Modular GPU Programming with Typed Privileges

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Abstract

1 Introduction

CUDA [15] is a low-level, imperative programming language for NVIDIA GPUs. These GPUs are organized into a hierarchy of compute resources. *Threads* are the basic unit of sequential execution; *blocks* are groups of threads that can cooperate through shared scratchpad memory; and the *grid* is the full collection of blocks launched for a GPU kernel. A CUDA program executes in parallel across this hierarchy, but is written from the *perspective of a single thread*.

While operations are specified per-thread, some operations are only semantically valid when executed *collectively* by a group of threads. The `__syncthreads()` primitive, for instance, synchronizes all threads within a block, and will cause a deadlock if executed by a subset of threads within that block. There are many such collective operations, including tensor core instructions , warp shuffle operations , and other kinds of synchronization primitives. These operations, therefore, require programmers to carefully marshal compute resources and coordinate which threads execute which lines of code. As a result, such collective operations *break the illusion* of threads executing independently.

The conceptual clash between regular statements (executed by a single thread), and collective operations (executed by a groups of threads) hinders compositional reasoning. In CUDA, function calls, like all other statements, are per-thread operations. However, a function may contain code that executes a collective operation, creating a gap between the per-thread syntax for its invocation and the cooperative semantics of its execution.

This gap creates tension between abstraction and correctness, a tension apparent even in widely used CUDA libraries that package common functionality via function interfaces . Consider the following snippet of documentation taken directly from CUB [14], a library of parallel primitives. The documentation describes the `BlockReduce` function [4], in which all threads in a block collaboratively apply a reduction operation, such as a maximum or prefix sum, over an array:

> Computes a block-wide reduction for thread$_0$ using the specified binary reduction functor. The first `num_valid` threads each contribute one input element.
>
> - The return value is undefined in threads other than thread$_0$.
> - A subsequent `__syncthreads()` threadblock barrier should be invoked after calling this method if the collective's temporary storage (e.g., `temp_storage`) is to be reused or repurposed.

Through informal documentation, CUB attempts to convey several assumptions implicit in `BlockReduce`'s implementation. First, the function is only well-defined when invoked by all threads in a block. Second, because it accesses memory shared among threads, any subsequent reuse of that storage requires synchronization to ensure all threads have completed their accesses.

In effect, CUB attempts to retrofit CUDA with information about the compute and memory requirements of collective operations. In fact, by prefixing functions with identifiers such as `Block`, CUB creates ad-hoc "namespaces" for different functions that assume similar invariants. However, without a type system to statically enforce these requirements in CUDA, correct usage of this function depends on the user carefully reading and interpreting the documentation.

In this work, we ask the following question: can we provide low-level access to collective operations while statically guaranteeing that they execute with the necessary configuration of compute resources? By reifying these configurations as type-level *privileges*—so named because they dictate which operations are permitted in which contexts—we find that we can. Our key insight is that GPU programmers naturally map computations onto different compute resources, and unlike CUDA, which obscures this mapping, we can expose it in a type system that tracks privileges. This tracking enforces that collective operations are executed with the necessary resources, while still allowing users to access low-level interfaces.

Our approach is a departure from previous work, which attempts to resolve the mismatch between per-thread syntax and collective execution by restricting access to low-level operations. Triton [21], a tile-based, multi-dimensional array language, limits users to programming only at the block level, which prevents programmers from writing the highest-performance kernels. A variety of functional languages, such as Futhark [12] , provide compile-time guarantees through their type systems but are machine-agnostic: they cannot expose low-level control over hardware . Some other efforts, like Descend [13], aim to provide a low-level, safe GPU systems programming language, but lack support for collective operations like tensor cores. As a result, CUDA remains the de-facto standard for writing high-performance kernels on modern GPUs .

We introduce Snap, a new low-level GPU programming language that guarantees correct usage of hardware resources by construction. Inspired by coeffect systems [16] and dependency calculi [1], Snap statically tracks the configurations of compute and memory resources with type-level privileges. We also develop Bundl, a core calculus underpinning Snap, provide formal rules for manipulating the privilege of both code and data, and prove *type-and-privilege safety*. This ensures that Bundl is sound, and that all threads are statically privileged to run the operations they attempt at run time.

A parallel goal, alongside to correctness, is performance. We incorporate modern GPU features such as tensor cores and asynchronous data movement into Snap, and demonstrate that Snap can achieve the same performance as hand-written, highly optimized code on an H100 and a 4070 Ti Super. Our contributions are:

(1) Snap, a low-level GPU programming language with type-level privileges that tracks compute and memory resources (Section 3);
(2) Bundl, a formal model of Snap that tracks privileges in both its type system and operational semantics, along with a soundness theorem guaranteeing that threads only execute operations for which they are statically privileged (Section 4); and
(3) An implementation of Snap (Section 5) that demonstrates that it can support modern GPU features like tensor cores and asynchronous data movement, achieving performance comparable to hand-optimized CUDA implementations (Section 6).

Section 7 discusses related work, and Section 8 discusses the limitations of our approach and outlines future work.

2 Background & Motivation

Before diving into Snap's design, we begin with an overview of the GPU's compute and memory hierarchies and outline the challenges posed by reasoning about them collectively.

2.1 Compute Hierarchy

In CUDA, programmers launch computations that run on thousands of threads. These threads are organized into a *compute hierarchy* that defines how work is distributed and scheduled on the GPU. At the top of this hierarchy is the *grid*, representing all threads launched as part of a single computation. The grid is divided into *blocks*, each containing a user-specified number of threads. A *thread*, in turn, is the finest unit of execution, and it is the machine's basic unit of sequential control.

Users define the behavior of their computation by writing a single program which is replicated across all threads. Users control the behavior of individual threads by reading built-in identifiers like `threadIdx.x`—to determine a thread's position within a block—and `blockIdx.x`—to determine

```
1   int tid = threadIdx.x;
2   if (tid >= 0 && tid< 31){
3     float A[4];
4     float B[2];
5     float C[4] = { 0 };
6    # Populate A with unique values
7     for (int i = 0; i < 4; i++)
8         A[i] = tid * 4 + 1;
9     # Populate B with unique values
10    for (int i = 0; i < 2; i++)
11        B[i] = tid * 4 + 1;
12    # Issue a warp-level tensor-core
13    # operation: D = A * B  + C
14   asm("mma.sync.aligned.m16n8k8..."
15      "{%0, %1, %2, %3},    " /*D*/
16      "{%4, %5, %6, %7},    " /*A*/
17      "{%8, %9},           " /*B*/
18      "{%10, %11, %12, %13};" /*C*/
19      : "=r"(C[0]),"=r"(C[1]), ...
20      : "r"(A[0]), "r"(A[1]), ...
21        "r"(B[0]), "r"(B[1]),
22        "r"(C[0]), "r"(C[1]), ...);}
```

Fig. 1. Invoking a warp-level tensor-core instruction in CUDA. We elide some typecasts necessary to invoke the assembly operation.

```
tid : int @ thread[1] = id();
with group(thread[32]):
  A : float[4] @ thread[1]
  B : float[2] @ thread[1]
  C : float[4] @ thread[1]

  # Populate A with unique values
  for i in range(0, 4, 1):
    A[i] = tid * 4 + 1
    C[i] = 0
  # Populate B with unique values
  for i in range(0, 2, 1):
    B[i] = tid * 4 + 1

  # Issue tensor-core op.
   intrinsic.mma(
     A[0], A[1], A[2], A[3],
     B[0], B[1],
     C[0], C[1], C[2], C[3],
     out=[A[0], A[1], A[2], A[3]])
```

Fig. 2. Invoking a warp-level tensor-core instruction in Snap.

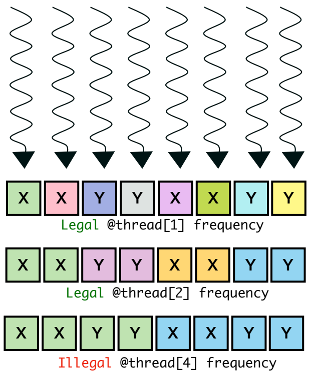
X X Y Y X X Y Y
Legal @thread[1] frequency
X X Y Y X X Y Y
Legal @thread[2] frequency
X X Y Y X X Y Y
Illegal @thread[4] frequency

Fig. 3. The @ syntax represents the frequency at which a variable varies across threads T1–T8. Each @ denotes a coloring of the variable; threads that "view" the variable with the same color must observe the same value.

the block's position within the grid—at runtime. Combining these two IDs, users can assign each thread to its share of the full computation.

A natural and tempting way to interpret these built-in identifiers is to think of them as indices into implicit "parallel `for`-loops" where each iteration is executing simultaneously. While this view is sufficient to understand CUDA's programming model when threads do independent work, it quickly breaks down in the presence of *collective operations*.

Unlike most instructions, which are executed by a single thread, collective operations must be executed collaboratively by a group of threads. For example, a "warp-level" tensor-core operation is only *meaningful* when invoked by a collection of 32 threads—a *warp*—acting together. Consider Figure 1 which shows a tensor-core operation being invoked by a warp on line **14**. For the call, each thread sets up its portion of the operands, and the operation is performed *once* for the entire warp, with the results scattered across participating threads. If the condition on line **2** were instead `tid > 0 && tid < 30`, making fewer than 32 threads reach line **14**, the instruction would be undefined. To make matters worse, the restriction is not only on the number of threads executing the operation, but also on their alignment. In this case, the starting ID of the group of 32 participating threads must be aligned to a multiple of 32. So, if the condition on line **2** were instead `tid > 1 && tid < 33`, the instruction would still be invalid, even though 32 threads would execute it.

Collective operations make reasoning about CUDA programs challenging because they force programmers to track the *convergence behavior* of threads. Specifically, programmers must reason both about how many threads reach a particular point in the program and how those threads are arranged, accounting for alignment. This difficulty is further amplified due to two reasons. First, programs may have multiple *points* of convergence, requiring programmers to mentally track the relative ID of a thread within a logical group as that group changes over the course of a program's execution. Second, threads may be participating in multiple *levels* of convergence within the same program. In our example, we only considered the warp-level tensor-core operation, but there are other collective operations that require convergence at different granularities like the *warp-group*-level tensor-core operations, which must be issued collectively by four warps, or the block-level `__syncthreads()` synchronization primitive, which must be executed by all threads within a block.

Reasoning about collective operations is already error-prone within the context of a single function, but becomes even more difficult when reasoning interprocedurally *across* functions. A callee may assume a certain number of threads or blocks and may structure its computation, including convergence behavior, around that assumption However, such assumptions are not visible in the callee's function interface, which only exposes the input and output types. To invoke that function correctly, users must read documentation, or worse, read the callee's implementation to understand its assumptions, breaking modular reasoning.

In this work, we make the programmer's implicit assumptions about a program's convergence behavior explicit in the program's source. Consider the example in Figure 2 which shows an equivalent rewrite of the CUDA program in Figure 1 using Snap. In Figure 2, the call to the `mma` intrinsic is legal only because it is executed within a `group` of 32 threads, as made explicit on line **2**. Since Snap's `group` construct already enforces both the size and alignment of participating threads, the legality of the `mma` operation is guaranteed by construction.

Moreover, Snap also tracks the *frequency* at which values diverge in *space*. The `@` syntax attached to each variable deceleration denotes the rate at which a value changes across compute resources on the machine. For example, a `thread[1]` value can diverge for every thread, while a `block[1]` can diverge for every block, but not for threads within that block. Intuitively, the `@` construct "colors" a variable across threads, and any threads that share a color are required to agree on the variable's value. Figure 3 illustrates some valid and invalid colorings of a variable.

Using this information, Snap enforces rules for reading from and writing to data. So, if we had attempted to introduce a condition based on `tid` *within* the `group`'s scope, Snap would reject that program at compile time. This is because code executing at a lower frequency cannot read variables that change at a higher frequency than it. We will explain the syntax and rules of legal Snap programs in greater detail in Section 3.

2.2 Memory Hierarchy

GPUs also have a *memory hierarchy*, which, like the compute hierarchy, is organized into three levels. All threads in a grid can read and write from *global memory*, where operands typically reside at the start of a computation and where results are eventually written. Each block has access to a limited amount of *shared memory*, a programmer-managed scratchpad typically used to stage data that will be repeatedly used. Finally, each thread maintains its private state in *registers* and *local memory*, which is used for storing the thread's stack.

This memory hierarchy has its own notion of convergence and divergence, mirroring that of compute. For example, when launching a kernel on the GPU, a pointer may initially belong to the grid because every thread sees the same pointer value at the start of the computation. To write to that memory, each thread computes an offset from the original base address. In doing so, the pointer *diverges* across threads within the larger memory region so each thread can write independently. After these writes, the memory must *reconverge*, ensuring that all threads have completed their updates before the memory can safely be returned to its original logical owner in the hierarchy.

```
int x = threadIdx.x;
if (x >= 0 && x< 31){
  float A[4];
  float B[2];
  float C[4] = { 0 };

  // k is A's stride
  A[0] = a_mem[(x/4)*k+(x%4)]);
  A[1] = a_mem[(x/4)*k+(x%4)+4]);
  A[2] = a_mem[(x/4+8)*k+(x%4)]);
  A[3] = a_mem[(x/4+8)*k+(x%4)+4]);

  // n is B's stride
  B[0] = b_mem[(x%4)*n+(x/4)]);
  B[1] = b_mem[(x%4+4)*n+(x/4)]);

  asm("mma.sync...");

  // Write back into c_mem
  // n is C 's stride
  c_mem[(x/4)*n+2*(x%4)] = C[0];
  c_mem[(x/4)*n+2*(x%4)+1] = C[1];
  c_mem[(x/4+8)*n+2*(x%4)] = C[2];
  c_mem[(x/4+8)*n+2*(x%4)+1]=C[3];}
```

Fig. 4. Invoking a warp-level tensor-core instruction in CUDA after loading data from memory.

CUDA does not explicitly model these memory spaces, nor does it track whether allocations in a given memory space, such as shared memory, exceed device limits. Snap, on the other hand, reifies these spaces in its type system by explicitly decorating each variable with its associated memory space and restricting shared memory to static allocations, throwing an error if an allocation exceeds device limits. We will discuss these aspects in detail in Section 3.5. For now, however, we focus on how memory diverges and reconverges in tandem with the compute hierarchy.

Let us reconsider the tensor-core example in Figure 1, this time initializing the operands of the tensor-core operation from pointers into memory. In Figure 4, A and B are now populated from data in global memory, a_mem and b_mem. After the tensor-core operation completes, the result is written back to c_mem, also in global memory.

In this program, c_mem is logically accessed from two different levels of the compute hierarchy. When entering this code, each thread in a group of 32 threads sees the same value for c_mem. Then, each thread within these 32 locates an offset within c_mem that it can write to (lines **21-24**). To restore c_mem back to its 32 thread-level ownership, all 32 threads must first synchronize to ensure all per-thread writes have completed. This is similar to the requirement we saw documented in **??**.

Similarly to the compute hierarchy (Section 2.1), CUDA programmers are responsible for tracking the logical owner of a piece of memory as it evolves over the course of a program, and for ensuring that appropriate synchronization occurs whenever that ownership changes.

Snap, by contrast, makes the evolution of a memory object's logical ownership explicit in the type system and *automatically* inserts the synchronization required to restore memory to its original ownership In Figure 5, we show how memory is lowered through the compute hierarchy for the same tensor-core operation in introduced in Figure 4. The lowering of c_mem is required in this program as Snap only permits writes when data is owned by a single thread. To lower c_mem, we use Snap's partition operation on line **31**. The operation takes the name of the memory to partition, c_mem, and lowers it to a single thread, and assigns it a new name, c_thrd; Snap will not allow references to the old name c_mem within the partition's scope. The partition function also takes an indexing function, and each time c_thrd is accessed, this indexing function is implicitly applied. Once the partition's scope ends, Snap will insert a synchronization before the first use of the original name, so that all per-thread writes have completed.

