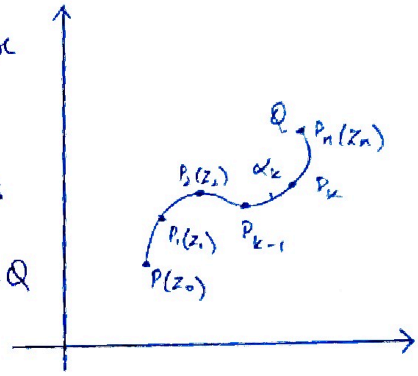
Complex Integration.

Complex line integrals.

Consider a continuous function $f(z)$ of the complex variable $z = x+iy$ defined at all points of a curve C extending from P to Q. Divide the curve C into n parts by arbitrarily taking points $P = P(z_0), P_1(z_1), P_2(z_2) \dots P_n(z_n) = Q$ on the curve C.



Let α_k be any point on the arc of the curve from P_{k-1} to P_k and let $\delta z_k = z_k - z_{k-1}$ where $k = 1, 2, 3, \dots n$.

Then $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(\alpha_k)\delta z_k$ where $\max |\delta z_k| \to 0$ as $n \to \infty$ is defined as the complex line integral along the path C usually denoted by $\int_C f(z)\,dz$.

If C is a simple closed curve the notation $\oint_C f(z)\,dz$ is also used.

Properties of Complex Integral.

- If $-C$ denotes the curve traversed from Q to P then

$$\int_{-C} f(z)dz = -\int_C f(z)\,dz.$$

- If C is split into a number of parts $C_1, C_2, C_3 \dots$ then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \dots$$

- If λ_1 and λ_2 are constants then

$$\int_C [\lambda_1 f_1(z) \pm \lambda_2 f_2(z)]\,dz = \lambda_1 \int_C f_1(z)\,dz \pm \lambda_2 \int_C f_2(z)\,dz$$

Line integral of complex valued function.

Let $f(z) = u(x,y) + iv(x,y)$ be a complex valued function defined over region R and C be a curve in the region. Then

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

Problems:

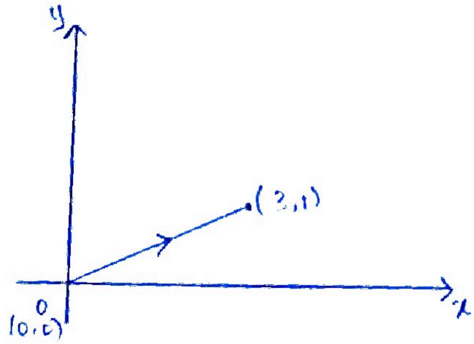
1. Evaluate $\int_C z^2\,dz$

(a) Along the straight line from $z=0$ to $z=3+i$

(b) Along the curve made up of two line segments, one from $z=0$ to $z=3$ and another from $z=3$ to $z=3+i$

Soln:

(a)
$$\int_C z^2\,dz = \int_0^{3+i} z^2\,dz$$



Here z varies from 0 to $3+i$ means that (x,y) varies from $(0,0)$ to $(3,1)$. The equation of the line joining $(0,0)$ and $(3,1)$ is given by

$$\frac{y-0}{x-0} = \frac{1-0}{3-0} \quad \text{or} \quad y = \frac{x}{3}$$

Further $z^2 = (x+iy)^2 = x^2-y^2+2ixy$ and $dz = dx + idy$.

$$\int_C z^2\,dz = \int_{(0,0)}^{(3,1)} (x^2-y^2+2ixy)(dx+idy)$$

$$= \int_{(0,0)}^{(3,1)} (x^2-y^2)\,dx - 2xy\,dy + i\int_{(0,0)}^{(3,1)} 2xy\,dx + (x^2-y^2)\,dy$$

We have $y = \frac{x}{3}$ or $x = 3y$. and so $dx = 3dy$

$$= \int_{y=0}^{1} (9y^2-y^2)\,3dy - 6y^2dy + i\int_{y=0}^{1} 18y^2dy + (9y^2-y^2)\,dy$$

$$= \int_{y=0}^{1} (24y^2 - 6y^2)\,dy + i\int_{y=0}^{1} 26\,y^2\,dy$$

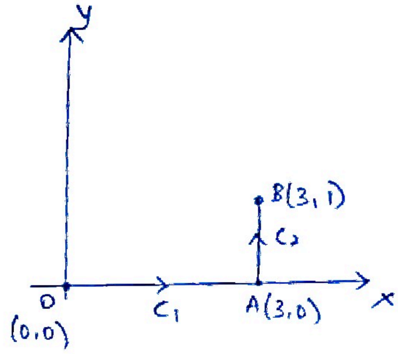
$$= 18\left[\frac{y^3}{3}\right]_0^1 + i\,26\left[\frac{y^3}{3}\right]_0^1$$

$$= 6 + \frac{26}{3}i$$

$\therefore$ Along the given path, $\int_C z^2\,dz = 6 + \frac{26}{3}i$

(b) Segments from $z=0$ to $z=3$ and $z=3$ to $z=3+i$ means that (x,y) varies from $(0,0)$ to $(3,0)$ and then from $(3,0)$ to $(3,1)$.

$$\int_c z^2 dz = \int_{c_1} z^2 dz + \int_{c_2} z^2 dz \quad \text{---} ①$$



Along C_1 : $y=0 \Rightarrow dy=0$

x varies from 0 to 3.

