{w_2}{w_3}-1\right)}{(w_1-w_2)w_3\left(1-\dfrac{w}{w_3}\right)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \qquad \frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$\Rightarrow \qquad w = \dfrac{1-z}{1+z}\left(\dfrac{1+i}{1-i}\right)$

$\Rightarrow \qquad w = i\left(\dfrac{1-z}{1+z}\right) \qquad \left[\because \dfrac{1+i}{1-i} = i\right]$

This is the required bilinear transformation.

Example 2.1.5: Find the bilinear transformation that maps the points 0, *i*, ã onto the points 1, J *i*, J1. And also find its fixed points.

Solution: **(VTU 2007, 2014, 2017)**

Let $z_1 = 0,\ z_2 = i,\ z_3 = \infty$ and $w_1 = 1,\ w_2 = -i,\ w_3 = -1$

Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

Substitute the given set of points in $\dfrac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \dfrac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$\Rightarrow \qquad \dfrac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \dfrac{(z-z_1)z_3\left(\dfrac{z_2}{z_3}-1\right)}{(z_1-z_2)z_3\left(1-\dfrac{z}{z_3}\right)}$

$\Rightarrow \qquad \dfrac{(w-1)(-i+1)}{(1+i)(-1-w)} = \dfrac{(z-0)(0-1)}{(0-i)(1-0)}$

$\Rightarrow \qquad \dfrac{(1-w)}{(1+w)}\left(\dfrac{1-i}{1+i}\right) = \dfrac{z}{i}$

$\Rightarrow \qquad \dfrac{(1-w)}{(1+w)}(-i) = \dfrac{z}{i} \qquad \left[\because \dfrac{1-i}{1+i} = -i\right]$

$\Rightarrow \qquad \dfrac{(1-w)}{(1+w)} = z$

$\Rightarrow \qquad 1-w = z+wz$

$\Rightarrow \qquad 1-z = w(1+z)$

$\Rightarrow \qquad w = \dfrac{1-z}{1+z}$

This is the required bilinear transformation.

Let z_0 be the fixed point of the transformation such that $z_0 = f(z_0)$.

We have, $w = f(z) = \dfrac{1-z}{1+z}$

$$\therefore \quad z_0 = \frac{1-z_0}{1+z_0}$$

$$\Rightarrow \qquad z_0 + z_0{}^2 = 1 - z_0$$

$$\Rightarrow \qquad z_0{}^2 + 2z_0 - 1 = 0$$

$$\Rightarrow \qquad z_0 = -1 \pm \sqrt{2}$$

Example 2.1.6: Find the bilinear transformation that maps the points 1, *i*, J1 onto the points 2, *i*, J2.

Solution: **(VTU 2006, 2011, 2016)**

Let $z_1 = 1,\ z_2 = i,\ z_3 = -1$ and $w_1 = 2,\ w_2 = i,\ w_3 = -2$

Let $w = \dfrac{az+b}{cz+d}$ be the required bilinear transformation.

Substitute the given set of points in $\dfrac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \dfrac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$$\Rightarrow \qquad \frac{(w-2)(i+2)}{(2-i)(-2-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\Rightarrow \qquad \frac{(w-2)}{(w+2)}\frac{(2+i)}{(2-i)} = \frac{(z-1)}{(z+1)}\frac{(1+i)}{(1-i)}$$

$$\Rightarrow \qquad \frac{(w-2)}{(w+2)} = \frac{(z-1)}{(z+1)}\frac{(3+i)}{(3-i)}$$

$$\Rightarrow \qquad \frac{(w-2)}{(w+2)} = \sim \qquad \text{where } \sim = \frac{(z-1)}{(z+1)}\frac{(3+i)}{(3-i)}$$

