Department of Mathematical Sciences

Bachelor of Science

(Mathematical Sciences Education)

Year 2

Probability & Statistics

Continuous Random Variables

By the end of this lesson you must be able to

- Understand the probability density function of a continuous random variable.
- Use the probability density function of a continuous random variable to find the probability of some event.
- Be able to distinguish between discrete and continuous random variables.
- Be able to compute and interpret the expected value, variance, and standard deviation for a continuous random variable.

Introduction

- A continuous random variable can take any value in an interval. Continuous random variables are used to model physical characteristics such as time, length, position, etc.
- For continuous random variables, the probability that X takes on any particular value x is 0. That is, $P(X=x)=0$. Instead, we will need to find the probability that X falls in some interval *(a, b)* by using a probability density function ("p.d.f.").
- For continuous random variables it does not matter whether we write "*less than*" or "*less than or equal to*" because the probability that X is precisely equal to x is 0.

Introduction…cont’d

- In principle variables such as height, weight, and temperature are continuous, in practice the limitations of our measuring instruments restrict us to a discrete (though sometimes very finely subdivided) world.
- However, continuous models often approximate real-world situations very well, and calculus is frequently easier to work with than mathematics of discrete variables and distributions.

Examples of continuous random variables

Experiment	Random Variable (x)	Possible Values for the Random Variable
Operate a bank	Time between customer arrivals in minutes	$x \geq 0$
Fill a soft drink can (max = 12.1 ounces)	Number of ounces	$0 \leq x \leq 12.1$
Construct a new library	Percentage of project complete after six months	$0 \leq x \leq 100$
Test a new chemical process	Temperature when the desired reaction takes place (min 150° F; max 212° F)	$150 \leq x \leq 212$

Probability Density Function

X is said to be a continuous random variable if there is a function *f(x)* so that for any constants *a* and *b* with $-\infty \le a \le b \le \infty$,

$$\boxed{P(a \le X \le b) = \int_a^b f(x)dx\,.}$$

The function $f(x)$ is called the probability density function (p.d.f).

Probability Density Function…cont'd

The following properties of *f(x)* are useful:

1. $\int_{-\infty}^{+\infty} f(x)dx = 1$

2. $f(x) > 0$

3. $P(a \leq X \leq b) = \int_a^b f(x)dx$ is the area under *f(x)* from *a* to *b* for any *a* and *b*.

Area under the curve

Figure 1 demonstrates the area under the curve which is the probability of X between *a* and *b*.

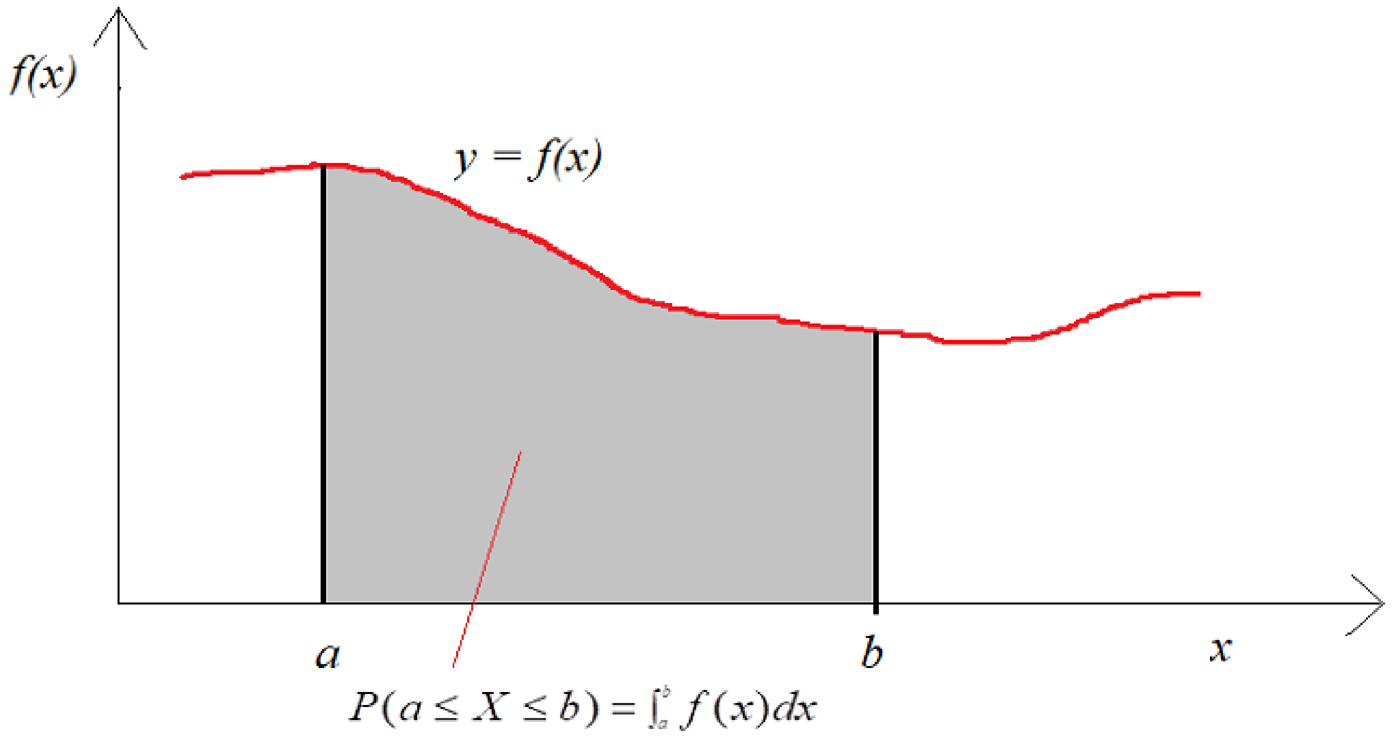

Figure 1: The probability of X between a and b.

