Discrete Mathematics: Week Beginning 22^{nd} April, 2024

0.1 Isomorphism of graphs

Definition 0.1.1. Graphs G_1 and G_2 are *isomorphic* if there is a one-to-one, onto function f (bijective function) from the vertices of G_1 and the vertices of G_2 and a one-to-one, onto function g from the edges of G_1 to the edges of G_2, so that an edge e is incident on v and w in G_1 if and only if the edge $g(e)$ is incident on $f(v)$ and $f(w)$ in G_2. The pair of functions f and g is called an *isomorphism* of G_1 onto G_2.

Thus graphs are isomorphic if they are the same as graphs, even though the names of the vertices and edges may be different.

Example 0.1.2. An isomorphism of graphs G_1 and G_2 below is defined by

$$f(a) = A,\ f(b) = B,\ f(c) = C,\ f(d) = D,\ f(e) = E,$$

$$g(x_i) = y_i,\ i = 1, \cdots, 5.$$

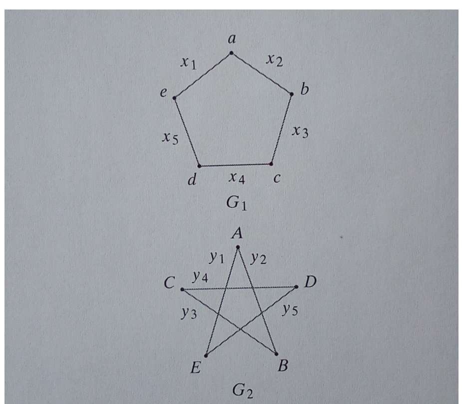

Figure 8.6.1
Isomorphic graphs.

We can think of functions f and g as 'renaming functions'.

Remark 0.1.3. If we define a relation R on a set of graphs by the rule $G_1 R G_2$ if G_1 and G_2 are isomorphic, R is an equivalence relation. Each equivalence class consists of a set of mutually isomorphic graphs.

Theorem 0.1.4. *Graphs* G_1 *and* G_2 *are isomorphic if and only if for some ordering of their vertices, their adjacency matrices are equal.*

Corollary 0.1.5. *Let* G_1 *and* G_2 *be simple graphs. The following are equivalent:*

(a) G_1 *and* G_2 *are isomorphic.*

(b) There is a one-to-one, onto function f *from the vertex set of* G_1 *to the vertex set of* G_2 *satisfying the following: Vertices* v *and* w *are adjacent in* G_1 *if and only if the vertices* $f(v)$ *and* $f(w)$ *are adjacent in* G_2.

Example 0.1.6. The adjacency matrix of graph G_1 in Example 0.1.2 relative to the vertex ordering a, b, c, d, e,

$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array},$$

is equal to the adjacency matrix of graph G_2 in Example 0.1.2 relative to the vertex ordering A, B, C, D, E,

$$\begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

We see again that G_1 and G_2 are isomorphic.

Example 0.1.7. The graphs G_1 and G_2 below are not isomorphic, since G_1 has seven edges and G_2 has six edges.

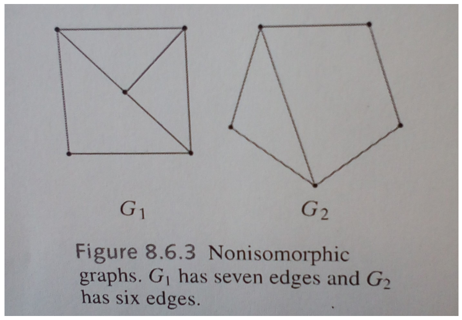

Figure 8.6.3 Nonisomorphic graphs. G_1 has seven edges and G_2 has six edges.

0.2 Hamiltonian Cycles and the Traveling Salesperson Problem

A cycle in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice, is called Hamiltonian cycle.

The cycle was named so in honor of Irish scholar Sir William Rowan Hamilton.

Example 0.2.1. The cycle (a, b, c, d, e, f, g, a) for the graph below is a Hamiltonian cycle.