$z^2 dz$ becomes $x^2 dx$

$$\int_{c_1} z^2 dx = \int_{x=0}^{3} x^2 dx = \left[\frac{x^3}{3}\right]_0^3 = 9$$

Along C_2 : $x=3 \Rightarrow dx=0$

y varies from 0 to 1.

$z^2 dz$ becomes $(3+iy)^2\, i dy$

$$\int_{c_2} z^2 dz = \int_0^1 (3+iy)^2\, i dy$$

$$= \int_0^1 \left[(9-y^2+6iy)\, i\right] dy$$

$$= i\left[9y - \frac{y^3}{3} + 6i\frac{y^2}{2}\right]_0^1$$

$$= i\left(9-\frac{1}{3}+3i\right) = 9i - \frac{i}{3} - 3$$

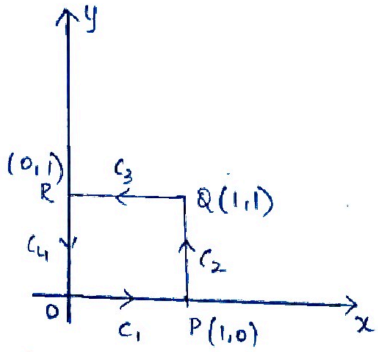
$\therefore$ ① becomes : $\displaystyle\int_c z^2 dz = 9 + 9i - \frac{i}{3} - 3$

$$= 6 + \frac{26}{3} i$$

$\therefore$ Along the given path, $\displaystyle\int_c z^2 dz = 6 + \frac{26}{3} i$.

2. Evaluate $\int_C |z|^2 dz$ where C is a square with the following vertices $(0,0), (1,0), (1,1), (0,1)$.

Soln:
$$\int_C |z|^2 dz = \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz \quad \text{---(1)}$$



We have $|z|^2 dz = (x^2+y^2)(dx+idy)$

Along OP : $y=0 \Rightarrow dy=0$
x varies from 0 to 1.
$|z|^2 dz = x^2 dx$

$$\int_{C_1} |z|^2 dz = \int_{x=0}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Along PQ : $x=1 \Rightarrow dx=0$
y varies from 0 to 1.
$|z|^2 dz = (1+y^2)\, i dy$

$$\int_{C_2} |z|^2 dz = i\int_{y=0}^{1} (1+y^2) dy = i\left[y+\frac{y^3}{3}\right]_0^1 = i\left(1+\frac{1}{3}\right)$$

Along QR : $y=1 \Rightarrow dy=0$
x varies from 1 to 0
$|z|^2 dz = (x^2+1)\, dx$

$$\int_{C_3} |z|^2 dz = \int_{x=1}^{0} (x^2+1) dx = \left[\frac{x^3}{3}+x\right]_1^0 = -\frac{1}{3}-1$$

Along RO : $x=0 \Rightarrow dx=0$
y varies from 1 to 0.
$|z|^2 dz = y^2\, i dy$

$$\int_{C_4} |z|^2 dz = i\int_{y=1}^{0} y^2 dy = i\left[\frac{y^3}{3}\right]_1^0 = i\left(-\frac{1}{3}\right)$$

$\therefore$ (1) : $$\int_C |z|^2 dz = \cancel{\frac{1}{3}} + i + \cancel{\frac{i}{3}} - \cancel{\frac{1}{3}} - 1 - \cancel{\frac{i}{3}}$$
$$= -1 + i \text{ along the given path.}$$

3. Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$ along the following paths

(a) the parabola $x=2t$, $y=t^2+3$

(b) the straight line from $(0,3)$ to $(2,4)$.

Soln: (a) x varies from 0 to 2 and hence

$$\left.\begin{array}{l} \text{if } x=0,\ 2t=0 \Rightarrow t=0 \\ \text{if } x=2,\ 2t=2 \Rightarrow t=1 \end{array}\right\} \Rightarrow t \text{ varies from } 0 \text{ to } 1.$$

$$I = \int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$$

We have : $x=2t$ $y=t^2+3$

$dx = 2dt$ $dy = 2t\,dt$

$$\therefore I = \int_0^1 \left(2(t^2+3)+4t^2\right) 2dt + \left(3(2t)-(t^2+3)\right) 2t\,dt$$

$$= \int_0^1 2(2t^2+6+4t^2) + \left((6t-t^2-3)2t\right) dt$$

$$= \int_0^1 (24t^2 - 2t^3 - 6t + 12)\,dt$$

$$= 24\left[\frac{t^3}{3}\right]_0^1 - 2\left[\frac{t^4}{4}\right]_0^1 - 6\left[\frac{t^2}{2}\right]_0^1 + 12\left[t\right]_0^1$$