The Cumulative Distribution Function

- Let X denote any random variable. The distribution function of X denoted by $\mathrm{F(x)}$, is such that

$$F(x) = P(X \leq x) \text{ for every } -\infty < x < \infty.$$

- So, *F(x)* is related to $f(x)$ through the following equation:

$$F(x) = \int_{-\infty}^{x} f(t)dt \ ,$$

where $F(.)$ is the probability function and t is used as the variable of integration.

Expected Value and Variance

- Let X be a continuous random variable with range [a, b] and probability density function $f(x)$. The expected value of X is defined by

$$\boxed{E(X) = \int_a^b x f(x)dx}$$

- Let X be a continuous random variable with range [a, b] and probability density function f (x). The expected value of X is defined by

$$\boxed{\begin{aligned}\sigma^2 &= Var(X) = E[(x-\mu)^2] \\ &= \int_a^b (x-\mu)^2 f(x)dx\end{aligned}}$$

Expected Value and Variance…cont'd

- The variance of a can also be expressed as

$$Var(X) = E(X^2) - \mu^2$$
$$= \left(\int_a^b x^2 f(x)dx\right) - \mu^2$$

where $\mu = E(X)$.

- The standard deviation is the square root of the variance

$$\sigma = \sqrt{Var(X)}$$

Example 1

The distribution of the amount of gravel (in tons) sold by a particular construction supply company in a given week is a continuous random variable X with pdf

$$f(x) = \begin{cases} \frac{3}{2}(1 - x^2) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $f(x)$ is a valid density function.

(b) What about the probability that X is greater than 0.75?

(c) What about $P(0.25 < X < 0.75)$?

Solution

(a) Refer to class notes!!

(b) The probability that X is greater than 0.75.

$$P(X > 0.75) = \int_{0.75}^{1} \left(\tfrac{3}{2} - \tfrac{3}{2}x^2\right)dx$$
$$= \left[\tfrac{3}{2}x - \tfrac{1}{2}x^3\right]_{0.75}^{1}$$
$$= \left[\tfrac{3}{2}(1) - \tfrac{1}{2}(1)^3\right] - \left[\tfrac{3}{2}(0.75) - \tfrac{1}{2}(0.75)^3\right]$$
$$= 0.0859375$$

(c) $$P(0.25 < X < 0.75) = \int_{0.25}^{0.75} \left(\tfrac{3}{2} - \tfrac{3}{2}x^2\right)dx$$
$$= \left[\tfrac{3}{2}x - \tfrac{1}{2}x^3\right]_{0.25}^{0.75}$$
$$= \left[\tfrac{3}{2}(0.75) - \tfrac{1}{2}(0.75)^3\right] - \left[\tfrac{3}{2}(0.25) - \tfrac{1}{2}(0.25)^3\right]$$
$$= 0.546875$$

Example 2

Let X be a random variable with PDF given by

$$f_X(x) = \begin{cases} cx^2 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.
b. Find EX and $\mathrm{Var}(X)$.
c. Find $P(X \geq \frac{1}{2})$.

Solution

(a)

$$\int_{-1}^{1} cx^2 dx = \frac{x^3}{3}c\Big|_{-1}^{+1}$$

$$= \frac{c}{3} + \frac{c}{3}$$

$$= \frac{2}{3}c$$

So, equating to 1:

$$c = \frac{3}{2}$$

(b)

$$E[X] = \int_{-1}^{1} x\left(\frac{3}{2}x^2\right)dx$$

$$= \frac{3}{2}\int_{-1}^{1} x^3 dx$$

$$= \frac{3x^4}{8}\Big|_{-1}^{+1}$$

$$= \frac{3}{8} - \frac{3}{8}$$

$$= 0$$

Solution…cont'd

(b) $V[X] = E[X^2] - \left(E[X]\right)^2$

But E[X]=0 then, we have

$$
\begin{aligned}
V[X] &= E[X^2] \\
&= \int_{-1}^{1} x^2\left(\frac{3}{2}x^2\right)dx \\
&= \frac{3}{2}\int_{-1}^{1} x^4 dx \\
&= \left.\frac{3x^5}{10}\right]_{-1}^{+1} \\
&= \frac{3}{5}
\end{aligned}
$$

(c) To find $P(X \geq \frac{1}{2})$

$$
\begin{aligned}
P\left(X \geq \frac{1}{2}\right) &= \frac{3}{2}\int_{\frac{1}{2}}^{1} x^2 dx \\
&= \frac{3}{2}\left[\frac{x^3}{3}\right]_{\frac{1}{2}}^{1} \\
&= \left[\frac{x^3}{2}\right]_{\frac{1}{2}}^{1} \\
&= \frac{7}{16}
\end{aligned}
$$

Activity

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate

a. $P(x > 0.5)$

b. $E[X]$

c. $V[X]$

Conclusion

- The probability distribution of a continuous random variable is represented by a probability density curve.
- The probability that the random variable takes a value in any interval of interest is the area above this interval and below the density curve.
- Random variables are used in all types of economic and financial decision making to carry out random experiments.
- Statistical tools and probability distribution are used to determine the probable outcomes in a given scenario, and thus facilitate decision making.

References

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