Discrete Mathematics: Week Beginning 15^{th} April, 2024

0.1 Notes on graph theory

Definition 0.1.1. A graph (or undirected graph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an ordered pair of vertices. If there is a unique edge e associated with the vertices v and w, we write $e = (v, w)$ or $e = (w, v)$. In this context, (v, w) denotes an edge between v and w in an undirected graph and not an ordered pair. Sometimes an edge between v and w is denoted $\{v, w\}$.

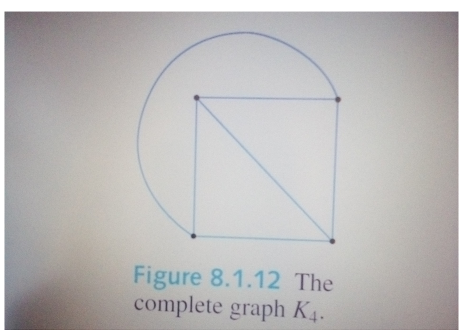
A directed graph (or digraph) G consists of a set V of vertices (or nodes) and a set E of edges (or arcs) such that each edge $e \in E$ is associated with an ordered pair of vertices. If there is a unique edge e associated with an ordered pair (v, w) of vertices, we write $e = (v, w)$, which denotes an edge from v to w.

If G is a graph (undirected or directed) with vertices V and edges E, we write $G = (V, E)$.

Unless specified otherwise, the sets E and V are assumed to be finite and V is assumed to be nonempty. A graph with neither loops nor parallel edges is called a **simple graph**.

Definition 0.1.2. The *complete graph on n vertices*, denoted K_n, is the simple graph with n vertices in which there is an edge between every pair of distinct vertices.

Example 0.1.3. The complete graph on 4 vertices, K_4, is shown below.



Definition 0.1.4. A graph $G = (V, E)$ is *bipartite* if there exist subsets V_1 and V_2 (either possibly empty) of V such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$, and each edge in E is incident on one vertex in V_1 and one vertex in V_2.

Example 0.1.5. The graph below is bipartite since if we let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5\}$, each edge is incident on one vertex in V_1 and one vertex in V_2.

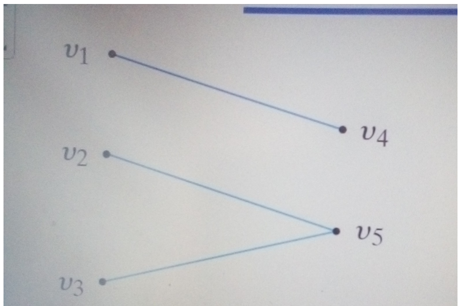


Figure 8.1.13 A bipartite graph.

Notice that Definition 0.1.4 states that if e is an edge in a bipartite graph, then e is incident on one vertex in V_1 and one vertex in V_2. It does not state that if v_1 is a vertex in V_1 and v_2 is a vertex in V_2, then there is an edge between v_1 and v_2. For example, the graph in Example 0.1.5 is bipartite since each edge is incident on one vertex in $V_1 = \{v_1, v_2, v_3\}$ and one vertex in $V_2 = \{v_4, v_5\}$. However, not all edges between vertices in V_1 and V_2 are in the graph. For example, the edge $\{v_1, v_2\}$ is absent.

Example 0.1.6. The graph below is not bipartite.

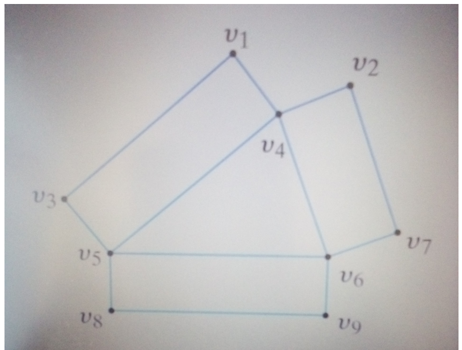


Figure 8.1.14 A graph that is not bipartite.

It is often easiest to prove that a graph is not bipartite by arguing by contradiction. Suppose that the graph is bipartite. Then the vertex set can be partitioned into two subsets V_1 and V_2 such that each edge is incident on one vertex in V_1 and one vertex in V_2. Now consider the vertices v_4, v_5 and v_6. Since v_4 and v_5 are adjacent, one is in V_1 and the other in V_2. We may assume that v_4 is in V_1 and that v_5 is in V_2. Since v_5 and v_6 are adjacent and v_4 is in V_1, v_6 is in V_2. But now v_6 is both in V_1 and V_2, which is a contradiction since V_1 and V_2 are disjoint. Therefore the graph is not bipartite.

Example 0.1.7. The complete graph K_1 on one vertex is bipartite. We may let V_1 be the set containing the one vertex and V_2 be the empty set. Then each edge (namely none!) is incident on one vertex in V_1 and one vertex in V_2.