$$= 8 - \frac{1}{2} - 3 + 12 = \frac{33}{2} \text{ along the given path.}$$

(b) Equation along straight line joining $(0,3)$ and $(2,4)$

$$\frac{y-3}{x-0} = \frac{4-3}{2-0} \Rightarrow \begin{array}{l} x = 2y-6 \\ dx = 2dy \end{array}$$

$$\therefore I = \int_{y=3}^{4} \left[2y+(2y-6)^2\right] 2dy + \left[3(2y-6)-y\right] dy$$

$$= \int_3^4 \left[4y + (4y^2+36-24y)2\right] dy + \left[6y-18-y\right] dy$$

$$= \int_3^4 (8y^2 - 44y + 72 + 6y - 18 - y)\,dy$$

$$= \int_3^4 (8y^2 - 39y + 54)\,dy$$

$$= 8\left[\frac{y^3}{3}\right]_3^4 - 39\left[\frac{y^2}{2}\right]_3^4 + 54\left[y\right]_3^4 = \frac{296}{3} - \frac{273}{2} + 54 = \frac{97}{6} \text{ along the given path.}$$

4. Evaluate : $\int_0^{2+i} (\bar{z})^2 \, dz$ along :

(a) the line $x = 2y$ (b) the real axis upto 2 and then vertically to $2+i$.

Soln: $I = \int_0^{2+i} (\bar{z})^2 \, dz$

We have $(\bar{z})^2 = (x-iy)^2 = (x^2-y^2) - i(2xy)$

$dz = dx + idy$

(a) Along $x = 2y$.

$dx = 2dy$

z varies from 0 to $2+i$.

$\Rightarrow (x,y)$ varies from $(0,0)$ to $(2,1)$

$$\therefore \quad I = \int_{y=0}^{1} \left((4y^2 - y^2) - i(4y^2)\right)(2dy + i\,dy)$$

$$= \int_{y=0}^{1} (3-4i)y^2 (2+i)\, dy$$

$$= \int_{y=0}^{1} (10-5i)\, y^2 dy = (10-5i)\left[\frac{y^3}{3}\right]_0^1 = \frac{10-5i}{3}$$ along the given path.

(b) $I = \int_{C_1} (\bar{z})^2 dz + \int_{C_2} (\bar{z})^2 dz$

Along OA : $y = 0 \Rightarrow dy = 0$

x varies from 0 to 2.

$(\bar{z})^2 dz = x^2 dx$

$$\int_{C_1} (\bar{z})^2 dz = \int_{x=0}^{2} x^2 dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

Along AB : $x = 2 \Rightarrow dx = 0$

y varies from 0 to 1.

$(\bar{z})^2 dz = \left((4-y^2) - 4iy\right) i dy$

$$\int_{C_2} (\bar{z})^2 dz = i \int_{y=0}^{1} (4 - y^2 - 4iy)\, dy$$

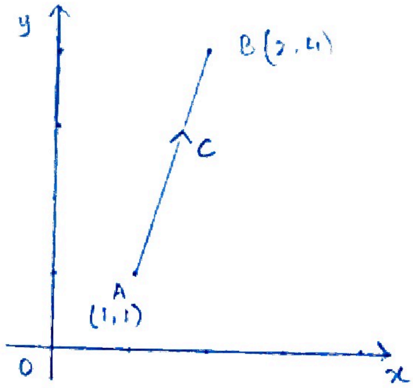
$$= 4i\left[y\right]_0^1 - i\left[\frac{y^3}{3}\right]_0^1 + 4\left[\frac{y^2}{2}\right]_0^1$$

$$= 4i - \frac{i}{3} + 2 = \frac{11i}{3} + 2$$

$$\therefore \quad I = \frac{8}{3} + \frac{11i}{3} + 2$$

$$= \frac{1}{3}(14 + 11i)$$

5. Evaluate $\int_A^B (x^2+ixy)\,dz$ where $A(1,1)$ and $B(2,4)$ along
(i) Straight line AB.
(ii) Curve $C : x = t,\ y = t^2$.



Soln: Equation of line joining $(1,1)$ and $(2,4)$.

$$\frac{y-1}{x-1} = \frac{4-1}{2-1}$$

$$y-1 = 3(x-1)$$

$$y = 3x-2.$$

(i) Straight line AB :

$$y = 3x-2$$

$$dy = 3dx$$

x varies from 1 to 2.

$$I = \int_A^B (x^2+ixy)\,dz = \int_A^B (x^2+ixy)(dx+idy)$$

$$= \int_{x=1}^{2} \left(x^2+ix(3x-2)\right)(dx+3i\,dx)$$

$$= \int_{x=1}^{2} (x^2+3x^2i-2ix)(1+3i)\,dx$$

$$= (1+3i)\left[\frac{x^3}{3}+x^3i-ix^2\right]_1^2$$

$$= (1+3i)\left[\left(\frac{8}{3}+8i-4i\right)-\left(\frac{1}{3}+i-i\right)\right]$$

$$= (1+3i)\left(\frac{7}{3}+4i\right) = \frac{7}{3}+4i+7i-12 = -\frac{29}{3}+11i$$ along the given path.