We would like to emphasize that preventing data races is not an one of Snap's goals. In Snap, data races *can* occur; however, since the data is eventually synchronized before it is reused, the last-writer wins. Out-of-bounds accesses are considered undefined behavior. We believe that prior work, such as Descend [13] , lays the foundation for reasoning about data-race free programs, and that a Descend-like memory model can be adapted to Snap. Snap is instead focused on the *interaction* between the compute and memory hierarchies, and on reasoning about them simultaneously to ensure that an operation is executed only when sufficient compute resources are available, a guarantee that prior work like Descend cannot provide.

```
1  @ccuda("device")
2  @requires(thread[32])
3  def simple_mma(
4      a: ptr(const(float)) @ thread[32],
5      b: ptr(const(float)) @ thread[32],
6      c: ptr(float) @ thread[32]):
7    x : int @ thread[1] = id()
8    with group(thread[32]):
9     A : float[4] @ thread[1]
10    B : float[2] @ thread[1]
11    C : float[4] @ thread[1]
12    # Reads do not need to be lowered.
13    A[0] = a_mem[(x/4)*k+(x%4)]);
14    A[1] = a_mem[(x/4)*k+(x%4)+4]);
15    A[2] = a_mem[(x/4+8)*k+(x%4)]);
16    A[3] = a_mem[(x/4+8)*k+(x%4)+4]);
17    B[0] = b_mem[(x%4)*n+(x/4)]);
18    B[1] = b_mem[(x%4+4)*n+(x/4)]);
19    # Skipping initialize C to 0...
20    intrinsic.mma(
21      A[0], A[1], A[2], A[3],
22      B[0], B[1],
23      C[0], C[1], C[2], C[3],
24      out=[C[0], C[1], C[2], C[3]])
25    # Must be at thread[1] to write.
26    idx : int @ thread[1] =
27     lambda ro, co: (x/4+ro)*n+2*(x%4)+co
28    with partition(c_mem,p=thread[32],f=idx)
   ↪  as c_thrd:
29      c_thrd[0, 0] = C[0]
30      c_thrd[0, 1] = C[1]
31      c_thrd[8, 0] = C[2]
32      c_thrd[8, 1)] = C[3]
33    # <-- Sync point inferred by name
```

Fig. 5. Invoking a warp-level tensor-core instruction in Snap after loading data from memory.

3 The Snap Language

Snap is an imperative, low-level programming language designed at a level of abstraction comparable to CUDA's. Snap's syntax explicitly represents how compute (grid, blocks, and threads) and data

```
1  @ccuda("global")
2  # Top-level privilege class and required shared memory usage.
3  @requires(grid[1], block[1], thread[32], smem=1280)
4  def mmaTF32NaiveKernel(A: ptr(const(float)) @ grid[1],
5            B: ptr(const(float)) @ grid[1],
6            C: ptr(float) @ grid[1],
7            M : int @ grid[1],
8            N : int @ grid[1],
9            K : int @ grid[1]):
10
11   # Replicate this code across 1 grid.
12   with group(grid[1]):
13     # @ grid[1] inferred from compute privelege
14     MMA_N  : constexpr(int) = 8
15     ...
16     num_blocks_n : const(int) = (N + MMA_N - 1) / MMA_N
17
18     # id() function returns the block id, inferred from
19     # @ block[1].
20     block_row : const(int) @ block[1] = id() / num_blocks_n
21     block_col : const(int) @ block[1] = id() % num_blocks_n
22
23
24     warp_row: const(int) @ block[1] = block_row * MMA_M
25     warp_col : const(int) @ block[1] = block_col * MMA_N
26
27     # Give each block an offset into C
28     offset = lambda x: warp_row * N + warp_col + x
29     with partition(C, p=block[1], f=offset) as C_blk:
30       with group(block[1]):
31
32        # SMEM declarations only allowed with block[1] privilege.
33         A_smem : shared(float[16 * 8]) @ block[1]
34         B_smem : shared(float[8 * 8])  @ block[1]
35         C_smem : shared(float[16 * 8]) @ block[1]
36         # Now, id() returns the thread id
37         idx : int @ thread[1] = id() * 4
38         # To write to C_smem, must drop to thread[1] privilege
39        with partition(C_smem,p=thread[1],offset=idx) as C_thrd:
40           for i in range(0, 4, 1):
41             with group(thread[1]):
42               C_thrd[i] = 0
43
44       for i in range(0, K_tiles, 1):
45         a_idx : int @ thread[1] = id() * 4
46         # <--- Name will insert a sync point
47         # because this partition is a parent partition
48         # of itself (back edge from for loop)
49         for j in range(0, 4, 1):
50           global_row : int @ thread[1] = warp_row + row
51           global_col: int @ thread[1] = i * MMA_K + col
52           with partition(A_smem, p=thread[1], f=..) as As_thrd:
53             with group(thread[1]):
54               As_thrd[0] = A[global_row * K + global_col]
55
56         b_idx : int @ thread[1] = id() * 2
57         # <--- Name will insert a sync point
58         # (back edge from for loop)
59         for j in range(0, 2, 1):
60           # Similar to write into C_smem ...
61           with partition(B_smem, p=thread[1], f=...) as Bs_thrd:
62             with group(thread[1]):
63               Bs_thrd[0] = B[global_row_b * N + global_col_b]
64
65           # <--- Name will insert a sync point
66           # (back edge from for loop)
67           # Give each warp an offset into C_smem
68           with claim(C_smem, p=thread[32]) as Cs_warp:
69               match split(thread):
70                 case 32:
71                   # Can pass A_smem and B_smem
72                   # as arguments to simple_mma for
73                   # those two are declared const.
74                   # "read up" is okay.
75                   simple_mma(A_smem, B_smem, Cs_warp)
76
77       # Write back final result to C
78       for j in range(0, 4, 1):
79         flat_idx_c : int @ thread[1] = id() * 4 + j
80         row_c : int @ thread[1] = flat_idx_c / MMA_K
81         col_c : int @ thread[1] = flat_idx_c % MMA_K
82         offset = lambda x: row_c * N + col_c + x
83         with partition(C_blk, p=thread[1], f=offset) as C_thrd:
84           with group(thread[1]):
85             C_thrd[0]  = C_smem[row_c * MMA_N + col_c]
86 return
```

Fig. 6. Tensor Float 32 matrix Multiplication in Snap.

(global, shared and local memory) resources are used and distributed, enforcing that programs only execute collective operations with sufficient resources.

As a running example, the program in Figure 6 computes a matrix multiplication between two float arrays, `A` and `B`, to produce an output matrix `C`. In this kernel, each block computes an independent tile of the output `C`. A block first locates its assigned tile index, then loads the corresponding rows and columns from `A` and `B` (line **20-21**). Since matrix multiplication offers ample opportunities for data reuse, to amortize the cost of loading data, we stage `A` and `B` in shared memory (lines **46-59**). Each thread within the block then *diverges* and loads different elements of `A` and `B`. Finally, the program invokes an Ampere warp-level tensor-core instruction to compute the output, requiring threads in a warp to *converge* (line **71**). We encapsulate this tensor-core instruction in a function, demonstrating how function composition works in Snap. This function's implementation is identical to the one shown in Figure 5.

3.1 Levels

Snap models the machine's compute hierarchy through *levels*. There are three levels in Snap—`grid`, `block` and `thread`—which are organized as expected: a `grid` consists of multiple units of the `block` level, which in turn consist of multiple units of the `thread` level. Levels have an order defined over them, with `thread` < `block` < `grid`.

There are two key differences between Snap's levels and those in CUDA. First, Snap does not model a three-dimensional grid or block structure, say through different levels for `grid_x`, `grid_y`, or `grid_z`. Second, there are two "levels", *warp* and *warp group*, noticeably absent from our hierarchy.

These differences are a deliberate design choice. On the hardware, the units of each level are arranged in a single linear order, and the three-dimensional structures are simply *interpretations* of this linear ordering, not distinct hardware resources. Similarly, warps and warp groups are organizational constructs defined in terms of existing levels. Namely, a warp is a group of 32 threads whose first thread ID is aligned to 32. A warp group, which only became a meaningful construct with the release of the Hopper architecture , consists of 128 threads aligned to 128.

Rather than embedding these interpretations into the language as special cases by introducing new levels, Snap provides these organizational constructs as aliases for groupings of 32 and 128 threads, respectively. Users can also define their own constructs that express multi-dimensional structures and custom groupings of specific sizes.

The primary mechanism for organizing units at each level is a *privilege*.

3.2 Privileges & Privilege Classes

Privileges are the central concept in Snap, allowing it to determine which compute resources are being requested by the programmer, whether they are available in the program's context, and, once provided, if those resources are sufficient for a given operation.

A *privilege* is a level—`grid`, `block`, or `thread`—parametrized by a static constant `n`, specifying the number of units at that level. A privilege is written as `level[n]` . For example, `thread[2]` denotes a privilege of two threads, `block[4]` denotes a scope of four blocks, and so on.

Privileges also carry *alignment information*: a privilege parameterized by `n` is aligned to `n`. In this way, a warp is simply a desugaring of `thread[32]`, and a warp group is a desugaring of `thread[128]`.

Finally, privileges have a *partial order* defined on them. We say that $\texttt{level}_2[\texttt{n}_2]$ is a *higher privilege* than $\texttt{level}_1[\texttt{n}_1]$ iff either $\texttt{level}_1 < \texttt{level}_2$, or, if $\texttt{level}_1 = \texttt{level}_2$, then $\texttt{n}_2 \ \% \ \texttt{n}_1 = 0$. The condition $\texttt{n}_2 \ \% \ \texttt{n}_1 = 0$ may appear somewhat mysterious at first, but we will see why it is required in Section 3.2.2.

In Snap, *both code and data* are bound to privileges . Code is bound to a set of privileges, called a *privilege class* meanwhile data is bound to a single privilege.

3.2.1 Privileges for Code, or Privilege Classes.

Privilege classes capture what compute resources a program has access to. Each function starts out with a top-level privilege class that defines the total available resources that it assumes are available to it. As a shorthand, when we refer to a code's privilege, we mean the highest privilege available to it in its class . In Figure 6, the privilege class is declared on line **2** using the notation `@requires(grid=..,block=..)` Starting from this top-level declaration, programmers shape the current privilege class in the function's body through two constructs, `group` and `split`.

Group. The `group` construct bundles privileges available in the current privilege class into "groups" of a particular granularity. Intuitively, `group` narrows the current privilege class and replicates it across collections of a finer granularity. For example, Figure 6, line **26** shows a grouping of a single block, while line **22** in Figure 5 shows a grouping of 32 warps , that will ultimately invoke a tensor-core operation.

An invocation of `group(level[n])` is allowed if and only if the privilege class contains privilege higher than `level[n]`. For example, `group(warp[32])` on **22** in Figure 5 is allowed because the privilege class of the function declares that the top level on line **22** starts out with at least 32 threads.

```
# Example 1
with group(thread[2]):
    # Illegal because block > thread
    with group(block[1]):
        pass
# Example 2
with group(block[1]):
    # Illegal because 1 % 2 != 0
    with group([block[2]]):
        pass
```

Fig. 7. Illegal uses of `group`.

Once `group(level[n])` is invoked, it modifies the current privilege class in two key ways. First, it removes all parents of `level[n]` from it. Second, the number of available units for `level` is set to exactly `n`. Combined, these rules eliminate malformed programs that would result in unsatisfiable requirements . We shown some examples of illegal group statements in Figure 7.

Split. Unlike group, which is used for replication over smaller, equal sized resource sets, split is used for sharding currently available resources into unequal sized sets. For example, line **63** in Figure 6 shows a split of a thread[32] privilege that is used to execute a warp-level tensor-core operation.

The split operation reflects unordered composition that is not mere replication, and all branches of the split execute in parallel. Once split(level) is invoked, the privilege classes of its branches diverge. A split construct will eventually compile to an if-statement during code generation.

```
# Example 1
with group(thread[32]):
 match split(thread):
   case 32:
     simple_mma(..)
   # Illegal! 32 + 1 > 33
   case 1:
       pass
```

Fig. 8. Illegal split that exceeds available privileges.

Therefore, to satisfy a split operation, Snap checks that the *cumulative sum* of the privileges requested by each branch of the split can be satisfied. For example, if we were to change the program in Figure 6 slightly (on line **63-69**), to the one shown in Figure 8, Snap would throw a compile-time error. Finally, because privileges assume alignment, every branch of the split must also be aligned. In other words, not all splits whose sizes sum to less than or equal to the available units are necessarily valid. Figure 9 shows an example of this constraint. This alignment constraint is also why we compare privileges by checking whether one quantity is a multiple of the other's.

```
# Example 2
with group(thread[33]):
 match split(thread):
   case 1:
     pass
   # Illegal! 32 + 1 <= 33,
   # but the second branch
   # is not aligned by 32.
   case 32:
     simple_mma(..)
```

Fig. 9. Illegal split that violates alignment.

Inside each branch that requests n units, all parent levels of level[n] are removed from its privilege class, and the number of available units for level is set to exactly n, and, within that scope, all resource requirements are checked against that privilege class.

3.2.2 *Privileges for Data.*

Not all reads from and writes to variables are allowed in Snap. To see why, consider a group of threads with the block[1] privilege, branching on a variable which has different values across those threads. In this case, different threads will take different paths, and the structure of the block[1] privilege will be lost. To prevent this, Snap must also track privileges associated with individual variables. The privilege on a variable is specified in its declaration: an integer variable v is declared with thread[1] privilege using the syntax v : int @ thread[1].[1] Intuitively, the data privilege of a variable tells Snap the *frequency* at which its values change in space . For example, a grid[1] value is the same for all threads in a grid, whereas a block[2] value changes every two blocks.

To enforce this frequency invariant, Snap restricts how variables can be written. Figure 10 shows an illegal write from a thread[1] variable x to a block[1] variable y. As x is constant across entire blocks, we cannot write y—which can vary from thread to thread—into it. More generally, the rule for writes is that programs may "write down".

```
x : int @ block[1]
y : int @ thread[1]
with group(thread[1]):
  x = y
```

Fig. 10. Illegal write of block[1] variable.

Reads from variables are dually constrained. Figure 11 gives an example of an illegal read. Because the variable flag varies across threads, branching on it causes a subset of threads in the block to reach the __syncthreads(), causing a deadlock, despite the fact that __syncthreads() should always be safe in block[1]. The rule for variable reads is that programs may "read up".

[1]If not explicitly annotated, Snap infers a variable's privilege to be the privilege of the code where it was declared. For example, MMA_N, num_blocks_n are all inferred to be at grid[1] scope on lines **13 and 15** in Figure 6.

```
flag : bool @ thread[1] = ...
with group(block[1]):
  if (flag)
    __syncthreads();
```

Fig. 11. Illegal read of `thread[1]` variable.

Together, the “read up” and “write down” rules ensure that information only flows from high privilege variables to low privilege ones. This, in turn, guarantees that all threads with the same privilege remain logically grouped during execution. In Figure 6, we can see rule “read up” in action on lines **45 and 46**. Meanwhile, “write down” is being used to set up variables on line **18-22**.

3.3 The `id()` Function

As opposed to exposing users to special hardware `blockIdx.x`, `threadIdx.x` variables directly, Snap provides the `id()` function instead. The `id` function returns the relative index, or a “label”, for a given compute privilege. The interpretation of the `id` function depends on both the privilege of the variable it is being written to, and the privilege class that the `id` function is being invoked in. In our example in Figure 6, we use a call to `id` in `grid[1]` scope at line **18 and 19** to locate the tile that each block is ultimately charge of computing. In Figure 2, however, a call to `id` in `thread[32]` scope at line **7** will not return the global thread ID of the threads launched in the kernel, but rather the *relative* thread ID within the 32 threads, ranging from 0 to 31.

3.4 Collective Operations

With privilege classes for code and data in place, we can now turn our attention to Snap’s reasoning about collective operations. By statically tracking privilege classes for code, Snap guarantees that collective operations are semantically valid by ensuring that they are invoked with sufficient privilege.

There are two ways to access collective operations in Snap. The first is to use intrinsics provided by the compiler. These intrinsics have built-in privilege classes attached to them, which the compiler can directly enforce. In Figure 2, the tensor-core call at line **22** is made using one of the intrinsics provided by our compiler. The second, more flexible approach allows users to extend these operations using the `unsafe` construct, explained in Section 3.8. Using unsafe code, users can inline a collective operation—often a single assembly instruction—into a Snap function and ensure that the function’s top-level requirements, specified with `@require`, faithfully describe the privileges required to execute that instruction. Then, Snap will check the call to the function as any other function call, enforcing the correct privilege. The rules for checking function calls are described in Section 3.7.