Definition 0.1.8. The *complete bipartite graph on m and n vertices*, denoted $K_{m,n}$, is the simple graph whose vertex set is partitioned into two sets V_1 with m vertices and V_2 with n vertices in which the edge set consists of all edges of the form (v_1, v_2) with $v_1 \in V_1$ and $v_2 \in V_2$.

Example 0.1.9. The complete bipartite graph on two and four vertices, $K_{2,4}$, is shown below.

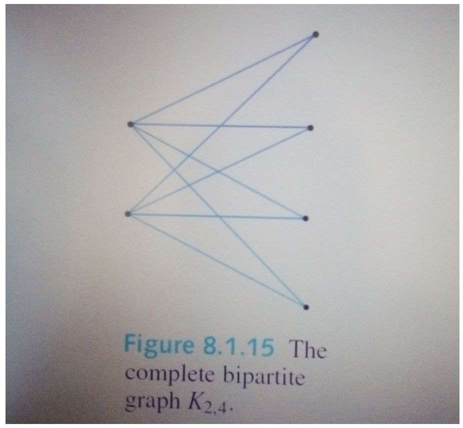


Figure 8.1.15 The complete bipartite graph $K_{2,4}$.

0.2 Paths and Cycles

Definition 0.2.1. Let v_0 and v_n be vertices in a graph. A *path* from v_0 to v_n of length n is an alternating sequence of $n+1$ vertices and n edges beginning with vertex v_0 and ending with vertex v_n,

$$(v_0, e_1, v_1, e_2, v_2, e_3, v_3, \cdots, v_{n-1}, e_n, v_n),$$

in which edge e_i is incident on vertices v_{i-1} and v_i for $i = 1, \cdots, n$.

The formalism in Definition 0.2.1 means: Start at vertex v_0; go along edge e_1 to v_1; go along edge e_2 to v_2; and so on.

Example 0.2.2. In the graph below,

$$(1, e_1, 2, e_2, 3, e_3, 4, e_4, 2) \tag{1}$$

is a path of length 4 from vertex 1 to vertex 2.

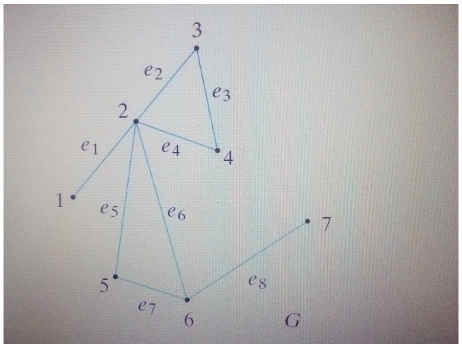


Figure 8.2.1 A connected graph with paths $(1, e_1, 2, e_2, 3, e_3, 4, e_4, 2)$ of length 4 and (6) of length 0.

Example 0.2.3. In the graph of Example 0.2.2, the path (1) consisting solely of vertex 6 is a path of length 0 from vertex 6 to vertex 6.

In the absence of parallel edges, in denoting a path we may suppress the edges. For example, the path of the graph in Example 0.2.2 may also be written $(1, 2, 3, 4, 2)$.

Definition 0.2.4. A graph G is *connected* if given any vertices v and w in G, there is a path from v to w.

Example 0.2.5. The graph G of Example 0.2.2 is connected since, given any vertices v and w in G, there is a path from v to w.

Example 0.2.6. The graph G below is not connected since, for example, there is no path from vertex v_2 to vertex v_5.

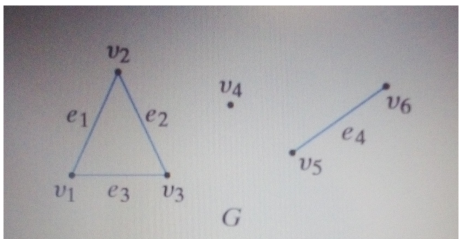


Figure 8.2.2 A graph that is not connected.

Definition 0.2.7. Let v and w be vertices in a graph G. A *simple path* from v to w is a path from v to w with no repeated vertices.
A *cycle (or circuit)* is a path of nonzero length from v to v with no repeated edges.
A *simple cycle* is a cycle from v to v in which, except for the beginning and ending vertices that are both equal to v, there are no repeated vertices.

Example 0.2.8. For the graph in Example 0.2.2, we have the following information.

Path	Simple path?	Cycle?	Simple Cycle?
$(6, 5, 2, 4, 3, 2, 1)$	No	No	No
$(6, 5, 2, 4)$	Yes	No	No
$(2, 6, 5, 2, 4, 3, 2)$	No	Yes	No
$(5, 6, 2, 5)$	No	Yes	Yes
(7)	Yes	No	No

Definition 0.2.9. (**Girth of a graph**) The girth of an undirected graph is the length of a shortest cycle contained in the graph.

If the graph does not contain any cycles, (that is, it i a forest), its girth is defined to be infinity.

Example 0.2.10. A 4-cycle (square) has girth 4.