(ii) Curve $C : x = t \quad y = t^2$

$$dx = dt \quad dy = 2t\,dt$$

when $\left.\begin{array}{l} x=1,\ t=1 \\ x=2,\ t=2 \end{array}\right\}$ t varies from 1 to 2.

$$I = \int_A^B (x^2+ixy)\,dz = \int_{t=1}^{2} \left(t^2+it(t^2)\right)(dt+2it\,dt)$$

$$= \int_1^2 (t^2+it^3)(1+2it)\,dt$$

$$I = \int_1^2 (t^2 + it^3)(1 + 2it)\,dt$$

$$= \int_1^2 \left(t^2 + it^3 + 2it^3 + 2i^2t^4\right) dt$$

$$= \int_1^2 \left(t^2 + 3it^3 - 2t^4\right) dt$$

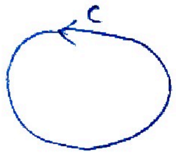
$$= \left[\frac{t^3}{3} + \frac{3it^4}{4} - \frac{2t^5}{5}\right]_1^2$$

$$= \left[\frac{8}{3} + 3i\,\frac{\cancel{16}^{\,4}}{\cancel{4}} - \frac{2}{5} \times 32\right] - \left[\frac{1}{3} + \frac{3i}{4} - \frac{2}{5}\right]$$

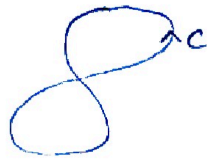
$$= -\frac{151}{15} + \frac{45}{4}i \text{ along the given path.}$$

Simple closed curve :

A curve C is said to be simple closed curve if it does not cross itself otherwise is said to be multiple closed curve.



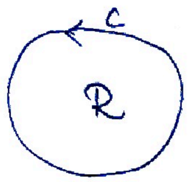
Simple closed Curve



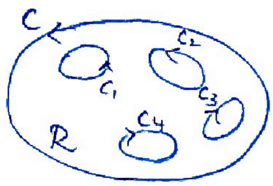
Multiple closed curve.

Simple connected region :

A region R is said to be simple connected region if it does not contain holes or breaks within it. Otherwise it is multiple connected region.

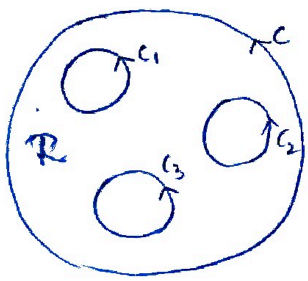


R: Simple connected region

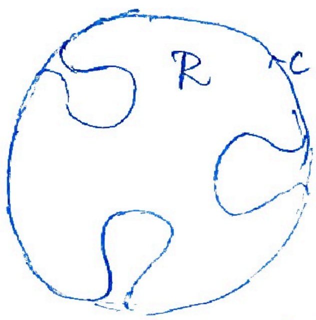


R : Multiple connected region.

<u>Note</u>: Multiple connected region can be converted into simple connected region by giving one or more cross cuts.



Multiple connected region



Simple connected region.

Cauchy's theorem.

Statement: If $f(z)$ is analytic at all points inside and on a simple closed curve c then $\int_c f(z)\,dz = 0$.

Proof: Let $f(z) = u + iv$

Then $\int_c f(z)\,dz = \int_c (u+iv)(dx + idy)$

$$\int_c f(z)\,dz = \int_c (udx - vdy) + i\int_c (vdx + udy) \quad \text{——} ①$$

Green's theorem states that in a plane if $M(x,y)$ and $N(x,y)$ are two real valued functions having continuous partial derivatives in a region R bounded by the curve c then

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Then $\int_c udx + (-v)dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

And $\int_c vdx + udy = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$

Substituting in ① :

$$\int_c f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

By Cauchy Riemann equations : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

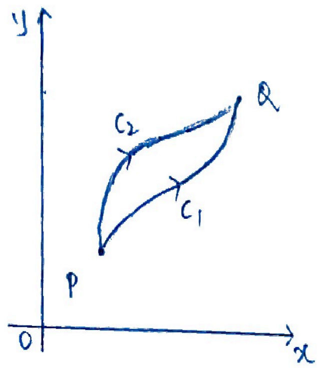
$$\int_c f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

$$\therefore \int_c f(z)\,dz = 0$$

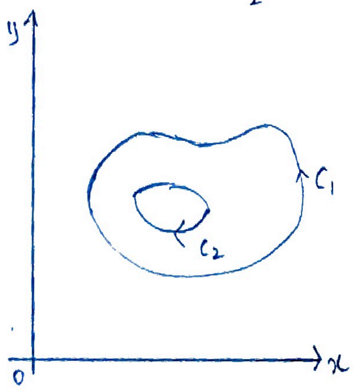
Consequences of Cauchy's theorem.