A quirk of some collective operations is that, although they need to be invoked with higher compute privileges, they often produce results with lower data privileges. For example, the tensor-core operation in Figure 2 is invoked at `thread[32]` granularity but generates output at the `thread[1]` level. That is, even though the call to the tensor call is a convergent operation requiring the `thread[32]` compute privilege, it produces divergent output with `thread[1]` data privilege. To be able to handle these operations be must be able to allow `thread[1]` state to coexist with the `thread[32]` privilege.

One potential solution is to artificially coarsen the granularity of the operation by having the intrinsic internally scatter the `thread[1]` output into a `thread[32]` variable. However, this approach is often unacceptable for low-level CUDA performance turning. Often, performance engineers need to manually schedule and interleave memory and arithmetic instructions to hide latency, with loads or stores occurring well in advance of their dependent operations and interleaved with other instructions. If we coarsen these operations, we will disallow these optimizations entirely.

In Snap, however, this behavior does not pose a problem. Because of the “write down” rule, registers can be set up in advance of the operation (e.g., line **14-20** in Figure 2) without violating correctness.

3.5 Memory

In Snap, there are three mechanisms for obtaining a pointer type. The first method is to directly allocate data. The second and third are to obtain offsets into existing data by using `partition` or claim. These constructs mirror the group and split constructs, and are used to lower the data privilege associated with memory.

3.5.1 Allocations in Different Memory Spaces.

In Section 2.2, we saw that the GPU has distinct memory spaces, like global, local, and shared memory. Local allocations reside in either the stack or registers, which Snap can distinguish upfront . Shared memory pointers are explicitly annotated with the shared keyword, while global pointers are unannotated.

The primary memory resource Snap needs to manage are shared memory allocations. Local memory allocations are just stack allocations, and don't need special treatment. We also assume that sufficient global memory has been allocated before the kernel launch and don't provide a mechanism for runtime allocations—this is typically the case in real-world programs.

In CUDA, shared memory is a limited resource, typically a few hundred kilobytes per block. Programmers manually manage this memory by bumping pointers to allocate offsets for different data. Each allocation's pointer must be carefully tracked throughout the program, and exceeding the available space will lead to a runtime error. Snap simplifies shared memory management by requiring all shared memory allocations have a static size. Since shared memory is only visible to all threads in a block, a shared memory allocation is only valid with block[1] privilege. An example of such an allocation is shown on lines **28-30** of Figure 6. During compilation, Snap will automatically handle the necessary pointer arithmetic to assign each allocation an appropriate offset within the shared memory space.

The @requires annotation specifies the amount of shared memory a function expects to have available, and Snap uses this information to ensure that a function's allocations do not exceed its declared limit, and checks whether there is enough shared memory available at its call sites. Figure 6 declares the amount of shared memory it will require on **2**. We use standard techniques to statically bound memory usage with Snap's type system.

3.5.2 Privileges for Memory.

In Section 3.2.2, we focused on tracking for primitive data types and stack allocations, where each thread maintained its own local copy of the data. In contrast, pointers into memory, such as global or shared memory, do not represent distinct copies, but instead refer to the same underlying storage that may be accessed simultaneously by multiple threads. To use them, we must first *lower* their privileges, allowing individual threads to write different values to different locations, and then *restore* them back to their original privilege before execution continues. .

As with variables in Section 3.2.2, lower-privilege data may read from higher-privilege data without restrictions. For writes, however, Snap enforces that all pointers into memory be written with thread[1] privilege; they therefore must be lowered before they can be safely updated.

Partition. The partition construct plays the same role for memory that group plays for compute. It is used to lower a memory object from a higher privilege to a lower one by assigning each lower privilege a distinct offset into the original pointer . It takes the original name of a memory object , base_name, an indexing function f, and a target privilege level[n] to which the memory is being lowered, and produces a new variable, new_name, pointing to the partitioned memory. In Figure 6, for example, we partition the C buffer to block[1] privilege on line **25** first, and then lower it further to thread[1] privilege on line **78** so that we can actually write to it.

Once a memory object has been partitioned, the original name referring to it goes out of scope, and the memory can only be accessed through the new name introduced by partition. Each use of this new name applies the indexing function f to compute the true offset of the access. Within the scope of the partition statement, the original name cannot be repartitioned. Because split denotes unordered composition, Snap prohibits sibling `split` branches from partitioning the same memory object. When the scope of the `partition` statement ends, Snap automatically inserts the necessary synchronization to restore the memory to its original privilege. The mechanism for inserting and optimizing these synchronization points is described in Section 5.1.

At this point, it is necessary to consider how different indexing functions affect the possibility of data races. If the indexing function is injective, each lower-privilege privilege receives a distinct offset into the underlying array, and the resulting partition is free of data races. When the indexing function is not injective, multiple privileges may write to the same location, introducing a potential data race. Out-of-bounds accesses are undefined behavior.

As we discussed in Section 2.2, achieving data-race freedom is *not* one of Snap's goals, and our guarantees continue to hold even when indexing functions are not injective. Because Snap automatically inserts the necessary synchronization before the original memory name is restored to its higher privilege, Snap ensures that all writes have completed before accessing the orginal name. The subsequent uses of that pointer will observe the value written by the last-writer.

We believe that extending Snap to reason about data races and out-of-bounds accesses is a promising direction for future work.

Claim. The `claim` construct is the analogue of `split` for memory . Using `claim`, the entire memory region can be moved to a lower privilege.

Unlike `partition`, `claim` does not take an indexing function. The `claim` construct takes the original memory's name, the target privilege to lower to, and a new name to assign to the lowered memory. The `new_name` is accessible only within a single, explicit `split` branch; sibling branches are not permitted to read or write from this memory. In Figure 6, we use the `claim` operation on line **63** to lower the memory to `thread[32]` privilege.

As with `partition`, a `split` construct introduces its own scope, and the original name is inaccessible within this scope.

3.6 Asynchrony

3.7 Function Composition

To check whether a function call is valid, Snap needs to ensure that the call site can provide the set of privileges that a function will attempt to use over its execution, and that the arguments passed to the function also respect the privileges declared for its input and output types. Both requirements are represented explicitly via the `@require` construct.

As mentioned in Section 3.2, each function carries a privilege class. We call this top-level privilege class the function's *privilege signature*. The signature specifies the top-level privilege class the function *assumes* will be provided to it. This signature represents the *minimum amount* of compute and memory resources the function must be called with. In the example above in Figure 6, line **2** defines the privilege signature. Up to this point, we have described programs written *inside* functions under the assumption that the resource signature can be satisfied. In Section 3.2, for example, we used this signature to set up the top-level privilege class for the function.

Compute privileges. At a function's call site, then, we need to ensure that the function's privilege signature can be satisfied. Snap compares the current compute privilege with the one required by the function, and verifies that only functions whose privilege signatures can be satisfied are called.

Per-argument checks. In Snap, functions have pass-by-value semantics. For arguments, we distinguish between primitive data types and pointers to memory. For primitive data types, like `int`, `bool`, and stack allocations, users can pass either arguments that live at or above the data privilege specified by the function's signature. For pointers, we allow passing pointers that live at higher data privilege if the pointer is marked as `const`, indicating that it will only be used for reading (line **4 and 5** in Figure 2). Otherwise, we require that pointers match the exact privilege of the function signature.

3.8 Unsafe

Snap also has an `unsafe` construct that lets users inline collective operations that have not been explicitly added as intrinsics in the compiler. After using `unsafe`, these operations can be wrapped in a function whose signature declares the compute privileges required to call it.

4 Formalism

Having introduced the full Snap language, we now describe Bundl, a core calculus that formalizes its most fundamental aspects by statically tracking compute and data privileges. We use Bundl to prove that well-typed Snap programs are not only type-safe, but will also never get stuck trying to execute operations for which they do not have the required compute resources.

In this section, we describe Bundl's type system and operational semantics—in particular how it manages compute and data privileges—and build up to a formal proof of type-and-privilege safety. By instrumenting Bundl's operational semantics with privileges, our safety theorem can guarantee that dynamically-realized privileges match the ones inferred by the type system.

4.1 Bundl Type System

The core idea in Bundl is to track, at the type-level, the program's current compute and data privilege. To achieve this, we borrow techniques from the literature on *coeffects* [16] and *dependency tracking* [1]. In particular, the compute privilege is tracked on the typing judgment, which has the form $\Gamma \vdash^{\pi} e : \tau$ for expressions and $\Gamma \vdash^{\pi} s$ for statements; the π over the $\vdash$ is the compute privilege of e and s.

The πs themselves are pairs of levels h of the GPU hierarchy at which each privilege lives (either `Thread`, `Block` or `Grid`), and numbers tracking the size of the logical group denoted by each privilege . Bundl's πs directly correspond to privileges in Snap.

Each variable in the typing context has a data privilege attached to it; data can only be read or written from a variable when its privilege matches that of the code interacting with it. This requirement is made manifest in the T-Var rule, which can be found in Figure 12. The premise of this rule also reveals that the context Γ maps variables to both types τ and also the privileges π with which they may be read and written.

Figure 12 also shows other key rules, and these rules fall into two main categories: rules for managing the current compute privilege and rules for managing data privileges.

4.1.1 Compute Privilege Management. In Snap, there are two operations which manage compute privileges: `group` and `split`. In Bundl, to better model the details of privilege attenuation , we introduce a third construct, `destruct`.

Bundl's `group` statement directly corresponds to Snap's, and is checked by the T-Group rule. Given some privilege that can be divided into q equally sized smaller privileges π, the statement `group` $q\,s$ will check with privilege $q \cdot \pi$ provided that s itself checks with privilege π. The starting privilege is of the form $q \cdot \pi$, and encodes Snap's alignment requirement.

The `split` statement, meanwhile, is checked by the T-Split rule, and functions like a binary version of the n-ary splitting construct in Snap. It enforces the same divisibility requirements to ensure that the privileges of code and data remain properly aligned, and then checks the two sub-statements s_1 and s_2 with the divided, smaller privileges.

The final rule for managing compute privileges is T-Destruct, which handles downward movement from one level of the GPU hierarchy to another and makes explicit in Bundl programs exactly where such movement occurs. The $\downarrow$ operation on πs "destructs" the privilege into a many smaller

$\boxed{\Gamma \vdash^{\pi} e : \tau}$ *(Expression typing)*

$$\frac{x :^{\pi} \tau \in \Gamma}{\Gamma \vdash^{\pi} x : \tau}\ \textsc{T-Var}$$

$\boxed{\Gamma \vdash^{\pi} s}$ *(Statement typing)*

$$\frac{\Gamma \vdash^{(h,n_1)} s_1 \quad \Gamma \vdash^{(h,n_2)} s_2 \quad n_1,n_2\ \textbf{align to}\ n}{\Gamma \vdash^{(h,n)} \texttt{split}(n_1,n_2)\{s_1\}\{s_2\}}\ \textsc{T-Split}$$

where $n_1,n_2\ \textbf{align to}\ n ::= (n_1+n_2 \leq n)\ \mathit{and}\ (n_1 | n)\ \mathit{and}\ (n_2 | n)\ \mathit{and}\ (n_2 | n_1 + n)$

$$\frac{\Gamma \vdash^{\pi} s}{\Gamma \vdash^{q \cdot \pi} \texttt{group}\ q\ s}\ \textsc{T-Group} \qquad \frac{\Gamma, y :^{\downarrow\pi} \tau[]^{l} \vdash^{\pi} s \quad l \neq \texttt{Local}}{\Gamma, x :^{\pi} \tau[]^{l} \vdash^{\pi} \texttt{lower}\ x\ \texttt{into}\ y\ \texttt{in}\ s}\ \textsc{T-Lower}$$

$$\frac{\Gamma, y :^{(h,n/c)} \tau[]^{l} \vdash^{(h,n)} s \quad c | n \quad l \neq \texttt{Local}}{\Gamma, x :^{(h,n)} \tau[]^{l} \vdash^{(h,n)} \texttt{partition}\ x\ \texttt{into}\ y\ \texttt{by}\ c\ \texttt{in}\ s}\ \textsc{T-Partition} \qquad \frac{\Gamma \vdash^{\downarrow\pi} s}{\Gamma \vdash^{\pi} \texttt{destruct in}\ s}\ \textsc{T-Destruct}$$

$$\frac{\Gamma, y :^{(h,n')} \tau[]^{l} \vdash^{(h,n')} s \quad n' \leq n \quad l \neq \texttt{Local}}{\Gamma, x :^{(h,n)} \tau[]^{l} \vdash^{(h,n)} \texttt{claim}\ x\ \texttt{into}\ y\ \texttt{at}\ n'\ \texttt{in}\ s}\ \textsc{T-Claim}$$

Fig. 12. Core typing rules of Bundl. The typing rules presented here are a simplified selection of the full rules, which can be found in Appendix A.2.

privileges at a lower level; $\downarrow(\texttt{Grid}, 1) = (\texttt{Block}, B)$ and $\downarrow(\texttt{Block}, 1) = (\texttt{Thread}, T)$, where B and T are parameters to a particular instantiation of Bundl to describe the number of blocks per grid and threads per block.[2] Because $\downarrow$ is only defined on $(\texttt{Grid},1)$ and $(\texttt{Block},1)$, the rule enforces that one can only `destruct` their compute privilege when they have privileges for exactly one grid or block.

4.1.2 Data Privilege Management. The mechanism for managing data privileges mirrors that of compute privileges, with each operation for data corresponding to an operation for code.

The `partition` operation is analogous to `group`-ing a compute privilege. The typing rule for this operation, T-Partition, requires that the data privilege of the variable to be partitioned, x, is the same as the current compute privilege. After partitioning, a fresh variable y is introduced with a new privilege π/c. We use π/c as a shorthand to denote a division of a privilege π into c equally-sized, smaller privileges. Within the body of the `partition` statement, we disallow any references to the original variable x. Then, we continue checking the body of the `partition` statement against the original compute privilege π because the `partition` has no effect on the current compute privilege.

Unlike `partition`, which divides up a piece of data equally into lower privileges, the `claim` operation, mirroring the `split` construct, gives the entire variable to exactly one, smaller privilege. Accordingly, Bundl needs to ensure that only one branch of a `split` operation, with the appropriate π, can refer to the claimed variable. To ensure that this is the case, the T-Claim rule links the data privilege of the variable to the compute privilege of the code claiming it by changing *both* at the same time. This represents a minor difference from Snap, which implements additional static analysis to ensure that a claimed variable is only accessed in a single `split` branch.

[2]Bundl is abstracted over these B and T values, so instead of tracking classes of privileges the way that Snap does, it only tracks the top-level privilege described in Section 3.2.

$$\frac{L(t),S(b),\Sigma,t,b,0 \vdash^{(\texttt{Grid},1)} s \rightsquigarrow s' \dashv \eta',\sigma',\Sigma' \quad P(t,b)=s}{L,S,\Sigma,P \rightsquigarrow L[t\mapsto \eta'],S[b\mapsto \sigma'],\Sigma',P[(t,b)\mapsto s']}\ \textsc{S-Program}$$

$$\frac{p<n_1 \quad n_1,n_2\ \textbf{align to}\ n \quad \eta,\sigma,\Sigma,t,b,p,\vdash^{(h,n_1)} s_1 \rightsquigarrow s_1' \dashv \eta',\sigma',\Sigma'}{\eta,\sigma,\Sigma,t,b,p \vdash^{(h,n)} \texttt{split}(n_1,n_2)\{s_1\}\{s_2\} \rightsquigarrow \texttt{split}(n_1,n_2)\{s_1'\}\{s_2\} \dashv \eta',\sigma',\Sigma'}\ \textsc{S-Split-Left}$$

$$\frac{p\geq n_1 \quad p<n_1+n_2 \quad n_1,n_2\ \textbf{align to}\ n \quad \eta,\sigma,\Sigma,t,b,p-n_1,\vdash^{(h,n_2)} s_2 \rightsquigarrow s_2' \dashv \eta',\sigma',\Sigma'}{\eta,\sigma,\Sigma,t,b,p \vdash^{(h,n)} \texttt{split}(n_1,n_2)\{s_1\}\{s_2\} \rightsquigarrow \texttt{split}(n_1,n_2)\{s_1\}\{s_2'\} \dashv \eta',\sigma',\Sigma'}\ \textsc{S-Split-Right}$$

$$\frac{\eta,\sigma,\Sigma,t,b,p\ \textbf{mod}\ n \vdash^{(h,n)} s \rightsquigarrow s' \dashv \eta',\sigma',\Sigma'}{\eta,\sigma,\Sigma,t,b,p \vdash^{(h,q\cdot n)} \texttt{group}\ q\ s \rightsquigarrow \texttt{group}\ q\ s'; \dashv \eta',\sigma',\Sigma'}\ \textsc{S-Group}$$

$$\frac{\eta,\sigma,\Sigma,t,b,t\ \textbf{mod}\ T \vdash^{(\texttt{Thread},T)} s \rightsquigarrow s' \dashv \eta',\sigma',\Sigma'}{\eta,\sigma,\Sigma,t,b,0 \vdash^{(\texttt{Block},1)} \texttt{destruct in}\ s \rightsquigarrow \texttt{destruct in}\ s' \dashv \eta',\sigma',\Sigma'}\ \textsc{S-Destruct-Block}$$

Fig. 13. Core semantic rules of Bundl. As with the typing rules, we present only a simplified selection of the full rules, which can be found in Appendix A.3.

