Summer Class: Machine Learning and Data Science

Shuaifeng Li

Department of Physics, University of Michigan

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Matrix as a Vector Transformer

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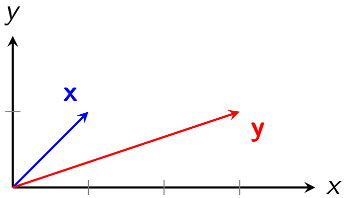

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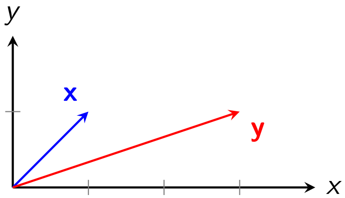

What does this mean geometrically?

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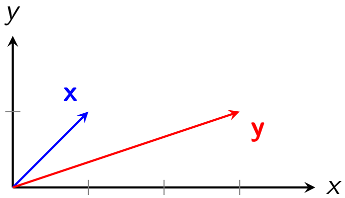
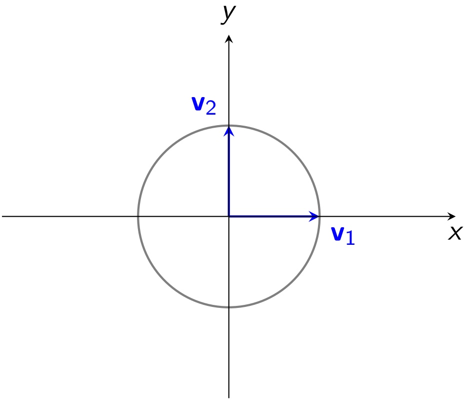


What does this mean geometrically?

- Changes length (scaling).
- Changes direction (rotation/skewing).
- May collapse dimensions.

Transforming a Set of Vectors

Input space: Unit Circle



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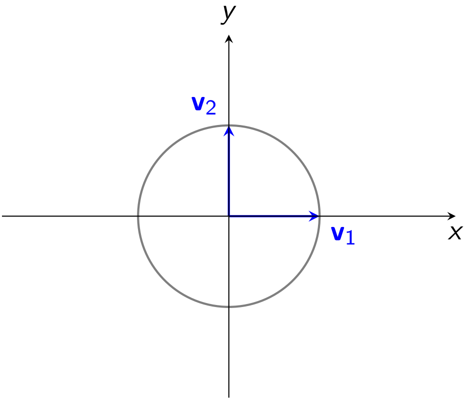
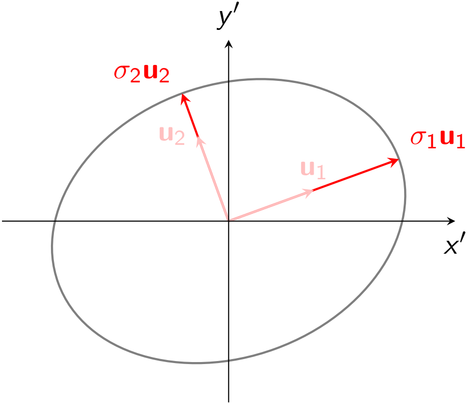


After A, Output space: Ellipse



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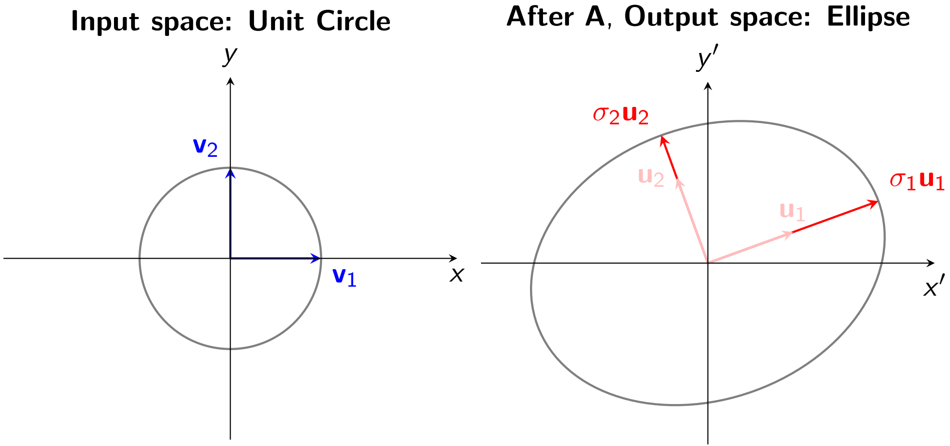