- If $f(z)$ is analytic in a region R and if P and Q are any two points in it then $\int_P^Q f(z)\,dz$ is independent of the path joining P and Q.

  That is $\int_P^Q f(z)\,dz$ is same for all curves joining P and Q.

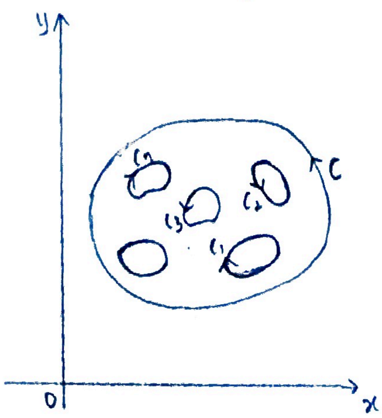


- If C_1, C_2 are two simple closed curves such that C_2 lies entirely within C_1 and if $f(z)$ is analytic on C_1, C_2 and in the region bounded by C_1, C_2 then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$



- If C is a simple closed curve enclosing non overlapping simple closed curves $C_1, C_2, C_3 \ldots C_n$ and if $f(z)$ is analytic in the region between C and those curves then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$



Cauchy's Integral Formula :

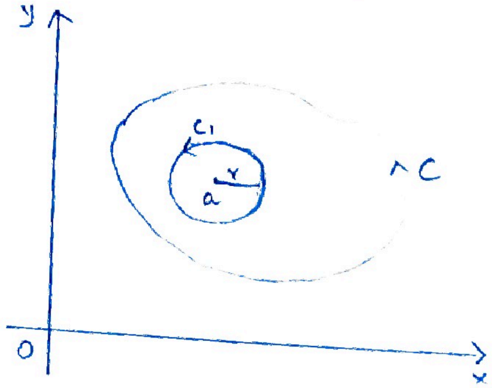
Statement : If $f(z)$ is analytic within and on a simple closed curve C and 'a' is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Proof: Given $f(z)$ is an analytic function inside and on a simple closed curve C except at 'a'.

Since 'a' is a point within C, we shall enclose it by a circle C_1 with $z=a$ as centre and r as radius such that C_1 lies entirely within C.

$\therefore$ The function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the region between C and C_1.



As a consequence of Cauchy's theorem,

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \quad \text{--- (1)}$$

The equation of C_1 can be written as :

$|z-a| = r$

$\Rightarrow z-a = r.e^{i\theta}$

or $z = a + re^{i\theta}$

$dz = rie^{i\theta}\,d\theta$

$r \to$ constant

$\theta \to$ varies from 0 to 2π.

Substituting in (1) :

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})}{r.e^{i\theta}}\,.rie^{i\theta}\,d\theta = i\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

As $r \to 0$

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta = i f(a)\left[\theta\right]_0^{2\pi} = 2\pi i f(a)$$

$$\therefore \boxed{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz}$$

Generalized Cauchy's Integral formula.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

where $f^{(n)}(a)$ denotes the nth derivative of $f(z)$ at $z=a$.

When $n=1$

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$$

When $n=2$

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz$$

When $n=3$

$$f'''(a) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-a)^4} dz$$

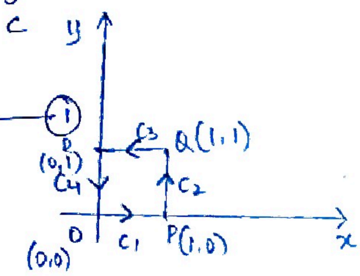
Problems:

1. Verify Cauchy's theorem for the function $f(z) = z^2$ where C is the square having vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

Soln: C is the square $OPQR$ and by Cauchy's theorem $\int_C f(z)\,dz = 0$

$\therefore$ We need to show:

$$\int_{C_1} z^2 dz + \int_{C_2} z^2 dz + \int_{C_3} z^2 dz + \int_{C_4} z^2 dz = 0 \qquad (1)$$



Along C_1: $y = 0 \Rightarrow dy = 0$

x varies from 0 to 1.

$z^2 dz = x^2 dx$

$$\int_{C_1} z^2 dz = \int_{x=0}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Along C_2: $x = 1 \Rightarrow dx = 0$

y varies from 0 to 1.

$z^2 dz = (1+iy)^2 i\,dy$

$$\int_{C_2} z^2 dz = i\int_{y=0}^{1} (1 - y^2 + 2iy)\,dy = i\left[y - \frac{y^3}{3} + iy^2\right]_0^1$$

$$= i\left(1 - \frac{1}{3} + i\right)$$

Along C_3: $y = 1 \Rightarrow dy = 0$

x varies from 1 to 0.

$z^2 dz = (x+i)^2 dx$

$$\int_{C_3} z^2 dz = \int_{x=1}^{0} (x+i)^2 dx = \int_{x=1}^{0} (x^2 - 1 + 2ix)\,dx$$

$$= \left[\frac{x^3}{3} - x + ix^2\right]_1^0 = -\frac{1}{3} + 1 - i$$

Along C_4: $x = 0 \Rightarrow dx = 0$

y varies from 1 to 0.