Lastly, the T-Lower rule mirrors the T-Destruct rule; it uses the $\downarrow$ operator to move a variable from one level of the hierarchy to another, distributing it equally among all the child privileges in the same manner as T-Partition.

4.2 Bundl Semantics

Having explained the key rules of the type system, we can move on to discuss Bundl's operational semantics. To reflect the fact that GPU programs execute in parallel across numerous threads, we model the semantics of Bundl in the style of Turon et al. [22], using a two-level small step judgment. We present the key rules of this semantics in Figure 13.

The top level (i.e., device-level) judgment has just one rule: S-Program. This rule acts as a "frame" for the lower level (i.e., thread-level) judgment, and steps a collection of thread-id-indexed local memories (L), a collection of block-id-indexed shared memories (S), a global memory (Σ), and a *thread pool* to an updated collection of memories and thread pool. The thread pool maps thread and block ids to code, intuitively representing the program being executed by each thread at the current moment. The S-Program non-deterministically chooses a thread id and block id and steps it according to the thread-level judgment. This allows the semantics to model the full range of non-deterministic behavior arising from the GPU's thread scheduler.[3]

The thread-level judgment has the shape $\eta,\sigma,\Sigma,t,b,p \vdash^{\pi} s \rightsquigarrow s' \dashv \eta',\sigma',\Sigma'$. The ηs in this judgment represent the thread's local memory, the σs represent the shared memory, the Σs represent global memory, and the s's represent the statement being executed by the thread. The primed versions of each of these are the output of the step judgment. Critically, notice that a π also appears on the thread-level judgment just as it does on the typing judgment. This is because the thread semantics *dynamically tracks and verifies privileges.* The same way a program can get stuck if a value does not have the right type, the semantics of Bundl also get stuck if code attempts to access data or invoke

[3]In reality, the GPU's thread scheduler guarantees that warps execute their threads in lockstep, but this modeling every thread as completely independent is both simpler and a conservative overestimate of the nondeterministic behavior of the GPU.

commands with the wrong privilege. This runtime privilege is present in Bundl, but is erased by Snap during compilation; we will later use it to prove that well-privileged programs will always execute with the same privilege that the type system checked them against.

Other variables in the judgment track the positions of threads within their scopes. The t variable represents the thread's id, b is the id of the block in which the thread lives, and p is the relative position of the thread in the compute privilege with which it is executing. For example, threads with a (`Thread`,4) privilege each have a unique p between 0 and 3, while threads with a (`Block`,16) privilege will have the same p within in each block, between 0 and 15 depending on the block.

The semantic rules for privileges involve manipulating p to track which threads take which code paths when privileges are split or grouped. Notice that in the S-Program rule, the thread stepping judgment always begins with privilege (`Grid`,1): all the privilege management rules are congruences, handling further evaluation with lower privileges as determined by the particular rule used.

These rules take great care to ensure that p always describes the relative position of a compute resource within its privilege; the payoff is that Bundl's semantics can later use this p value to model the way that Snap automatically adjusts indices into data when partitioning a data privilege.

Beyond these key rules for compute privilege management, we have modeled many of the other features of Snap, such as asynchronous operations and thread synchronization, in Bundl. To handle features like these we equip the operational semantics with additional structure, including sets of semaphores [10] for thread syncs and a stack of effect handlers for modeling deferred asynchronous computations in the style of Ahman and Pretnar [2]. We have elided these details here for simplicity, but interested readers can see further details in Appendix A.3.

4.3 Soundness Theorem

Together, the type system and operational semantics help us prove the following syntactic soundness theorem, which says that Bundl programs are type safe and do not get stuck trying to execute operations for which they lack the required privilege:

Theorem 4.1. *(Type-and-Privilege Safety). For any program s such that* $\Gamma \vdash^{\pi} s$*, either:*

(1) s is `skip`*, or*
(2) for any well-typed environments η*,* σ*, and* Σ*, there is an* s'*,* η'*,* σ'*, and* Σ' *such that* $\eta,\sigma,\Sigma,t,b,p \vdash^{\pi} \rightsquigarrow s' \dashv \eta',\sigma',\Sigma'$ *where* η'*,* σ'*, and* Σ' *are well-typed by an extension of* Γ'*.*

Proof. Via the usual progress and preservation lemmas. The full proof of these lemmas can be found in Appendix A.4. □

It is worth noting that this soundness theorem is a safety theorem, not a liveness theorem. The theorem does guarantees that if all threads that can reach a point in the program with statically-determined privilege π do reach that point, then those threads will be logically grouped according to that same privilege π. It does not, however, guarantee that all threads that *can* reach the point will eventually do so. Indeed, in the presence of nontermination, liveness does not hold: a subset of the threads could `split` off and loop forever. While we believe the liveness version of this theorem holds for a terminating fragment of Bundl, it is beyond the scope of this paper—such proofs are notoriously challenging and are often research contributions [3, 11, 22] in and of themselves. We hope to tackle this proof for Bundl in future work.

5 Implementation

Snap is implemented as an embedded language in Python. Once a program type checks, Snap lowers it to a CUDA file. All privilege information is erased during this step, and the generated CUDA contains no run-time checks. The file can then be compiled by `nvcc`, NVIDIA's closed-source compiler, to produce an executable. Because Snap operates at roughly the same abstraction level as CUDA, there is a one-to-one mapping between most language constructs and their CUDA counterparts. A notable change is the addition of three parameters to each device function: the thread's relative ID, the block's ID, and an offset into shared memory that it can use for allocations.

While most of Snap's implementation is straightforward, inserting synchronization points is more involved, which we discuss now.

5.1 Inserting Synchronization

As described in Section 3.5, once data has been partitioned, Snap is responsible for inserting the appropriate synchronization when the partition's scope ends.

To determine where these synchronization points need to be inserted, Snap constructs a *data-control-flow graph* from the program. Each node corresponds to a partition, and each edge captures program-order precedence: the parent partition must complete before the child begins. The graph can have backedges introduced through loops. In this graph, each partition is categorized as a *read* or a *write* partition. A synchronization point is inserted before entering a partition if either of two conditions holds:

(1) If the parent partition is a write partition, a synchronization point is inserted to ensure that the current partition observes the most recent data.
(2) If the current partition is a write partition, a synchronization point is inserted to ensure that all preceding reads on the same memory have completed.

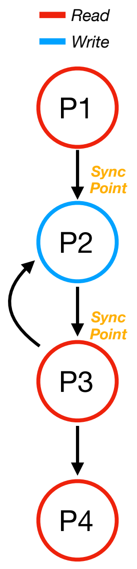

Fig. 14. An example data-control-flow graph lifted by Snap with its inferred synchronization points.

Figure 14 shows an example graph with the synchronization points derived from these two conditions. The inferred synchronization points in Figure 6 have also been marked on lines **TODO**.

Using this information, Snap emits *wait* operations before a partition begins (to wait for its parent) and *arrive* operations when a partition ends (to signal completion). CUDA exposes a general split-barrier primitive that we use to implement these waits and arrives. For special cases, such as synchronizing an entire block or warp, Snap instead uses primitives like `__syncthreads()` or `__syncwarp()`.

Synchronization for asynchronous data movement is handled in exactly the same way and uses the same underlying graph. CUDA allows asynchronous load operations to be associated with a split barrier, so Snap binds each asynchronous transfer to the appropriate wait–arrive pair inferred from the graph. Certain features, such as commit-group–style synchronization (), require additional reasoning, and Snap performs further static analysis to insert the necessary synchronization around these operations.

Given this setup, naively inserting synchronization immediately after each partition scope would be correct but prohibitively slow. To avoid this, Snap applies two optimization passes. A wait-motion pass pushes waits downward toward the first use of the partitioned name, and an arrive-motion pass pulls arrives upward toward its last use, reducing unnecessary stalls while preserving correctness.

6 Evaluation

We evaluate Snap in the context of the four main questions:

RQ1 Can Snap express a diversity of CUDA programs?
RQ2 Can Snap express programs that use advanced GPU features?
RQ3 Can Snap match the performance of existing, speed-of-light CUDA code?

Although the following question is not central to our paper's main claims—since library design ultimately reflects choices made by programmers, and our language allows abstraction design much like any other language—we were, nevertheless, curious to explore it based on empirical and anecdotal observation:

RQ4 Can Snap help build compositional libraries that users can use with confidence?

To perform this evaluation, we use two GPUs. The first is the NVIDIA H100 SXM5, a server-grade chip that supports tensor-core operations and a dedicated hardware copy engine, the Tensor Memory Accelerator. Notably, the H100 introduces a new *logical* level called the *warp group* (collection of 4-aligned warps), and we show that our programming model can accommodate this new level. Moreover, because the H100 has historically served as the primary GPU for large-scale AI training, many CUDA kernels on this hardware are already highly optimized and achieve near–speed-of-light performance, providing a rigorous baseline for comparison. To ensure our results generalize beyond the H100, we additionally test programs on a second GPU, the NVIDIA 4070 SuperTi GPU, a consumer-grade chip.

Like mentioned in Section 5, when name typechecks a program, it produces a CUDA file. We compile the CUDA file with `nvcc` version with flags . To measure performance of our benchmark, we set up the values from an input using a random number generator. We use the average running time sampled across 10 iterations, with a warm-up of 5 iterations.

6.1 RQ1: Can Snap express a diversity of CUDA programs?

To answer this research question, we evaluate Snap on two benchmarks: a sequence of matrix multiplication kernels that grow in complexity and a single-pass parallel prefix scan with decoupled look-back that requires programmers to think carefully about multiple points of convergence.

Matrix Multiplication. We chose matrix multiplication as our first benchmark for two main reasons. First, there exist several implementations that achieve near-peak performance, providing a strong baseline. Second, matrix multiplication allows for a range of increasingly sophisticated implementations that stress different parts of the language, making it ideal for evaluating expressiveness.

To undertake this exploration, we adapt the codebase from to implement a matrix multiplication on the `float` datatype, also known as `sgemm`, in Snap. As this is a `float` benchmark, it *does not need to* use advanced GPU features like asynchrony or tensor cores to achieve speed-of-light performance. We will discuss these advanced features in Section 6.2.

We implement several variants of the `sgemm` benchmark, reproduced in Appendix . These include a naive baseline, a version that exploits memory coalescing, 2D blocking with shared-memory staging, 2D blocking with vectorized loads, and a warp-tiling strategy that effectively introduces an additional level of tiling from the perspective of a warp. We find that the runtime performance of our Snap implementations closely matches that of the original versions. This is expected, since the generated code is nearly identical to the hand-written versions, aside from small differences such as hoisted expressions and a few index calculations. The downstream compiler (`nvcc`) readily inlines hoisted expressions that are used only once, so the performance-critical inner loops remain effectively unchanged.

Single-pass Parallel Prefix Scan with Decoupled Look-Back. We also implement scan, a widely used parallel primitive, in Snap. We focus on the prefix-sum scan, which computes, for each position in an array, the sum of all elements up to that position. Prefix-sum sits in a different corner of the GPU design space from matrix multiplication: it is memory intensive, requires careful attention to the convergence behavior of different threads, and traditionally requires multiple passes over data.

We implement the single-pass parallel prefix scan with decoupled look-back, introduced by Merrill and Garland , an elegant algorithm that does not require multiple passes over the input data. In this algorithm, each block computes a local prefix sum over its assigned region of the array. Once finished, it writes the final element of its region into a global array. Each block then *looks back* to accumulate the contributions of prior blocks, allowing them to independently determine the global prefix without a full sweep over memory. This decoupled look-back mechanism lets blocks progress at different speeds while still producing correct global results.

The algorithm involves several distinct points of convergence. Within each block, work is decomposed into fine-grained thread-level and warp-level scans. After producing the local result, blocks publish their partial prefix to global memory. Finally, each block waits until enough prefix information from earlier blocks becomes available, at which point it accumulates the value and completes its section of the global scan.

We implement this full strategy in Snap, available in Appendix . Our implementation uses the unsafe feature to implement a global-memory spinlock that lets blocks check when the preceeding block's data is ready.

6.2 RQ2: Can Snap express programs that use advanced GPU features?

To test whether Snap can express programs that use advanced features of modern GPUs, we write a matrix multiplication for the `bf16` datatype for the H100.

This benchmark is an acid test of our language, as the H100 `bf16` matrix multiplication pushes several language features to the extreme. To write a matrix multiplication that can hit peak throughput on an H100, we need to write a warp-specialized kernel that uses the Tensor Memory Accelerator (TMA) , an asynchronous hardware copy engine that can move tiles of data at a time, and uses the warp-group–level tensor core instructions, or `wgmma`, specifically introduced for the Hopper architecture.

The implementation works as follows. First, we assign each block on our machine a logical tile of the output to compute. The block is then divided into a producer warp-group and a set of consumer warp-groups, where the producer loads data for multiple consumers, and both must signal to each other when one is done loading and the other is ready to compute. In this way, the benchmark overlaps computation with data movement by pipelining loads.

The implementation in Snap looks different from normal CUDA code, particularly in how pipelining is expressed. Since Snap uses *names* and subsequent `partition` or `claim` constructs to determine the synchronization each region requires, when pipelining, we cannot dynamically change the pipeline slot simply by maintaining an index variable that wraps around based on the pipeline length. Instead, each pipeline slot must be given a separate name so that Snap can track them independently and actually overlap compute with data-movement. This leads to pipeline slots that must be individually named and forces the load logic to be effectively "unrolled," since we can no longer iterate over a pipeline-slot index. In turn, this forces that all pipelines in Snap be statically sized. In practice. these pipelines are statically sized in CUDA anyway because they occupy shared memory, which is a small, finite resource that must be explicitly managed.

Notably, for this benchmark, to get the `wgmma` instruction to work we did not need to add a special privilege in our language; we could simply use `thread[128]`. We did, however, need to add a special TMA asynchronous data-movement construct, since Snap will eventually need to synchronize these transfers.

6.3 RQ3: Can Snap match the performance of existing, speed-of-light CUDA code?

In Section 6.1 and Section 6.2, we examined programs that expressed the same computation in multiple ways, expressed operations that rely on multiple points of convergence, and used advanced GPU features. We now discuss the performance of these programs.

The performance of the matrix multiplication benchmarks is shown in , where we demonstrate that Snap is competitive with cuBLAS [5].

For the prefix scan, we compare our performance to , the library introduced earlier in Section 1, and show that we can reach approximately % of CUB's peak bandwidth.

Finally, we evaluate our H100 implementation and show that it is competitive with cuBLAS on square sizes, falling between %. We emphasize that this is an exacting benchmark, and achieving this level of performance requires writing large kernels with precise control over low-level features.

6.4 Case Study: Can Snap help with library design?

Over the course of our evaluation, we found ourselves developing a small library of functions—similar in spirit to the CUB library—that we could call to execute our operations. Based on our experience, we would like to study, qualitatively, if Snap can help programmers design libraries that they can compose with confidence.