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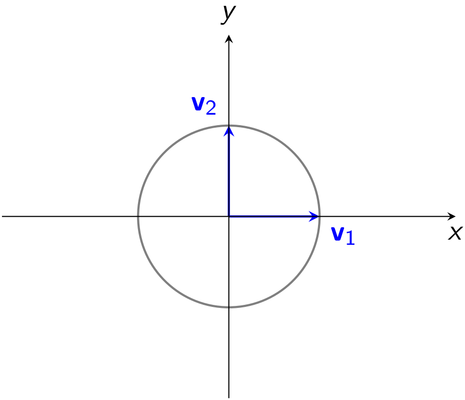

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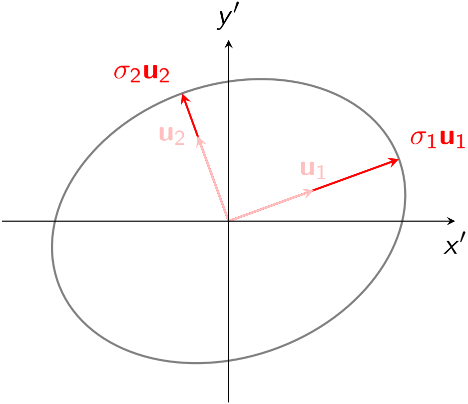

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$$Av_i = \sigma_i u_i \Rightarrow AV = U\Sigma \Rightarrow A = U\Sigma V^*.$$

What is Singular Value Decomposition?

- The Singular Value Decomposition (SVD) is a fundamental matrix factorization in linear algebra. It asserts that any real or complex matrix $A \in \mathbb{C}^{m \times n}$ can be decomposed into the product of three specific matrices.

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- $U \in \mathbb{C}^{m \times m}$: An orthogonal (or unitary for complex) matrix whose columns are the left singular vectors.
- $\Sigma \in \mathbb{C}^{m \times n}$: A diagonal matrix (with non-negative real numbers in decreasing order on the diagonal) containing the singular values σ.
- $V \in \mathbb{C}^{n \times n}$: An orthogonal (or unitary for complex) matrix whose columns are the right singular vectors.

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  - If $m > n$, $\Sigma = \begin{bmatrix} \hat{\Sigma} \in \mathbb{C}^{n\times n} \\ 0 \end{bmatrix}$. If $m < n$, $\Sigma = \left[\hat{\Sigma} \in \mathbb{C}^{m\times m}, 0\right]$. If $m = n$, $\Sigma = \hat{\Sigma} \in \mathbb{C}^{m\times n}$.

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   - Σ is a diagonal matrix. It scales the rotated vectors along these new axes.
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3. Rotation by U:
   - U is an orthogonal matrix. It rotates the scaled vectors into their final positions in the codomain (output space $\mathbb{R}^m$).
   - It aligns the scaled vectors with the basis vectors of the range (u_i).

Visualizing the Transformation

Let's start with an ellipse and apply the transformation matrix:

$$A = \begin{pmatrix} 0.4330 & -0.7500 \\ 1.2500 & -0.4330 \end{pmatrix}$$

.