$z^2 dz = (iy)^2 i\,dy$

$$\int_{C_4} z^2 dz = -i\int_{y=1}^{0} y^2 dy = -i\left[\frac{y^3}{3}\right]_1^0 = -i\left(-\frac{1}{3}\right) = \frac{i}{3}$$

Substituting in LHS of (1):

$$\frac{1}{3} + i\left(1 - \frac{1}{3} + i\right) + \left(-\frac{1}{3} + 1 - i\right) + \frac{i}{3}$$

$$= \cancel{\frac{1}{3}} + \cancel{i} - \cancel{\frac{i}{3}} - 1 - \cancel{\frac{1}{3}} + \cancel{1} - \cancel{i} + \cancel{\frac{i}{3}}$$

$$= 0.$$

$$\therefore \int_C z^2 dz = 0.$$

Hence Cauchy's theorem is verified.

2. Evaluate : $\int_C \frac{e^z}{z+i\pi}\,dz$ over each of the following contours C:

(a) $|z| = 2\pi$ (b) $|z-1| = 1$. (c) $|z| = \pi/2$.

Soln: $\int_C \frac{e^z}{z+i\pi}\,dz = \int_C \frac{e^z}{z-(-i\pi)}\,dz$ which is of the form $\int_C \frac{f(z)}{z-a}\,dz$

Here $f(z) = e^z$ $a = -i\pi$

(a) $|z| = 2\pi$ is a circle with centre $(0,0)$ and radius 2π.

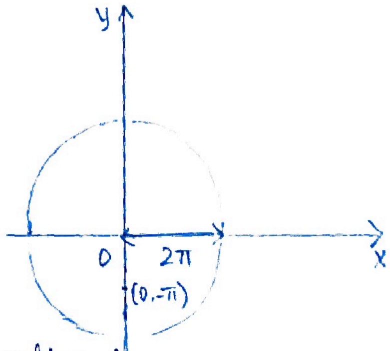
The point $z = a = -i\pi$ is a point $(0,-\pi)$ lies within the circle $|z| = 2\pi$.

Cauchy's integral formula $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$

$$\therefore \int_C \frac{e^z}{z+i\pi}\,dz = 2\pi i\, f(-i\pi) = 2\pi i \cdot e^{-i\pi}$$

$$= 2\pi i\,(\cos\pi - i\sin\pi)$$

$$= -2\pi i$$



(b) $|z-1| = 1$ is a circle with centre at $z = a = 1$ and radius 1.

Circle with centre $(1,0)$ and radius 1.

The point $P(0,-\pi)$ lies outside the circle $|z-1| = 1$ and hence by

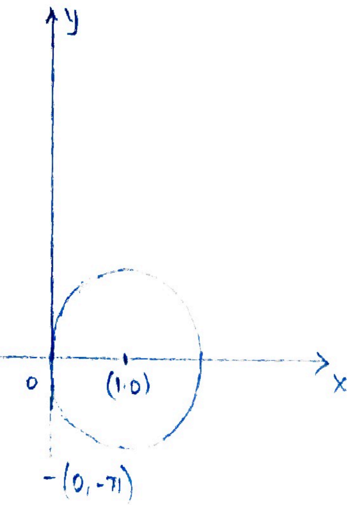
Cauchy's theorem $\int_C \frac{e^z}{z+i\pi}\,dz = 0$.

(c) $|z| = \pi/2$ is a circle with centre at Origin and radius $\pi/2$.

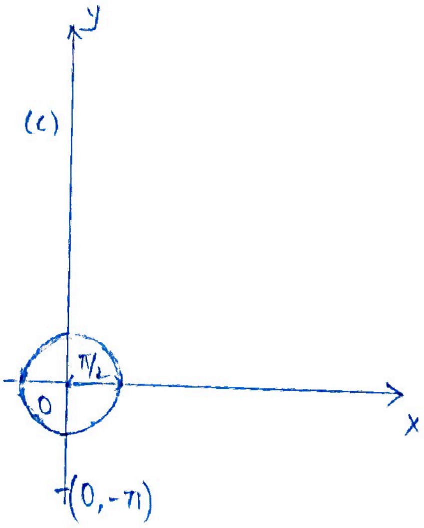
The point $P(0,-\pi)$ lies outside the circle $|z| = \pi/2$ and $\frac{e^z}{z+i\pi}$ is analytic inside and on the circle $|z| = \pi/2$.

By Cauchy's theorem $\int_C \frac{e^z}{z+i\pi}\,dz = 0$

(b)

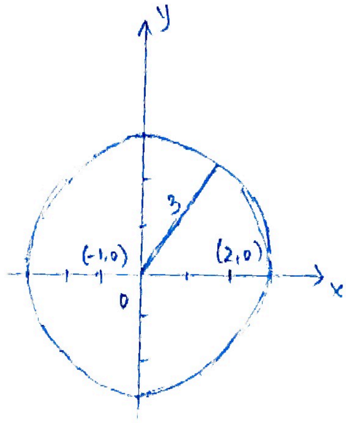


(c)



3. Evaluate $\int_C \frac{e^{2z}}{(z+1)(z-2)}\,dz$ where C is the circle $|z| = 3$.

Soln: The points $a=-1$, $a=2$ i.e. $(-1,0)$, $(2,0)$ lies inside $|z|=3$.