As mentioned in Section 1, CUB occupies a unique design point in the GPU library ecosystem. Unlike many other libraries such as cuBLAS [5], cuDNN [6], and cuSPARSE [7], which provide global interfaces that users can call and configure, CUB provides a device-side library organized into different levels. It makes these levels apparent by prefixing each of its functions with `Device`, `Block`, and `Thread`. The single-scan prefix sum in Section 6.1 used these functions in Snap already.

Let's turn our attention to a particular CUB function—the store function—–and examine how it is equivalently expressed in Snap, and how Snap's type system reifies the assumptions implicit in CUB.

In CUB, the store function is implemented as a class, as shown below:

```
template<typename T, int BlockDimX,
       int ItemsPerThread, BlockLoadAlgorithm Algorithm = BLOCK_LOAD_DIRECT,
       int BlockDimY = 1, int BlockDimZ = 1>
class BlockStore:
```

To use it, users must first instantiate an object of this class, and then call it with shared memory.

```
using BlockLoad = cub::BlockLoad<int, 128, 4, BLOCK_LOAD_DIRECT>;
// Allocate shared memory for BlockLoad
__shared__ typename BlockLoad::TempStorage temp_storage;
int thread_data[4]; // Thread local data
BlockLoad(temp_storage).Load(d_data, thread_data);
```

CUB exposes a leaky abstraction, where information about the number of threads, block sizes, and other details seeps through:

(1) The CUB documentation needs to specify the number of threads that the function can assume to be available, because within the function, each thread must locate itself in the computation and use its `threadIdx.x` accordingly. If the starting `threadIdx.x` is not 0, the function must compute its relative ID internally.
(2) The "item per thread" design is interesting and serves two purposes. The first is related to performance: if loops have constant bounds, they can be unrolled, enabling downstream optimizations. The second is related to correctness: the function relies on the assumption that all threads call the function with an equal number of values to load.
(3) The CUB documentation also needs to specify that `thread_data` can be data local to each thread.

(4) Finally, the CUB documentation specifies that if shared memory is being overwritten, a `__syncthreads()` call must be made to ensure that all reads have completed.

On the other hand, the same function has a completely different interface in Snap:

```
@ccuda("device")
@require(block[1], thread[1])
def load(src: ptr(const(shared(int))) @ block[1],
        item_per_thread : int @ block[1],
        total_size: int @ block[1],
        thread_data : ptr(int) @ thread[1]):
```

In Snap, we are able to encapsulate code effectively, reducing the need to communicate numerous implementation details through documentation:

(1) We do not need to pass in the number of threads at all. Whenever Snap calls a function, it threads the relative ID through, so each function can be written locally as if it were running alone, rather than having to determine where the thread resides in the global array.
(2) We do not need to make `item_per_thread` a template argument for *correctness*. Its frequency is set at the function signature, so Snap will never allow a function to be called with a lower-privilege value.
(3) In our interface, `thread_data` is explicitly set to a thread-local value. Since it is not marked as `const`, Snap conservatively assumes it may be written to, and enforces at compile time that only `thread[1]` values are passed in.
(4) Finally, using our synchronization pass outlined in Section 5.1, a `__syncthreads()` call will be inserted automatically if `src` is going to be used for writing.

7 Related Work

GPU Programming Languages.

Theoretic Foundations. The design of Bundl is heavily inspired by existing work on coeffect systems [16, 17] and dependency tracking [1]. Coeffects allow type systems to talk about how programs depend on their environments, and have achieved widespread use in Rust [20] in the form of linear types. In Snap and Bundl, privileges act like coeffects by describing what compute resources are necessary for programs to execute. Dependency tracking calculi, meanwhile, allow type systems to track how data and code depend on each other, and have commonly been used to implement secure information flow analyses [9]. In Bundl, low privilege data is unable flow into high privilege contexts, and we use dependency tracking to capture this restriction in Bundl's type system.

8 Conclusion, Limitations, and Future Work

We have presented Snap, a new low-level GPU language that statically guarantees safe usage of compute resources by construction, and have demonstrated that it is possible to achieve this safety without sacrificing performance. We hope our design can inform the implementation of existing libraries like CUB that attempt to achieve the same goals without static enforcement.

We believe Snap is ripe for extension with additional features and capabilities to further improve its expressivity and usefulness to GPU programmers. Currently, Snap does not have support for explicit pointers; programmers must use built-in constructs in order to interact with memory. However, prior work like Descend [13] provides a promising template for extending Snap with more permissive pointer usage. In particular, were we to restrict Snap's `partition` construct to a specific set of injective functions identified by Descend, we believe we could guarantee data-race freedom in Snap.

Another promising area for future work is expanding the set of architectures on which Snap and its design principles can be used. While Snap currently only supports programming the GPU, we believe we can extend the core ideas cleanly to the boundary with the CPU and in general to other heterogeneous hardware architectures. Indeed, we do not think that the idea of compute privileges is useful only on the GPU, and we would like to investigate how we might adapt Snap to other hierarchical programming systems. Conversely, we also believe it is possible for Snap to support more GPU architectures featuring even coarser-grained tensor-core operations like Blackwell [8].

We also would like to improve the ergonomics of programming with Snap, in particular by improving the pipelining experience and providing users with more control over synchronization. As we saw in Section 6.2, Snap requires pipeline slots to be explicitly named and all pipelines to be statically sized. We believe we can improve upon this by adding explicit language constructs for organizing pipelines to reduce boilerplate code in this case. Similarly, all syncs are currently handled by Snap's compiler, but we would like to extend the language with user-level synchronization operations, leveraging the existing privilege system to avoid deadlocking.

On the theoretical side, we are interested in further exploration of Bundl. In particular, we would like to examine a terminating fragment of the calculus and prove the liveness property discussed in Section 4.3: that every thread in the logical grouping denoted by privilege π will always reach the parts of a program with that code privilege. We believe this is equivalent to showing that all threads with the same privilege always execute the same code and observe the same data, and are optimistic that this can be proved via logical relation, ollowing in the footsteps of Turon et al. [22] and Spies et al. [18, 19].

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A Complete Bundl Type System, Semantics, and Syntactic Soundness Proofs

A.1 Basic Definitions

$$
\begin{aligned}
\text{Hierarchy Levels } h &::= \texttt{Grid} \mid \texttt{Block} \mid \texttt{Thread} \\
\text{Memory Kinds } l &::= \texttt{Local} \mid \texttt{Shared} \mid \texttt{Global} \\
\text{Privileges } \pi &: h \times \mathbb{N} \\
\text{Base Types } b &::= \texttt{bool} \mid \texttt{int} \mid \texttt{float} \\
\text{Types } \tau &::= b \mid b[\,]^l \mid \textbf{Fun}(\Gamma, \pi, m) \mid \texttt{async } \tau \\
\text{Contexts } \Gamma &::= \cdot \mid \Gamma, x :^{\pi} \tau \\
\text{Shared Memory Remaining } m &: \mathbb{N}
\end{aligned}
$$

Privileges π are part of an algebra parameterized over some constant values T (the number of threads per block) and B (the number of blocks per grid). With these values, we have two isomorphisms:

$$(\texttt{Block}, 1) \cong (\texttt{Thread}, T)$$

and

$$(\texttt{Grid}, 1) \cong (\texttt{Block}, B)$$

The `group` and `split` operations of Snap allows us to move along this isomorphism, from left to right. For clarity, in Bundl we split these operations into three: a `split` operation that can split privileges into multiple smaller ones, and a `destruct` operation that directly moves us along the isomorphism, and a `group` that divides our current privilege into equal sized parts.

Privileges π are also lexicographically ordered in the obvious way. hs are ordered such that $\texttt{Thread} \leq \texttt{Block} \leq \texttt{Grid}$, and $(h_1, n_1) \leq (h_2, n_2)$ iff $n_1 \leq n_2$ and $h_1 \leq h_2$.

We define scalar multiplication $i \times h$ of natural numbers with hs: $i \times (h, n) = (h, in)$.

We also define division of privileges and hierarchy levels of type $\pi \times \pi \to \mathbb{N}$. $\texttt{Grid}/\texttt{Block} = B$ and $\texttt{Block}/\texttt{Thread} = T$. We lift this to privileges like so: $(h_1, n_1)/(h_2, n_2) = ((h_1/h_2) \cdot n_1)/n_2$.

Lastly we define a partial $\downarrow$ operator on hs such that $\downarrow \texttt{Block} = \texttt{Thread}$ and $\downarrow \texttt{Grid} = \texttt{Block}$. Note that $\downarrow \texttt{Thread}$ is undefined. This operator lifts to πs whose second component is 1 and encodes the leftward component of the isomorphism above: $\downarrow (\texttt{Block}, 1) = (\texttt{Thread}, T)$ and $\downarrow (\texttt{Grid}, 1) = (\texttt{Block}, B)$.

Note also that the presentation of these rules in the main body of the paper elide the m portion, which tracks the maximum amount of memory a given computation is allowed to use. In the full system presented here, both the typing rules and the operational semantics carry an additional piece of information tracking allocated memory.

A.2 Typing Rules

A.2.1 Expressions.

$$
\frac{x :^{\pi} \tau \in \Gamma}{\Gamma \vdash^{\pi} x : \tau}\ \textbf{T-Var}
\qquad
\frac{}{\Gamma \vdash^{\pi} n : \texttt{int}}\ \textbf{T-Int}
\qquad
\frac{}{\Gamma \vdash^{\pi} f : \texttt{float}}\ \textbf{T-Float}
$$

$$
\frac{}{\Gamma \vdash^{\pi} b : \texttt{bool}}\ \textbf{T-Bool}
\qquad
\frac{\pi < (\texttt{Grid}, 1)}{\Gamma \vdash^{\pi} \texttt{partition\_id} : \texttt{int}}\ \textbf{T-Partition-Id}
$$

$$
\frac{\Gamma \vdash^{\pi'} e_1 : \tau[\,]^l \quad \Gamma \vdash^{\pi} e_2 : \texttt{int} \quad l = \texttt{Global or } l = \texttt{Local} \quad \pi \leq \pi'}{\Gamma \vdash^{\pi} e_1[e_2] : \tau}\ \textbf{T-Arr-Access}
$$

$$\frac{\Gamma \vdash^{\pi'} e_1 : \tau[\,]^{\texttt{Shared}} \quad \Gamma \vdash^{\pi} e_2 : \texttt{int} \quad \pi \leq (\texttt{Block},1) \quad \pi \leq \pi'}{\Gamma \vdash^{\pi} e_1[e_2] : \tau}\ \textbf{T-Arr-Access-Shared}$$

$$\frac{\Gamma \vdash^{\pi} e_1 : \texttt{int} \quad \Gamma \vdash^{\pi} e_2 : \texttt{int}}{\Gamma \vdash^{\pi} e_1 \ \texttt{bop}\ e_2 : \texttt{int}}\ \textbf{T-Bop} \qquad \frac{\Gamma \vdash^{\pi} e_1 : \texttt{int} \quad \Gamma \vdash^{\pi} e_2 : \texttt{int}}{\Gamma \vdash^{\pi} e_1 \ \texttt{cmp}\ e_2 : \texttt{bool}}\ \textbf{T-Cmp}$$

A.2.2 Statements.

$$\frac{f :^{\pi} \textbf{Fun}(x_i : \tau_i, \pi, m') \in \Gamma \quad \Gamma \vdash^{\pi} e_i : \tau_i \quad m' \leq m}{\Gamma \vdash^{\pi}_{m} f(e_1, \ldots, e_n)}\ \textbf{T-Function-Call}$$

$$\frac{\Gamma \vdash^{(h,n_1)}_{m} s_1 \quad \Gamma \vdash^{(h,n_2)}_{m} s_2 \quad n_1, n_2 \ \textbf{align to}\ n}{\Gamma \vdash^{(h,n)}_{m} \texttt{split}(n_1,n_2)\{s_1\}\{s_2\}}\ \textbf{T-Split} \qquad \frac{\Gamma \vdash^{\downarrow\pi}_{m} s}{\Gamma \vdash^{\pi}_{m} \texttt{destruct in}\ s}\ \textbf{T-Destruct}$$

$$\frac{\Gamma \vdash^{(h,n)}_{m} s}{\Gamma \vdash^{(h,q\cdot n)}_{m} \texttt{group}\ q\ s}\ \textbf{T-Group} \qquad \frac{x :^{\pi'} \tau \in \Gamma \quad \Gamma \vdash^{\pi} e : \tau \quad \pi' \leq \pi \quad \pi' \mid \pi}{\Gamma \vdash^{\pi}_{m} x = e}\ \textbf{T-Assn}$$

$$\frac{}{\Gamma \vdash^{\pi}_{m} \texttt{init}_{\psi}}\ \textbf{T-Sync-Init} \qquad \frac{}{\Gamma \vdash^{\pi}_{m} \texttt{dec}_{\psi}}\ \textbf{T-Sync-Dec}$$

$$\frac{}{\Gamma \vdash^{\pi}_{m} \texttt{wait}_{\psi}}\ \textbf{T-Sync-Wait} \qquad \frac{}{\Gamma \vdash^{\pi}_{m} \texttt{skip}}\ \textbf{T-Skip} \qquad \frac{n \leq m}{\Gamma \vdash^{\pi}_{m} \texttt{free}\ n}\ \textbf{T-Free}$$

$$\frac{\Gamma \vdash^{\pi} e : \tau \quad \Gamma, x :^{\pi'} \tau \vdash^{\pi}_{m} s \quad \pi' \leq \pi \quad \pi' \mid \pi \quad \tau \ \text{not an array type}}{\Gamma \vdash^{\pi}_{m} x : \tau \ \texttt{@}\ \pi' = e \ \texttt{in}\ s}\ \textbf{T-Decl}$$

$$\frac{\Gamma \vdash^{\pi'} e_1 : \tau[\,]^{l} \quad \Gamma \vdash^{\pi} e_2 : \texttt{int} \quad \Gamma \vdash^{\pi} e_3 : \tau \quad l = \texttt{Global}\ \texttt{or}\ l = \texttt{Local} \quad \pi' \leq \pi \quad \pi' \mid \pi}{\Gamma \vdash^{\pi}_{m} e_1[e_2] = e_3}\ \textbf{T-Arr-Assn}$$

$$\frac{\Gamma \vdash^{\pi'} e_1 : \tau[\,]^{\texttt{Shared}} \quad \Gamma \vdash^{\pi} e_2 : \texttt{int} \quad \Gamma \vdash^{\pi} e_3 : \tau \quad \pi \leq (\texttt{Block},1) \quad \pi' \leq \pi \quad \pi' \mid \pi}{\Gamma \vdash^{\pi}_{m} e_1[e_2] = e_3}\ \textbf{T-Arr-Assn-Shared}$$

$$\frac{\Gamma \vdash^{\pi} e : \texttt{bool} \quad \Gamma \vdash^{\pi}_{m_1} s_1 \quad \Gamma \vdash^{\pi}_{m_2} s_2}{\Gamma \vdash^{\pi}_{\textbf{max}(m_1,m_2)} \texttt{if}\ e\ \texttt{then}\ s_1\ \texttt{else}\ s_2}\ \textbf{T-If}$$

$$\frac{\Gamma \vdash^{\pi} e : \texttt{bool} \quad \Gamma \vdash^{\pi}_{m} s}{\Gamma \vdash^{\pi}_{m} \texttt{while}\ e\ \texttt{do}\ s}\ \textbf{T-While} \qquad \frac{\Gamma \vdash^{\pi}_{m_1} s_1 \quad \Gamma \vdash^{\pi}_{m_2} s_2}{\Gamma \vdash^{\pi}_{\textbf{max}(m_1,m_2)} s_1 ; s_2}\ \textbf{T-Seq}$$

$$\frac{\Gamma, x :^{\pi} \tau[\,]^{l} \vdash^{\pi}_{m} s \quad l = \texttt{Global}\ \texttt{or}\ l = \texttt{Local}}{\Gamma \vdash^{\pi}_{m + n \cdot \texttt{size}(\tau)} x := \texttt{alloc}\ l\ \tau\ n\ \texttt{in}\ s}\ \textbf{T-Alloc}$$