$$\Rightarrow \qquad w - 2 = w\sim + 2\sim$$

$$\Rightarrow \qquad w - w\sim = 2 + 2\sim$$

$\Rightarrow \qquad w(1-\sim)=2(1+\sim)$

$$\Rightarrow \qquad w=\frac{2(1+\sim)}{(1-\sim)}$$

$$\Rightarrow \qquad w=\frac{2\left(1+\dfrac{(z-1)}{(z+1)}\dfrac{(3+i)}{(3-i)}\right)}{\left(1-\dfrac{(z-1)}{(z+1)}\dfrac{(3+i)}{(3-i)}\right)}=\frac{2\left[(z+1)(3-i)+(z-1)(3+i)\right]}{(z+1)(3-i)-(z-1)(3+i)}$$

$$\Rightarrow \qquad w=\frac{2\left[3z-zi+3-i+3z+zi-3-i\right]}{3z-zi+3-i-3z-zi+3+i}$$

$$\Rightarrow \qquad w=\frac{2\left[6z-2i\right]}{-2zi+6}=\frac{6z-2i}{3-zi}$$

$$\Rightarrow \qquad w=\frac{6z-2i}{3-zi}\times\frac{3+zi}{3+zi}=\frac{18z+6z^2i-6i+2z}{3+z^2}$$

$$\Rightarrow \qquad w=\frac{20z+6i\left(z^2-1\right)}{3+z^2}$$

This is the required bilinear transformation.

EXERCISE 2.1

1. Discuss the transformation $w = e^z$ **(VTU 2011, 2013, 2019)**
2. Discuss the transformation $w = z^2$ **(VTU 2011, 2013, 2015)**
3. Discuss the transformation $w = z + \frac{1}{z}, z \neq 0$ **(VTU 2010, 2012, 2014)**
4. Find the bilinear transformation which maps the points 1, i,-1 from z- plane to 2, i,-2 into w-plane. Also fixed the points. **(VTU 2010, 2012, 2014)**
5. Find the bilinear transformations that transform the points $z_1 = i, z_2 = 1, z_3 = -1$ on to the points $w_1 = 1, w_2 = 0, w_3 = \infty$ **(VTU 2011, 2013, 2015)**
6. Find the bilinear transformations that transform the points $z_1 = \infty,\ \ z_2 = i, z_3 = 0$ onto the points $w_1 = -1,\ w_2 = -i, w_3 = 1$. **(VTU 2011, 2019, 2015)**

7. Find the bilinear transformation which maps the points $z = 0,1,\infty$ onto the points $\omega = -5,-1,3$ respectively. Also find the invariant point.
(VTU 2011, 2013, 2015)
8. Find the bilinear transformation that transforms the points $z_1 = -1, z_2 = 0,\ z_3 = 1$ onto the points $w_1 = 0, w_2 = i, w_3 = 3i$. **(VTU 2018)**
9. Find the bilinear transformation which maps the points $z = 1,1,-1$ on to the points $\omega = i, 0, -i$ **(VTU 2010, 2012, 2014)**
10. Find the bilinear transformation which maps $Z = \infty, i, 0$ into $\omega = -1. -i, 1$. Also find the fixed points of the transformation. **(VTU 2012, 2014, 2017)**

2.5 Applications of Bilinear transformation

- The bilinear transform is used in digital signal processing and discrete-time control theory to transform continuous-time system representations to discrete-time and vice versa.
- The bilinear transform is a special case of a conformal mapping, often used to convert a transfer function of a linear, time-invariant (LTI) filter in the continuous-time domain (often called an analog filter) to a transfer function of a linear, shift-invariant filter in the discrete-time domain (often called a digital filter although there are analog filters constructed with switched capacitors that are discrete-time filters).
- Application of the bilinear transformation to broad-band microwave amplifier design. A microwave amplifier will give maximum gain when it is fed from a signal source which presents impedance to the amplifier which is, in general, different from the characteristic impedance of the feeder. A similar situation arises at the output. Further, if noise figure is to be minimized, input and output impedances are required which again are not the characteristic impedances of the input or output lines nor are they the same impedances required to maximize the gain. It is well known that reflection coefficients along a lossless transmission line system are related by bilinear transformations. In such transformations, circles transform into other circles or (exceptionally) into straight lines. Therefore it is to be expected that the unknown loci will be another family of circles.

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