By partial fractions:

$$\frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z+1)$$

when $z=2$ $\quad 1 = 3B \quad B = 1/3$

when $z=-1$ $\quad 1 = -3A \quad A = -1/3$

Hence $\frac{1}{(z+1)(z-2)} = \frac{-1}{3}\frac{1}{z+1} + \frac{1}{3}\frac{1}{z-2}$

$$\therefore \frac{e^{2z}}{(z+1)(z-2)} = \frac{-1}{3}\frac{e^{2z}}{z+1} + \frac{1}{3}\frac{e^{2z}}{z-2}$$

$$\int_C \frac{e^{2z}}{(z+1)(z-2)}\,dz = \frac{1}{3}\left[\int_C \frac{e^{2z}}{z-2}\,dz - \int_C \frac{e^{2z}}{z+1}\,dz\right] \quad \text{--- (1)}$$

Cauchy's Integral formula:

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\int_C \frac{e^{2z}}{z-2}\,dz = 2\pi i\, f(2) = 2\pi i\, e^4$$

$$\int_C \frac{e^{2z}}{z+1}\,dz = 2\pi i\, f(-1) = 2\pi i\, e^{-2}$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)(z-2)}\,dz = \frac{1}{3}\left[2\pi i\, e^4 - \frac{2\pi i}{e^2}\right]$$

$$= \frac{2\pi i}{3}\left[e^4 - \frac{1}{e^2}\right]$$

4. Evaluate $\int_C \frac{dz}{z^2-4}$ over the following curves C.

(a) $C : |z| = 1$ (b) $C : |z| = 3$ (c) $C : |z+2| = 1$.

Soln:

$$\frac{1}{z^2-4} = \frac{1}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$$

$$1 = A(z+2) + B(z-2)$$

when $z = 2$ $1 = A(4)$. $A = 1/4$

when $z = -2$ $1 = B(-4)$ $B = -1/4$

$$\therefore \int_C \frac{dz}{z^2-4} = \frac{1}{4}\left[\int_C \frac{dz}{z-2} - \int_C \frac{dz}{z+2}\right] \quad \text{—} ①$$

(a) $C : |z| = 1$ is a circle with centre at origin and radius 1.

$\therefore$ $a = 2, -2$ lie outside C.

By Cauchy's theorem $\int_C \frac{dz}{z^2-4} = 0$

(b) $C : |z| = 3$ is a circle with centre at origin and radius 3.

$a = 2, -2$ lie inside C.

Cauchy's integral formula : $\int_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$

$$\int_C \frac{1}{z-2}\, dz = 2\pi i\, f(2) = 2\pi i \cdot 1 = 2\pi i$$

$$\int_C \frac{1}{z+2}\, dz = 2\pi i\, f(-2) = 2\pi i \cdot 1 = 2\pi i$$

Substituting in ① : $\int_C \frac{dz}{z^2-4} = 0$.

(c) $C : |z+2| = 1$ is a circle with centre $(-2, 0)$ and radius 1.

$a = 2$ $(2,0)$ lies outside the circle and $a = -2$ $(-2,0)$ lies inside the circle.

By Cauchy's theorem $\int_C \frac{dz}{z-2} = 0$

By Cauchy's integral formula : $\int_C \frac{dz}{z-(-2)} = 2\pi i\, f(-2) = 2\pi i(1) = 2\pi i$.

Substituting in ① :

$$\int_C \frac{dz}{z^2-4} = \frac{1}{4}\left[0 - 2\pi i\right] = -\frac{\pi i}{2}$$

5. Evaluate : $\int_C \frac{e^{3z}}{z^2} dz$ over $C : |z| = 1$.

Soln: The point $z=0$ lies within the circle $|z|=1$.

By Generalised Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Taking $f(z) = e^{3z}$ $a=0$ $n=1$ $f'(z) = 3e^{3z}$

$$\int_C \frac{e^{3z}}{z^2} dz = \frac{2\pi i}{1!} f'(0) = 2\pi i\,(3e^{0z}) = 6\pi i.$$

6. Show that $\int_C \frac{\sin^2 z}{(z-\pi/6)^3} dz = \pi i$ where $C : |z| = 1$.

Soln: The point $z = \pi/6$ ($\simeq 0.52$) lies within the circle $|z| = 1$.

By Generalised Cauchy's integral formula.

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Taking $f(z) = \sin^2 z$ $n=2$ $a = \pi/6$

$f'(z) = 2 \sin z \cos z = \sin 2z$

$f''(z) = 2 \cos 2z$ $f''(\pi/6) = 2 \cos \pi/3 = 1$.

$$\therefore \int_C \frac{\sin^2 z}{(z-\pi/6)^3} dz = \frac{2\pi i}{2!} (1) = \pi i.$$

7. Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z| = k > 1$.