$$\frac{\Gamma, x :^{(\texttt{Block},1)} \tau[\,]^{\texttt{Shared}} \vdash^{\pi}_{m} s}{\Gamma \vdash^{(\texttt{Block},1)}_{m + n \cdot \texttt{size}(\tau)} x := \texttt{alloc Shared}\ \tau\ n\ \texttt{in}\ s}\ \textbf{T-Alloc-Shared}$$

$$\frac{\Gamma, y :^{(h,n/c)} \tau[\,]^{l} \vdash^{(h,n)}_{m} s \quad c \mid n \quad l \neq \texttt{Local}}{\Gamma, x :^{(h,n)} \tau[\,]^{l} \vdash^{(h,n)}_{m} \texttt{partition}_{\psi}\ x\ \texttt{into}\ y\ \texttt{by}\ c\ \texttt{in}\ s}\ \textbf{T-Partition}$$

$$\frac{\Gamma, y :^{(h,n')} \tau[\,]^{l} \vdash^{(h,n')}_{m} s \quad n' \leq n \quad l \neq \texttt{Local}}{\Gamma, x :^{(h,n)} \tau[\,]^{l} \vdash^{(h,n)}_{m} \texttt{claim}_{\psi}\ x\ \texttt{into}\ y\ \texttt{at}\ n'\ \texttt{in}\ s}\ \textbf{T-Claim}$$

$$\frac{\Gamma, y :^{\downarrow\pi} \tau[]^{l} \vdash^{\pi}_{m} s \quad l \neq \texttt{Local}}{\Gamma, x :^{\pi} \tau[]^{l} \vdash^{\pi}_{m} \texttt{lower}_{\psi}\ x\ \texttt{into}\ y\ \texttt{in}\ s}\ \textbf{T-Lower}$$

$$\frac{\Gamma, y :^{(\text{Thread},1)} \texttt{async}\ \tau[]^{l} \vdash^{(\text{Thread},1)}_{m} s}{\Gamma, x :^{(\text{Thread},1)} \tau[]^{l} \vdash^{(\text{Thread},1)}_{m} \texttt{async\_partition}_{\phi}\ x\ \texttt{into}\ y\ \texttt{in}\ s}\ \textbf{T-Async-Partition}$$

$$\frac{}{\Gamma, x :^{(\text{Thread},1)} \texttt{async}\ \tau[]^{l}, y :^{(\text{Thread},1)} \tau[]^{l'} \vdash^{(\text{Thread},1)}_{m} \texttt{async\_mempcy}(x,y)}\ \textbf{T-Async-Memcpy}$$

$$\frac{}{\Gamma, x :^{\pi} \tau[]^{l}, y :^{\pi} \tau[]^{l'} \vdash^{\pi}_{m} \texttt{memcpy}(x,y)}\ \textbf{T-Memcpy}$$

A.3 Complete Bundl Semantics

A.3.1 Definitions.

$$\text{Global Memory } \Sigma ::= \cdot \mid \Sigma, n \mapsto^{\pi} v$$
$$\text{Shared Memory } \sigma ::= \cdot \mid \sigma, n \mapsto^{\pi} v$$
$$\text{Local Memory } \eta ::= \cdot \mid \eta, n \mapsto^{\pi} v$$

$$\text{Block Memory Map } S ::= \forall n \in B, n \mapsto \sigma$$
$$\text{Thread Memory Map } L ::= \forall n \in T, n \mapsto \eta$$

$$\text{Synchronization Map } \Psi : \psi \to \mathbb{N} \to \mathbb{N}$$
$$\text{Deferred Computations Map } \Phi : \phi \to \{s\}$$

In real GPUs, thread ids are only unique within their block. However, in this calculus for simplicity we assume thread ids are global. One can convert back and forth between this abstracted notion of a thread id and a block-unique id via addition modulo T. That is, $t_{\text{real}} = t_{\text{simplified}}\ \textbf{mod}\ T$ and $t_{\text{simplified}} = t_{\text{real}} + b \cdot T$.

By convention the names for local and shared and global memory do not conflict, as on the GPU they will be separate pointer spaces. Additionally, we freely interchange between using names for variables and integer locations.

In the main body of the paper, for simplicity we elide the synchronization map and the deferred computation map from the operational semantics, as our theorems do not make any guarantees about non-interference. However, as they are part of the full semantics, we include them here for completeness. By convention the synchronization and deferred computation maps are a total functions, initialized to map to $\lambda\_.0$ for ψs and $\lambda\_.\{\}$ for ϕs not explicitly initialize.

The shape of the judgment for a single thread is $\eta, \sigma, \Sigma, t, b, p, \Psi \vdash^{\pi}_{m} s \rightsquigarrow s' \dashv_{m'} \eta', \sigma', \Sigma', \Psi'$. The t here is the thread id, the b is the block id, and the p is the partition id. The last of these three is modified and managed by the rules for `split`, `group` and `destruct`, and tracks the relative position of the thread within a group. This semantics is in a small step style.

The shape of the judgment for expressions is $\eta, \sigma, \Sigma \vdash^{\pi}_{\pi'} e \Downarrow v$. The two πs represent the ambient compute context (i.e., the context in which resources are being read), while π', represents the target compute context (i.e, the compute context of the variable into which the result of the expression is going to be written. This is relevant for computing the value of `partition_id`, which divides the two contexts. As a shorthand, we can divide a privilege by a scalar value like so: $(h,n)/c = (h,n)/(h,c)$.

The overall evaluation of a program is expressed as

$$L,S,\Sigma,P,\Psi,\Phi \rightsquigarrow L',S',\Sigma',P',\Psi',\Phi'.$$

In this judgment P serves as a *thread pool*, mapping pairs of thread and block ids (which don't change) to statements and memory (which can be updated by stepping). One can think of P as tracking which program is running on each thread. This steps according to the following rule:

$$\frac{L(t),S(b),\Sigma,t,b,0,\Psi,\Phi \vdash_m^{(\texttt{Grid},1)} s \rightsquigarrow s' \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi' \qquad P(t,b)=(s,m)}{L,S,\Sigma,P,\Psi,\Phi \rightsquigarrow L[t\mapsto\eta'],S[b\mapsto\sigma'],\Sigma',P[(t,b)\mapsto(s',m')],\Psi',\Phi'}\ \textbf{S-Program}$$

For simplicity of notation, we define an **update** operation that searches the three environments for the one that contains the variable being used (by convention, there is no conflict between the environments, as in reality they exist in three separate address spaces). We also define a similar **get** operation that retrieves a variable from memory, and a **rename** operation that remaps a variable with the same value but under a different name.

$$\begin{aligned}
\textbf{update}(\eta,\sigma,\Sigma,x,v) &= (\eta[x\mapsto^{\pi} v],\sigma,\Sigma)\ \textit{when}\ x\in^{\pi}\eta\\
\textbf{update}(\eta,\sigma,\Sigma,x,v) &= (\eta,\sigma[x\mapsto^{\pi} v],\Sigma)\ \textit{when}\ x\in^{\pi}\sigma\\
\textbf{update}(\eta,\sigma,\Sigma,x,v) &= (\eta,\sigma,\Sigma[x\mapsto^{\pi} v])\ \textit{when}\ x\in^{\pi}\Sigma
\end{aligned}$$

$$\begin{aligned}
\textbf{get}(\eta,\sigma,\Sigma,x) &= \eta(x)\ \textit{when}\ x\in^{\pi}\eta\\
\textbf{get}(\eta,\sigma,\Sigma,x) &= \sigma(x)\ \textit{when}\ x\in^{\pi}\sigma\\
\textbf{get}(\eta,\sigma,\Sigma,x) &= \Sigma(x)\ \textit{when}\ x\in^{\pi}\Sigma
\end{aligned}$$

$$\begin{aligned}
\textbf{rename}(\eta,\sigma,\Sigma,x,y,\pi') &= (\eta[y\mapsto^{\pi'}\eta(x)],\sigma,\Sigma)\ \textit{when}\ x\in^{\pi}\eta\\
\textbf{rename}(\eta,\sigma,\Sigma,x,y,\pi') &= (\eta,\sigma[y\mapsto^{\pi'}\sigma(x)],\Sigma)\ \textit{when}\ x\in^{\pi}\sigma\\
\textbf{rename}(\eta,\sigma,\Sigma,x,y,\pi') &= (\eta,\sigma,\Sigma[y\mapsto^{\pi'}\Sigma(x)])\ \textit{when}\ x\in^{\pi}\Sigma
\end{aligned}$$

A.3.2 *Privilege Management Rules.*

$$\frac{p<n_1 \quad n_1,n_2\ \textbf{align to}\ n \quad \eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{(h,n_1)} s_1 \rightsquigarrow s_1' \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{(h,n)} \texttt{split}(n_1,n_2)\{s_1\}\{s_2\} \rightsquigarrow \texttt{split}(n_1,n_2)\{s_1'\}\{s_2\} \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}\ \textbf{S-Split-Left}$$

$$\frac{p<n_1 \quad n_1,n_2\ \textbf{align to}\ n}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{(h,n)} \texttt{split}(n_1,n_2)\{\texttt{skip}\}\{s_2\} \rightsquigarrow \texttt{skip} \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Split-Left-Done}$$

$$\frac{p\geq n_1 \quad p<n_1+n_2 \quad n_1,n_2\ \textbf{align to}\ n \quad \eta,\sigma,\Sigma,t,b,p-n_1,\Psi,\Phi \vdash_m^{(h,n_2)} s_2 \rightsquigarrow s_2' \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{(h,n)} \texttt{split}(n_1,n_2)\{s_1\}\{s_2\} \rightsquigarrow \texttt{split}(n_1,n_2)\{s_1\}\{s_2'\} \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}\ \textbf{S-Split-Right}$$

$$\frac{p\geq n_1 \quad p<n_1+n_2 \quad n_1,n_2\ \textbf{align to}\ n}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{(h,n)} \texttt{split}(n_1,n_2)\{s_1\}\{\texttt{skip}\} \rightsquigarrow \texttt{skip} \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Split-Right-Done}$$

$$\frac{p\geq n_1+n_2 \quad n_1,n_2\ \textbf{align to}\ n}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{(h,n)} \texttt{split}(n_1,n_2)\{s_1\}\{s_2\} \rightsquigarrow \texttt{skip} \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Split-None}$$

$$\frac{\eta,\sigma,\Sigma,t,b,t \ \mathbf{mod}\ T,\Psi,\Phi \vdash_m^{(\mathtt{Thread},T)} s \rightsquigarrow s' \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}{\eta,\sigma,\Sigma,t,b,0,\Psi,\Phi \vdash_m^{(\mathtt{Block},1)} \mathtt{destruct\ in}\ s \rightsquigarrow \mathtt{destruct\ in}\ s' \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}\ \textbf{S-Destruct-Block}$$

$$\frac{\eta,\sigma,\Sigma,t,b,b \ \mathbf{mod}\ B,\Psi,\Phi \vdash_m^{(\mathtt{Block},B)} s \rightsquigarrow s' \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}{\eta,\sigma,\Sigma,t,b,0,\Psi,\Phi \vdash_m^{(\mathtt{Grid},1)} \mathtt{destruct\ in}\ s \rightsquigarrow \mathtt{destruct\ in}\ s' \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}\ \textbf{S-Destruct-Grid}$$

$$\frac{}{\eta,\sigma,\Sigma,t,b,0,\Psi,\Phi \vdash_m^{\pi} \mathtt{destruct\ in\ skip} \rightsquigarrow \mathtt{skip} \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Destruct-Done}$$

$$\frac{\eta,\sigma,\Sigma,t,b,p \ \mathbf{mod}\ n,\Psi,\Phi \vdash_m^{(h,n)} s \rightsquigarrow s' \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{(h,q\cdot n)} \mathtt{group}\ q\ s \rightsquigarrow \mathtt{group}\ q\ s'; \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}\ \textbf{S-Group}$$

$$\frac{}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{\pi} \mathtt{group}\ q\ \mathtt{skip} \rightsquigarrow \mathtt{skip}; \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Group-Done}$$

A.3.3 Thread Synchronization. We define a **size** operation on privileges to compute the size of a partition (the number of individual threads contained within it). The operation is defined as follows:

$$\begin{aligned}\mathbf{size}(\mathtt{Thread},n) &= n\\ \mathbf{size}(\mathtt{Block},n) &= n\cdot T\\ \mathbf{size}(\mathtt{Grid},n) &= n\cdot B\cdot T\end{aligned}$$

$$\frac{\Psi(\psi)(p)=0}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{\pi} \mathtt{wait}_\psi \rightsquigarrow \mathtt{skip} \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Sync-Wait-Done}$$

$$\frac{\Psi(\psi)(p)\neq 0}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{\pi} \mathtt{wait}_\psi \rightsquigarrow \mathtt{wait}_\psi \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Sync-Wait-Spin}$$

$$\frac{\Psi' = \Psi(\psi)[p \mapsto \Psi(\psi)(p)-1]}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{\pi} \mathtt{dec}_\psi \rightsquigarrow \mathtt{skip} \dashv_m \eta,\sigma,\Sigma,\Psi',\Phi}\ \textbf{S-Sync-Dec}$$

$$\frac{\Psi(\psi)(p)=0 \quad \Psi' = \Psi(\psi)[p \mapsto \mathbf{size}(\pi)]}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{\pi} \mathtt{init}_\psi \rightsquigarrow \mathtt{skip} \dashv_m \eta,\sigma,\Sigma,\Psi',\Phi}\ \textbf{S-Sync-Init-Zero}$$

$$\frac{\Psi(\psi)(p)\neq 0}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{\pi} \mathtt{init}_\psi \rightsquigarrow \mathtt{skip} \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Sync-Init-Nonzero}$$

A.3.4 Asynchrony.

$$\frac{\mathbf{rename}(\eta,\sigma,\Sigma,x,y,(\mathtt{Thread},1)),t,b,p,\Psi,\Phi \vdash_m^{(\mathtt{Thread},1)} s \rightsquigarrow s' \dashv_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}{\begin{array}{c}\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_{m'}^{(\mathtt{Thread},1)} \mathtt{async\_partition}_\phi\ x\ \mathtt{into}\ y\ \mathtt{in}\ s \rightsquigarrow\\ \mathtt{async\_partition}_\phi\ x\ \mathtt{into}\ y\ \mathtt{in}\ s' \dashv_m \eta',\sigma',\Sigma',\Psi',\Phi'\end{array}}\ \textbf{S-Async-Partition-Congr}$$

$$\frac{\Phi = \Phi'[\phi \mapsto \Phi'(\phi)\cup\{s\}]}{\begin{array}{c}\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash_m^{(\mathtt{Thread},1)} \mathtt{async\_partition}_\phi\ x\ \mathtt{into}\ y\ \mathtt{in\ skip} \rightsquigarrow\\ (\mathtt{async\_partition}_\phi\ x\ \mathtt{into}\ y\ \mathtt{in}\ s) \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi'\end{array}}\ \textbf{S-Async-Partition-Unwind}$$

$$\frac{\Phi(\phi)=\emptyset}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{(\mathsf{Thread},1)} \texttt{async\_partition}_\phi\ x\ \texttt{into}\ y\ \texttt{in}\ \texttt{skip}\rightsquigarrow \texttt{skip}\dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Async-Partition-Done}$$

$$\frac{}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{(\mathsf{Thread},1)} \texttt{async\_memcpy}(x,y)\rightsquigarrow \texttt{skip}\dashv_m \eta,\sigma,\Sigma,\Psi,\Phi[\phi\mapsto\Phi(\phi)\cup\{\texttt{memcpy}(x,y)\}]}\ \textbf{S-Async-Memcpy}$$

$$\frac{(\eta',\sigma',\Sigma')=\mathbf{update}(\eta,\sigma,\Sigma,x,\mathbf{get}(\eta,\sigma,\Sigma,y))}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{\pi} \texttt{memcpy}(x,y)\rightsquigarrow \texttt{skip}\dashv_m \eta',\sigma',\Sigma',\Psi,\Phi}\ \textbf{S-Memcpy}$$

A.3.5 Variables and Memory.