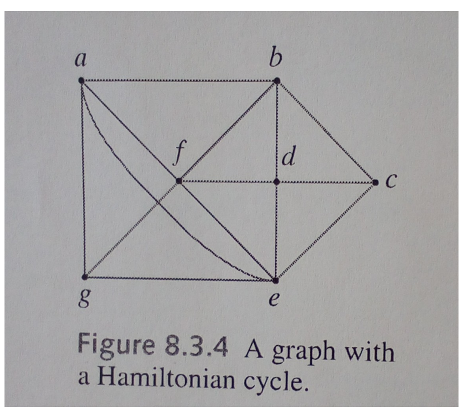

Figure 8.3.4 A graph with a Hamiltonian cycle.

Example 0.2.2. Show that the graph K below does not contain a Hamiltonian cycle.

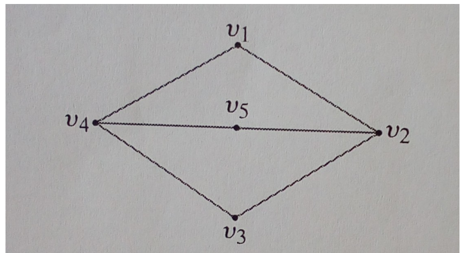

Figure 8.3.5 A graph with no Hamiltonian cycle.

Proof. Since there are five vertices, a Hamiltonian cycle must have five edges. Suppose that we could eliminate edges from the graph, leaving just a Hamiltonian cycle. We would have to eliminate one edge incident at v_2 and one edge incident at v_4, since each vertex in a Hamiltonian cycle has degree 2. But this leaves only four edges-not enough for a Hamiltonian cycle of length 5. Therefore, the graph K does not contain a Hamiltonian cycle. $\square$

Example 0.2.3. Show that the graph G below does not contain a Hamiltonian cycle.

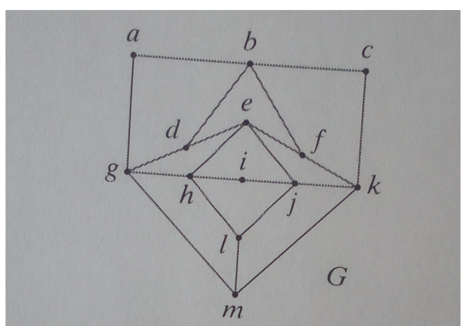

Figure 8.3.7 A graph with no Hamiltonian cycle.

Proof. Suppose that G has a Hamiltonian cycle H. The edges (a, b), (a, g), (b, c) and (c, k) must be in H since each vertex in a Hamiltonian cycle has degree 2. Thus edges (b, d) and (b, f) are not in H. Therefore, edges (g, d), (d, e), (e, f) and (f, k) are in H. The edges now known to be in H form a cycle C. Adding an additional edge to C will give some vertex in H degree greater than 2. This contradiction shows that G does not have a Hamiltonian cycle. □

The *traveling salesperson problem* is related to the problem of finding a Hamiltonian cycle in a graph. The problem is: Given a weighted graph G, find a minimum length Hamiltonian cycle in G.

If we think of the vertices in a weighted graph as cities and the edge weights as distances, the traveling salesperson problem is to find a shortest route in which the salesperson can visit each city one time, starting and ending at the same city.

Example 0.2.4. The cycle $C = (a, b, c, d, a)$ is a Hamiltonian cycle for the graph G below.

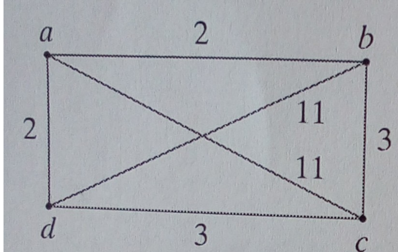

Figure 8.3.8 A graph for the traveling salesperson problem.

Replacing any of the edges in C by either of the edges labeled 11 would increase the length of C; thus C is a minimum-length Hamiltonian cycle for G. Thus C solves the traveling salesperson problem for G.