Soln: The point $z=-1$ lies within the circle.

By Generalised Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Here $f(z) = e^{2z}$ $a=-1$ $n=3$

$f'(z) = 2e^{2z}$ $f''(z) = 4e^{2z}$ $f'''(z) = 8e^{2z}$

$f'''(-1) = 8e^{-2}$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} (8e^{-2}) = \frac{8\pi i}{3e^2}$$

8. Evaluate $\int_C \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, where C is the circle $|z| = 3$

Soln: Here $f(z) = \sin \pi z^2 + \cos \pi z^2$

$|z| = 3$ is circle with centre as origin and radius 3.

So the points $z = 1$ and $z = 2$ lie within C.

By partial fractions,

$$\frac{1}{(z-1)^2 (z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

$$1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

when $z = 1$ $\quad 1 = B(-1) \quad B = -1$

when $z = 2$ $\quad 1 = C(1) \quad C = 1$

Equating co-eff of z^2 on both sides

$0 = A + C$ or $A = -C \Rightarrow A = -1$.

$$\therefore \frac{1}{(z-1)^2 (z-2)} = \frac{-1}{z-1} + \frac{-1}{(z-1)^2} + \frac{1}{z-2}$$

$$\therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz = -\int_C \frac{f(z)}{z-1} dz - \int_C \frac{f(z)\,dz}{(z-1)^2} + \int_C \frac{f(z)}{z-2} dz$$

$$= I_1 + I_2 + I_3 \quad \text{—— (1)}$$

$I_1 = -\int_C \dfrac{f(z)}{z-1} dz = -2\pi i\, f(1)$ (By Cauchy's integral formula)

$= -2\pi i (\sin \pi + \cos \pi) = 2\pi i$

$I_2 = -\int_C \dfrac{f(z)}{(z-1)^2} dz = -\dfrac{2\pi i}{1!} f'(1)$ (By Generalised Cauchy's integral formula)

$= -2\pi i \left[(\cos \pi z^2 - \sin \pi z^2)\, 2\pi z \right]_{at\ z=1}$

$= -2\pi i (-1) = 4\pi^2 i$

$I_3 = \int_C \dfrac{f(z)}{z-2} dz = 2\pi i\, f(2)$ (By Cauchy's integral formula)

$= 2\pi i (\sin 4\pi + \cos 4\pi) = 2\pi i$

$$\therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz = 2\pi i + 4\pi^2 i + 2\pi i$$

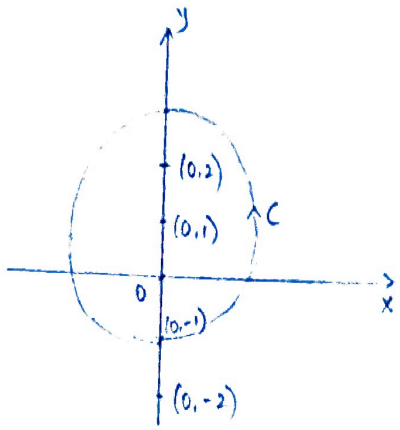
$$= 4\pi i + 4\pi^2 i = 4\pi i (1 + \pi)$$

9. Evaluate $\int_C \frac{dz}{(z^2+4)^2}$ where $C: |z-i| = 2$ by Cauchy's integral formula.

Soln: $C: |z-i| = 2 \rightarrow$ Circle with centre $(0,1)$ and radius 2

$$\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$$

$P_1 = (0,-2)$ $\quad$ $P_2 = (0,2)$



The point $(0,-2)$ does not lie within C.

The point $(0,2)$ lies within C.

$\therefore \frac{1}{(z+2i)^2(z-2i)^2}$ can be written as $\frac{1/(z+2i)^2}{(z-2i)^2}$

$\therefore f(z) = \frac{1}{(z+2i)^2}$

$f'(z) = \frac{-2}{(z+2i)^3}$ $\qquad f'(2i) = \frac{-2}{(4i)^3} = \frac{1}{32i}$

By Generalised Cauchy's integral formula,

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

$$\therefore \int_C \frac{dz}{(z^2+4)^2} = 2\pi i\, f'(2i) = 2\pi i \times \frac{1}{32i} = \frac{\pi}{16}$$

10. Evaluate $\int_C \frac{\sin^6 z}{(z-\pi/6)^3}dz$ where C is the circle $|z| = 1$.

Soln: By Generalised Cauchy's integral formula,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Here $f(z) = \sin^6 z$ $\qquad a = \pi/6 \qquad n = 2$

$f'(z) = 6\sin^5 z \cos z$

$f''(z) = -6\sin^6 z + 30\sin^4 z\cos^2 z$ $\qquad f''(\pi/6) = -6\left(\frac{1}{2}\right)^6 + 30\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{21}{16}$

$$\therefore \int_C \frac{\sin^6 z}{(z-\pi/6)^3}dz = \frac{2\pi i}{2} f''(\pi/6) = \pi i\,\frac{21}{16} = \frac{21\pi i}{16}$$