$$\frac{\eta,\sigma,\Sigma\vdash_{\pi'}^{\pi} e\Downarrow v}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{\pi} x:\tau\ \texttt{@}\ \pi':=e\ \texttt{in}\ s\rightsquigarrow s\dashv_m \eta[x\mapsto^{\pi'} v],\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Decl}$$

$$\frac{}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{\pi} \texttt{free}\ n\dashv_{m-n} \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Free}$$

$$\frac{}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{\pi} x:=\texttt{alloc Local}\ \tau\ n\ \texttt{in}\ s\rightsquigarrow s;\ \texttt{free}\ (n\cdot\mathsf{size}(\tau))\dashv_{m+n\cdot\mathsf{size}(\tau)} \eta[x\mapsto^{\pi}\langle x,n\rangle],\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Alloc-Local}$$

$$\frac{\pi=(\mathsf{Block},1)}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{\pi} x:=\texttt{alloc Shared}\ \tau\ n\ \texttt{in}\ s\rightsquigarrow s;\ \texttt{free}\ (n\cdot\mathsf{size}(\tau))\dashv_{m+n\cdot\mathsf{size}(\tau)} \eta,\sigma[x\mapsto^{\pi}\langle x,n\rangle],\Sigma,\Psi,\Phi}\ \textbf{S-Alloc-Shared}$$

$$\frac{}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{\pi} x:=\texttt{alloc Global}\ \tau\ n\ \texttt{in}\ s\rightsquigarrow s;\ \texttt{free}\ (n\cdot\mathsf{size}(\tau))\dashv_{m+n\cdot\mathsf{size}(\tau)} \eta,\sigma,\Sigma[x\mapsto^{\pi}\langle x,n\rangle],\Psi,\Phi}\ \textbf{S-Alloc-Global}$$

$$\frac{x\in^{\pi'}\eta,\sigma,\Sigma\qquad \eta,\sigma,\Sigma\vdash_{\pi'}^{\pi} e\Downarrow v\qquad (\eta',\sigma',\Sigma')=\mathbf{update}(\eta,\sigma,\Sigma,x,v)\qquad \pi'\mid\pi}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi,\Phi\vdash_m^{\pi} x=e\rightsquigarrow \texttt{skip}\dashv_m \eta',\sigma',\Sigma',\Psi,\Phi}\ \textbf{S-Assn}$$

$$\frac{\begin{array}{cc}\eta,\sigma,\Sigma,t,b,p\vdash_{\pi}^{\pi} e_1\Downarrow\langle l,n\rangle & i<n\quad \pi'\mid\pi\\ \eta,\sigma,\Sigma,t,b,p\vdash_{\pi}^{\pi} e_2\Downarrow i & x\in^{\pi'}\eta,\sigma,\Sigma\\ \eta,\sigma,\Sigma\vdash_{\pi'}^{\pi} e_3\Downarrow v & (\eta',\sigma',\Sigma')=\mathbf{update}(\eta,\sigma,\Sigma,l+i,v)\end{array}}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{\pi} e_1[e_2]=e_3\rightsquigarrow \texttt{skip}\dashv_m \eta',\sigma',\Sigma',\Psi,\Phi}\ \textbf{S-Arr-Assn}$$

$$\frac{s'=s[(y+c\cdot p)/y]\qquad (\eta',\sigma',\Sigma')=\mathbf{rename}(\eta,\sigma,\Sigma,x,y,\pi/c)}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{\pi} partition\psi xycs\rightsquigarrow \texttt{init}_\psi;s';\texttt{dec}_\psi;\texttt{wait}_\psi\dashv_m \eta',\sigma',\Sigma',\Psi,\Phi}\ \textbf{S-Partition}$$

$$\frac{(\eta',\sigma',\Sigma')=\mathbf{rename}(\eta,\sigma,\Sigma,x,y,(h,n_1))}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{(h,n_1+n_2)} \texttt{claim}_\psi\ x\ \texttt{into}\ y\ \texttt{at}\ n_1\ \texttt{in}\ s\rightsquigarrow \texttt{init}_\psi;\texttt{split}(n_1,n_2)\{s'\}\{\texttt{skip}\};\texttt{dec}_\psi;\texttt{wait}_\psi\dashv_m \eta',\sigma',\Sigma',\Psi,\Phi}\ \textbf{S-Claim}$$

$$\frac{(\eta',\sigma',\Sigma')=\mathbf{rename}(\eta,\sigma,\Sigma,x,y,\downarrow\pi)}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi\vdash_m^{\pi} \texttt{lower}_\psi\ x\ \texttt{into}\ y\ \texttt{in}\ s\rightsquigarrow \texttt{init}_\psi;s;\texttt{dec}_\psi;\texttt{wait}_\psi\dashv_m \eta',\sigma',\Sigma',\Psi,\Phi}\ \textbf{S-Lower}$$

A.3.6 Control Flow.

$$\frac{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi} e \Downarrow \texttt{true}}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m} \texttt{if}\ e\ \texttt{then}\ s_1\ \texttt{else}\ s_2 \rightsquigarrow s_1 \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-If-True}$$

$$\frac{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi} e \Downarrow \texttt{false}}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m} \texttt{if}\ e\ \texttt{then}\ s_1\ \texttt{else}\ s_2 \rightsquigarrow s_2 \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-If-False}$$

$$\frac{}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m} \texttt{while}\ e\ \texttt{do}\ s \rightsquigarrow \texttt{if}\ e\ \texttt{then}\ (s;\ \texttt{while}\ e\ \texttt{do}\ s)\ \texttt{else}\ \texttt{skip} \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-While}$$

$$\frac{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m} s_1 \rightsquigarrow s_1' \dashv^{\pi'}_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m} s_1;s_2 \rightsquigarrow s_1';s_2 \dashv^{\pi'}_{m'} \eta',\sigma',\Sigma',\Psi',\Phi'}\ \textbf{S-Seq-First}$$

$$\frac{}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m} \texttt{skip};s_2 \rightsquigarrow s_2 \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Seq-Done}$$

$$\frac{\Sigma(f)=\{[x_1:\tau_1,\ldots,x_n:\tau_n],s\} \quad \sigma,\Sigma \vdash^{\pi}_{\pi} e_i \Downarrow v_i \quad m' \le m}{\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m} f(e_1,\ldots,e_n) \rightsquigarrow s \dashv_m \eta[x_i \mapsto v_i],\sigma,\Sigma,\Psi,\Phi}\ \textbf{S-Function-Call}$$

A.3.7 Expressions.

$$\frac{\pi < (\texttt{Grid},1)}{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} \texttt{partition\_id} \Downarrow \pi/\pi' - 1}\ \textbf{E-Partition-Id}$$

$$\frac{}{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} x \Downarrow \textbf{get}(\eta,\sigma,\Sigma,x)}\ \textbf{E-Var}$$

$$\frac{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} e_1 \Downarrow \langle l,n \rangle \quad \eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} e_2 \Downarrow i \quad i<n \quad \pi' \le \pi}{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} e_1[e_2] \Downarrow \textbf{get}(\eta,\sigma,\Sigma,l+i)}\ \textbf{E-Arr-Access}$$

$$\frac{}{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} \vdash n \Downarrow n}\ \textbf{E-Int} \qquad \frac{}{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} \vdash b \Downarrow b}\ \textbf{E-Bool}$$

$$\frac{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} \vdash e_1 \Downarrow v_1 \quad \eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} e_2 \Downarrow v_2 \quad v = v_1\ \texttt{bop}\ v_2}{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} \vdash e_1\ \texttt{bop}\ e_2 \Downarrow v}\ \textbf{E-Bop}$$

$$\frac{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} \vdash e_1 \Downarrow v_1 \quad \eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} e_2 \Downarrow v_2 \quad v = v_1\ \texttt{cmp}\ v_2}{\eta,\sigma,\Sigma \vdash^{\pi}_{\pi'} \vdash e_1\ \texttt{cmp}\ e_2 \Downarrow v}\ \textbf{E-Cmp}$$

A.4 Theorems and Proofs

Note that in this section we assume no out of bounds array accesses. In general Snap (and by extension Bundl) makes no guarantees about array out of bounds.

A.4.1 More definitions. As a premise to our type safety theorems, we need to assume we have a well-typed environment, written $\Gamma \vdash \eta,\sigma,\Sigma$. We define what this means inductively

$$\frac{}{\eta,\sigma,\Sigma \vdash n : \texttt{int}}\ \textbf{V-Int} \qquad \frac{}{\eta,\sigma,\Sigma \vdash b : \texttt{bool}}\ \textbf{V-Bool} \qquad \frac{}{\eta,\sigma,\Sigma \vdash f : \texttt{float}}\ \textbf{V-Float}$$

$$\frac{\forall i<n, \eta,\sigma,\Sigma \vdash \textbf{get}(\eta,\sigma,\Sigma,x+i) : \tau}{\eta,\sigma,\Sigma \vdash \langle x,n \rangle : \tau[]^l}\ \textbf{V-Array} \qquad \frac{\Gamma, x_i :^{\pi} \tau_i \vdash^{\pi}_{m} s \quad \Gamma \vdash \cdot,\cdot,\textbf{fns}\ \Sigma}{\eta,\sigma,\Sigma \vdash \{x_i : \tau_i, s\} : \textbf{Fun}(x_i : \tau_i, \pi, m)}\ \textbf{V-Function}$$

$$\frac{}{\cdot \vdash \eta,\sigma,\Sigma}\ \textbf{G-Empty} \qquad \frac{\eta(x) =^{\pi} v \quad \eta,\sigma,\Sigma \vdash v : \texttt{int}}{\Gamma, x :^{\pi} \texttt{int} \vdash \eta,\sigma,\Sigma}\ \textbf{G-Int}$$

$$\frac{\eta(x) =^\pi v \quad \eta,\sigma,\Sigma \vdash v : \texttt{bool}}{\Gamma, x :^\pi \texttt{bool} \vdash \eta,\sigma,\Sigma}\ \textbf{G-Bool} \qquad \frac{\eta(x) =^\pi v \quad \eta,\sigma,\Sigma \vdash v : \texttt{float}}{\Gamma, x :^\pi \texttt{float} \vdash \eta,\sigma,\Sigma}\ \textbf{G-Float}$$

$$\frac{\eta(x) =^\pi v \quad \eta,\sigma,\Sigma \vdash v : \tau[]}{\Gamma, x :^\pi \tau[]^{\texttt{Local}} \vdash \eta,\sigma,\Sigma}\ \textbf{G-Local} \qquad \frac{\sigma(x) =^\pi v \quad \eta,\sigma,\Sigma \vdash v : \tau[]}{\Gamma, x :^\pi \tau[]^{\texttt{Shared}} \vdash \eta,\sigma,\Sigma}\ \textbf{G-Shared}$$

$$\frac{\Sigma(x) =^\pi v \quad \eta,\sigma,\Sigma \vdash v : \tau[]}{\Gamma, x :^\pi \tau[]^{\texttt{Global}} \vdash \eta,\sigma,\Sigma}\ \textbf{G-Global} \qquad \frac{\Sigma(x) =^\pi v \quad \eta,\sigma,\Sigma \vdash v : \textbf{Fun}(\Gamma',\pi,m)}{\Gamma, x :^\pi \textbf{Fun}(\Gamma',\pi,m) \vdash \eta,\sigma,\Sigma}\ \textbf{G-Function}$$

We can prove a couple simple lemmas about well-typed environments under operations like **rename**, **update**, and **get**.

Lemma A.1. *(Well-typed **get**) If $\Gamma \vdash \eta,\sigma,\Sigma$ and $x :^\pi \tau \in \Gamma$ then $\eta,\sigma,\Sigma \vdash \textbf{get}(\eta,\sigma,\Sigma,x) : \tau$.*

Lemma A.2. *(Well-typed **rename**) If $\Gamma, x :^\pi \tau \vdash \eta,\sigma,\Sigma$ then $\Gamma, x :^\pi \tau, y :^{\pi'} \tau \vdash \textbf{rename}(\eta,\sigma,\Sigma,x,y,\pi')$.*

Lemma A.3. *(Well-typed **update**) If $\Gamma \vdash \eta,\sigma,\Sigma$ and $\eta,\sigma,\Sigma \vdash v : \tau$ then $\Gamma, x :^\pi \tau \vdash \textbf{update}(\eta,\sigma,\Sigma,x,v)$.*

We also define a well-formedness precondition on p with respect to π:

$$(h,n) \vdash p ::= p < n$$

We also define well-formedness for the async stack:

$$\Gamma \vdash \Phi ::= \forall \phi, s \in \Phi(\phi), \Gamma \vdash_m^{(\texttt{Thread},1)} s$$

A.5 Proofs

Lemma A.4. *(Expression Safety) If $\Gamma \vdash^\pi e : \tau$ and $\Gamma \vdash \eta,\sigma,\Sigma$ and $\pi \vdash p$, then there is some v such that $\eta,\sigma,\Sigma \vdash_{\pi'}^{\pi} e \Downarrow v$ and $v : \tau$.*

Proof. This proof proceeds by induction on the typing relation for expressions. Despite the fact that this property implies termination, we do not need a logical relation to prove it because the expression language is very simple.

The **T-Int**, **T-Float**, and **T-Bool** cases are trivial, using the rules **E-Int**, **E-Bool** and **E-Float** to compute values. In the case for **T-Partition-Id**, the π premises of the typing rules match the premises of the evaluation rules, so these rules are simple as well.

The cases for **T-Bop** and **T-Cmp** follow directly from the inductive hypotheses, assuming a valid and correctly implemented set of binary operators and comparators.

The only interesting cases are **T-Arr-Access** and **T-Arr-Access-Shared**.

In both cases our inductive hypotheses and inversion give us that $\pi' \leq \pi$, and e_1 evaluates to a $\langle x,n \rangle$, and that all the values between x and $x+n$ in the appropriate environment are typed at τ. We also know that e_2 evaluates to an integer i. We assume that all array accesses are in bounds, so $i < l$, which is sufficient to use the **E-Arr-Access** rule to complete this case, and the proof. □

Lemma A.5. *(Expression Determinism) If $\eta,\sigma,\Sigma,t,b,p \vdash_{\pi'}^{\pi} e \Downarrow v_1$ and $\eta,\sigma,\Sigma,t,b,p \vdash_{\pi'}^{\pi} e \Downarrow v_2$ then $v_1 = v_2$.*

Proof. Straightforward by induction on the semantic derivation. □

Lemma A.6. *(Expression Well-Typedness) If $\Gamma \vdash_{\pi'}^{\pi} e : \tau$ and $\Gamma \vdash \eta,\sigma,\Sigma$ and $\pi \vdash p$, and $\eta,\sigma,\Sigma,t,b,p \vdash_{\pi'}^{\pi} e \Downarrow v$, then $v : \tau$.*

Proof. By our expression safety lemma our well-typed expression must evaluate to a well-typed value v'. By our determinism lemma v' must be the same as v, so v is well-typed. □

LEMMA A.7. *(Statement Progress) If $\Gamma \vdash \eta,\sigma,\Sigma$ and $\Gamma \vdash^{\pi}_{m} s$ and $\pi \vdash p$, then either s is* `skip` *or there is some s' such that $\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m'} s \rightsquigarrow s' \dashv_{m''} \eta',\sigma',\Sigma,\Psi',\Phi'$*

PROOF. This proceeds by induction on the typing derivation.

Privilege Management Rules.

- Case **T-Split**:
  In this case we have by our assumption that $\pi \vdash p$ that $p < n$. We also have that n_1,n_2 **align to** n, so $n_1 + n_2 \leq n$. There are three cases to consider, then: when $p < n_1$, when $p \geq n_1$ and $p < n_1 + n_2$, and when $p \geq n_1 + n_2$.
  In the first case, we have by our inductive hypothesis that s is either `skip` or that it can step in an (h,n_1) context. In the former case we can use the **S-Split-Left-Done** rule and in the latter we can use the **S-Split-Left rule**.
  The second case is almost symmetric. The only additional work we have to do is to argue that $(h,n_2) \vdash p - n_1$, or equivalently that $p - n_1 < n_2$. This, however, is immediate from our assumption that $p < n_1 + n_2$.
  In the last case, we just use the **S-Split-None** rule to step to `skip`.
- Case **T-Destruct**
  By our inductive hypothesis, we know that s can step at $\downarrow\pi$ if $\downarrow\pi \vdash p$. The $\downarrow$ operation is only defined at $(\texttt{Block},1)$ or $(\texttt{Grid},1)$, so we only need to consider the cases where π is one of those.
  In the former case p becomes t **mod** T while $\downarrow\pi$ is $(\texttt{Thread},T)$. t **mod** $T < T$ for any t so this satisfies the requirement that $\pi \vdash p$, which lets us use our inductive hypothesis: s is either `skip` or can step. If it can step, we can use this to satisfy the premise of **S-Destruct-Block** to step in this case. If it is `skip`, then we use the rule **S-Destruct-Done** to step instead.
  The latter case is the same, except using the fact that b **mod** $B < B$ and the **S-Destruct-Grid** rule.
- Case **T-Group**
  In this case we have by assumption that $(h,q \cdot n) \vdash p$, i.e., that $p < q \cdot n$.
  In this case we have our IH that if $(h,n) \vdash p'$ for some p', then s is either **skip** or steps with (h,n) privilege with p' as our partition id.
  We choose p' to be p **mod** n. This is always $< n$, so $(h,n) \vdash p'$. This lets us use our IH to get that s is either `skip` (in which case we can use the **S-Group-Done** rule to step) or itself steps, which lets us use the **S-Group** rule to step.

Thread Synchronization Rules.

- Case **T-Sync-Wait**
  $\Psi(\psi)(p)$ is either zero or it is not. In the former case we use the **S-Sync-Wait-Done** rule and in the latter we use **S-Sync-Wait-Spin**.
- Case **T-Sync-Dec**
  We use the **S-Sync-Dec** rule to step.
- Case **T-Sync-Init**
  We use the **S-Sync-Init-Zero** or **S-Sync-Init-Nonzero** rules depending on whether $\Psi(\psi)(p)$ is zero or not.

Asynchrony Rules.

- Case **T-Async-Partition**
  In this case, we have via our IH that if $\Gamma, y :^{(\texttt{Thread},1)} \texttt{async}\ \ \tau[\,]^{l} \vdash \eta',\sigma',\Sigma'$, then we can either step s with $(\texttt{Thread},1)$ privilege under environments η', σ', and Σ', or s is `skip`.

We have by assumption that $\Gamma, x :^{(\texttt{Thread},1)} \texttt{async}\ \tau[\,]^l \vdash \eta, \sigma, \Sigma$. By our environment renaming lemma, this gives us what we need to use our IH with $(\eta', \sigma', \Sigma')$ as $\mathbf{rename}(\eta, \sigma, \Sigma, x, y, (\texttt{Thread},1))$. Thus s either steps or is `skip`. In the former case we can step with **S-Async-Partition-Congr**, and in the latter we can use either **S-Async-Partition-Unwind** or **S-Async-Partition-Done** depending on whether $\Phi(\phi)$ is empty or not.

- Case **T-Async-Memcpy**
  Immediate via use of the **S-Async-Memcpy** rule.
- Case **T-Memcpy**
  Immediate via use of the **S-Memcpy** rule.

Memory Rules.

- Case **T-Decl**
  Via our lemma about expression type safety and our hypothesis that e is well-typed, we obtain the premises necessary to use the **S-Decl** rule to step.
- Case **T-Arr-Assn**
  Each of e_1, e_2 and e_3 must evaluate to a well-typed value by the expression type safety lemma. In particular, both e_1 evaluates to some $\langle l, n\rangle$ and e_2 evaluates to some i. We assume all array accesses are in bounds, so this is sufficient to use the **S-Arr-Assn** rule to step.
- Case **T-Arr-Assn-Shared**
  Same as previous case.
- Case **T-Free**
  Trivial via the **S-Free** rule.
- Case **T-Partition**
  Trivial via the **S-Partition** rule.
- Case **T-Claim**
  Trivial via the **S-Claim** rule.
- Case **T-Lower**
  Trivial via the **S-Lower** rule.
- Case **T-Alloc**
  We assume that l is not `Shared`, so we can use the **S-Alloc-Local** or **S-Alloc-Global** rule, depending on whether l is `Local` or `Global`.
- Case **T-Alloc-Shared**
  Trivial via the **S-Alloc-Shared** rule.

Control Rules.

- Case **T-Skip**
  Trivial
- Case **T-While**
  Trivial, all while loops step via the **S-While** rule
- Case **T-If**
  By our proof of expression type safety, the expression e steps to either the boolean value true or false. We can thus use either the **S-If-True** or **S-If-False** rules to step.
- Case **T-Seq**
  In this case we know by our IH that s_1 is either `skip` or can step. In the former case we use the `S-Seq-Done` rule and in the latter we use the `S-Seq-First` rule.
- Case **T-Function-Call**
  In this case we know by our expression safety lemma that each of the arguments will evaluate to a well-typed value. We also have by assumption that f has a function type, which by inversion

on the **V-Function** rule tells us that it is a closure type. Additionally our assumption that $\Gamma \vdash \eta,\sigma,\Sigma$ tell us that Σ contains f at the same type that Γ does. These premises are sufficient to use the **S-Function-Call** rule.

□

Lemma A.8. *(Statement Preservation) If $\Gamma \vdash \eta,\sigma,\Sigma$ and $\Gamma \vdash^{\pi}_{m} s$ and $\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m'} s \rightsquigarrow s' \dashv_{m''} \eta',\sigma',\Sigma,\Psi',\Phi'$ and $\pi \vdash p$ and $m \geq m'$ and $\Gamma \vdash \Phi$, then there is some Γ' such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \vdash^{\pi}_{m} s'$ and $\Gamma' \vdash \eta',\sigma',\Sigma'$ and $m \geq m''$ and $\Gamma' \vdash \Phi'$.*

Proof. We proceed by induction on the derivation of $\Gamma \vdash^{\pi}_{m} s$.

Privilege Management Rules.

- Case **T-Split**
  In this case we have by assumption that $(h,n) \vdash p$, $\Gamma \vdash \eta,\sigma,\Sigma$, and $\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m'}$ `split`$(n_1,n_2)\{s_1\}\{s_2\} \rightsquigarrow s' \dashv_{m''} \eta',\sigma',\Sigma,\Psi',\Phi'$. Our inductive hypotheses give us that if for any p, if $(h,n_1) \vdash p$ and s_1 steps with partition id p, or if $(h,n_2) \vdash p$ and s_2 steps with partition id p then their results are well typed.
  By inversion on our semantic derivation, we are in one of 5 cases.
  In the **S-Split-Left** case $p < n_1$ and s_1 steps to s_1'. This is sufficient to tell us that s_1' is well-typed and the output environments of that relation η',σ', and Σ' are all well-typed by $\Gamma' \supseteq \Gamma$, and that the memory is properly bounded by the typing rules.
  We can thus use the **T-Split-Left** rule to conclude that the result of this case is well-typed.
  The **S-Split-Left-Done** rule is trivial via the **T-Skip** rule.
  The **Right** cases are symmetric, with the observation that when $p \geq n_1$ and $p < n_1 + n_2$ then $p - n_1 < n_2$.
  The last **S-Split-None** rule is trivial via the **T-Skip** rule.
- Case **T-Destruct**
  In this case we have by assumption that $\downarrow \pi$ is defined, so π is either (`Block`,1) or (`Grid`,1). We also assume that $\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m'}$ `destruct in` $s \rightsquigarrow s' \dashv_{m''} \eta',\sigma',\Sigma,\Psi',\Phi'$. We also have by our inductive hypothesis that for any p such that $\downarrow \pi \vdash p$ and s'' such that s steps to s'' at p, then that step preserved well-typedness.
  By inversion on the step relation, we are in one of three cases.
  If the rule used **S-Destruct-Block**, then we know that s steps to s'' and p is t **mod** T. (`Thread`,T) $\vdash t$ **mod** T for any t, so we can use our inductive hypothesis to conclude that the s'' stepped to by s is well-typed, as are its environments and memory usage. The **T-Destruct** rule then gives us our desired goal.
  The **S-Destruct-Grid** case proceeds similarly, while the **S-Destruct-Done** case is trivial.
- Case **T-Group**
  In this case we have by assumption that
  $\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{(h,q\cdot n)}_{m'}$ `group` q $s \rightsquigarrow s' \dashv_{m''} \eta',\sigma',\Sigma,\Psi',\Phi'$. We have by our inductive hypothesis that s steps to s'' at some partition id p' and $(h,n) \vdash p'$, then s'' is well typed, as are the other outputs of that step.
  By inversion on the step relation, we are in one of two cases. The **S-Group-Done** case is trivial, so we shall focus on the **S-Group** case. In this case we have that s steps to s'' at partition id p **mod** n. It is always the case that $(h,n) \vdash p$ **mod** n for any p, so we can use our inductive hypothesis to conclude that s'' is well-typed, as are its output environments and memory usage. From there, it is a simple application of the **T-Group** rule to conclude that `group` q s' is well-typed, and to finish the case.

Thread Synchronization Rules. These rules are all trivial: with one exception all thread synchronization primitives step to skip without changing environment or memory, and are thus obviously well-typed. **S-Sync-Wait-Spin** does not produce skip, but it steps to the same statement as we already assumed typechecks in the premise of the lemma, so is straightforward nonetheless.

If we wanted to say something about deadlock freedom we'd have more work here, but we aren't doing that, so these rules are easy.

Asynchrony Rules.

- Case **T-Async-Partition** In this case we have that s is well-typed in a context where x has been renamed into y, with $(\texttt{Thread},1)$ privilege. We also have that $\Gamma, x :^{(\texttt{Thread},1)} \tau[]^l \vdash \eta,\sigma,\Sigma$ and $\Gamma, x :^{(\texttt{Thread},1)} \tau[]^l \Phi$.
  By inversion, we are in one of three cases.
  In the **S-Async-Partition-Done** case, we are done.
  In the **S-Async-Partition-Congr case**, our inductive hypothesis gives us that there is some Γ' such that $\Gamma, y :^{(\texttt{Thread},1)} \texttt{async}\ \ \tau[]^l \subseteq \Gamma'$ and $\Gamma' \vdash \Phi'$ and $\Gamma' \vdash \textbf{rename}(\eta,\sigma,\Sigma,x,y,(\texttt{Thread},1))$ via our well-typed renaming lemma. This lets us use the **T-Async-Partition** rule to check this case, with a choice of Γ' as $\Gamma', x :^{(\texttt{Thread},1)} \tau[]^l$.
  In the **S-Async-Partition-Unwind case**, our assumption that Φ is well-typed tells us that $\Gamma, y :^{(\texttt{Thread},1)} \vdash^{\{} (\texttt{Thread},1)_m s$. Thus, we can use the **T-Async-Partition** rule to type this case.
- Case **T-Async-Memcpy**
  In this case the statement and environment typing are trivial, we need only to show that the async stack remains well typed.
  In this case we have that x and y have the same type at $(\texttt{Thread},1)$. This is sufficient for us to check $x = y$ at $(\texttt{Thread},1)$, meaning that adding that instruction to the stack maintains its well-typedness.
- Case **T-Memcpy**
  Immediate via use of the **S-Memcpy** rule. We just need to show that the environment remains well typed, which we know via our lemmas about **update** and **get**.

Memory Rules.

- Case **T-Decl**
  By inversion, we are using **S-Decl** rule for evaluation. Our well-typed expression lemma gives us that v is well-typed, so it follows from our assumptions and our lemmas about extending environments that the extended η and s are well-typed by $\Gamma, x :^{\pi} \tau$.
- Case **T-Free**
  Trivial.
- Case **T-Alloc**
  By inversion we are either in the **S-Alloc-Local** or **S-Alloc-Global** rules. In either case, we assume that $\Gamma, x :^{\pi} \tau[]^l$ checks s, meaning we can use our extended environment lemmas and the **T-Seq** and **T-Free** rules to check these cases.
- Case **T-Alloc-Shared**
  Same as previous case.
- Case **T-Partition**
  In this case we have by assumption that $\Gamma, y :^{(h,n/c)} \tau[]^l s$ and l is not local and c divides n. By inversion on our step relation we must be in the **S-Partition** case, so we have $\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m}$ *partition*$\psi xycs \rightsquigarrow \texttt{init}_{\psi};s';\texttt{dec}_{\psi};\texttt{wait}_{\psi} \dashv_m \eta',\sigma',\Sigma',\Psi,\Phi$ where $s' = s[(y + c \cdot p)/y]$ and $(\eta',\sigma',\Sigma') = \textbf{rename}(\eta,\sigma,\Sigma,x,y,\pi/c)$. Via our well-typed renaming lemma contexts we know that we can check the renamed environments in the extended environment $\Gamma, y :^{\pi/c} \tau[]^l, x :^{\pi}$

$\tau[]^l$, and this context is also sufficient to check s' (via a substitution-preserves-typing lemma that is obvious). Using the **T-Skip** rule this is exactly what we need to show to complete this case, as the thread sync primitives check trivially via their typing rules.

- Case **T-Claim**
  Essentially the same as **T-Partition**.
- Case **T-Lower**
  Essentially the same as **T-Partition**.

Control Rules.

- Case **T-If**
  We have by assumption that `if` e `then` s_1 `else` s_2 steps to some s', and by inversion we know that either e evaluates to `true` and s' is s_1, or e evaluates to `false` and s' is s_2.
  In either case, our inductive hypotheses is sufficient to tell us that these are well-typed. In particular, in both cases our IHs tell us that the amount of memory used by stepping each branch of the `if` is less than the amount of memory computed by the type system for each branch. Because the whole `if` expression checks using the greater of the memory usage of m_1 or m_2 (i.e., the memory usage on each branch), the resulting usage for the whole conditional is also bounded by the type system.
- Case **T-Skip**
  Trivial: `skip` does not step
- Case **T-Seq**
  In this case we have that s_1 and s_2 are both well-typed (with m_1 memory and m_2 memory respectively), and $m = \mathbf{max}(m_1, m_2)$. We also have by inversion that s_1 either steps to `skip` or s_1', and our inductive hypothesis tells us that s_1' is well-typed.
  In the former case we can use the **S-Seq-Done** rule to trivially finish the case. In the latter, our IH allows us to finish the case, since m_1 is always $\leq \mathbf{max}(m_1, m_2)$
- Case **T-While**
  By inversion, we have that

$$\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m} \texttt{while } e \texttt{ do } s$$
$$\rightsquigarrow \texttt{if } e \texttt{ then } (s;\ \texttt{while } e \texttt{ do } s) \texttt{ else skip} \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi.$$

  We also have by assumption that e and s are well-typed. With this information, through use of the **T-If**, **T-Seq**, **T-While**, and **T-Skip** rules, we can conclude that the result of this rule is also well-typed.
- Case **T-Function-Call**
  By inversion we have that

$$\eta,\sigma,\Sigma,t,b,p,\Psi,\Phi \vdash^{\pi}_{m} f(e_1,\ldots,e_n)$$
$$\rightsquigarrow \texttt{call } s \texttt{ with } (x_i :^{\pi_i} \tau_i \mapsto v_i)@\{\eta,\sigma,\Sigma,m'\} \dashv_m \eta,\sigma,\Sigma,\Psi,\Phi,$$

  and also that $\Sigma(f) = \{[x_1 : \tau_1,\ldots,x_n : \tau_n], s\}$, and $\eta,\sigma,\Sigma,t,b,p \vdash^{\pi} e_i \Downarrow v_i$, and $m' \leq m$.
  We also have from the premises of our case that $f :^{\pi} \mathbf{Fun}(x_1 :^{\pi_1} \tau_1,\ldots,x_n :^{\pi_n} \tau_n, \pi, m') \in \Gamma$, and $\Gamma \vdash^{\pi} e_i : \tau_i$, and $m' \leq m$.
  Our lemma for expression well-typedness tells us that that each v_i is a well-typed value, and our assumption that $\Gamma \vdash \eta,\sigma,\Sigma$ tells us that $\Sigma(f)$ is a well-typed function and thus that **fns** $\Gamma, x_i :^{\pi_i} \tau_i \vdash^{\pi}_{m'} s$. We can take the union of this with Γ to produce $\Gamma, x_i :^{\pi_i} \tau_i \vdash^{\pi}_{m'} s$, which clearly checks s and the output environments.

□