- **Rotation (V^T):** Rotation by 30°.

$$V^T = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix} \approx R(30^\circ)$$

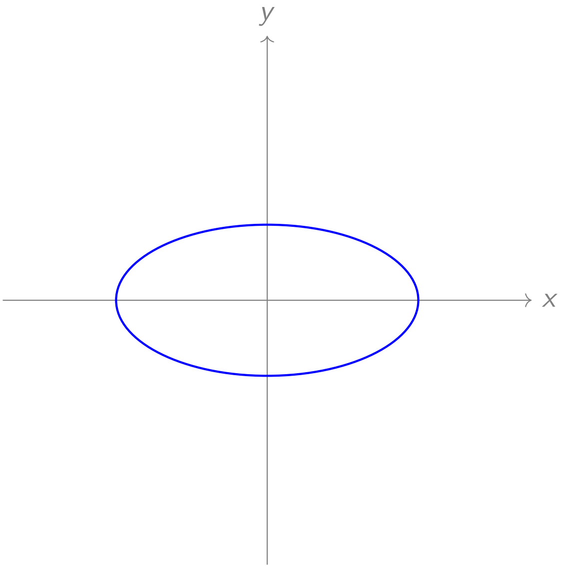
- **Scaling (Σ):**

$$\Sigma = \begin{pmatrix} 1.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

- **Rotation (U):** Rotation by 60°.

$$U = \begin{pmatrix} 0.5 & -0.866 \\ 0.866 & 0.5 \end{pmatrix} \approx R(60^\circ)$$

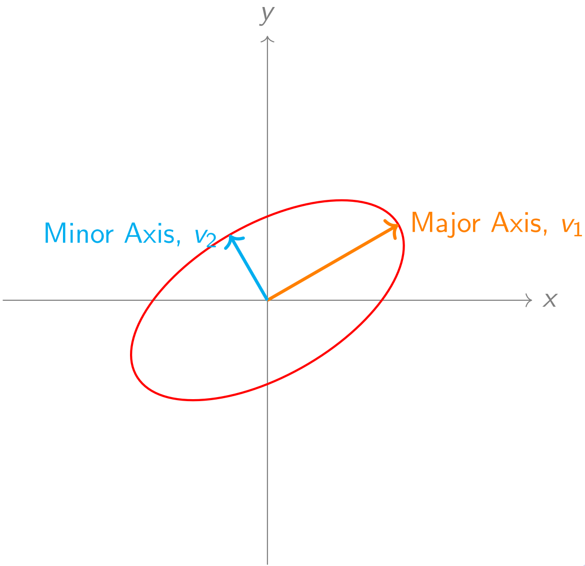
Visualizing the Transformation



Start: Ellipse

$x_0 \in \mathbb{R}^2$

Visualizing the Transformation



Step 1: Apply $V^\top$

$$x_1 = V^\top x_0$$

$V^\top$ rotates the entire coordinate system so that the directions we want to stretch and shrink are perfectly aligned with the main x and y axes (major and minor axes are rotated to lie along the 30° and 120° lines).

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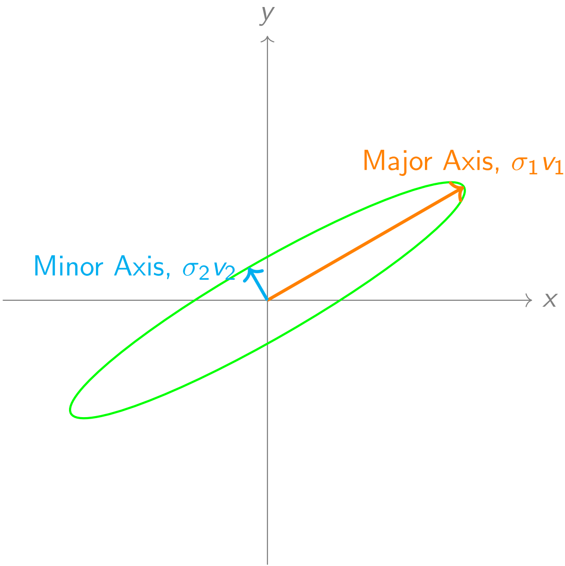
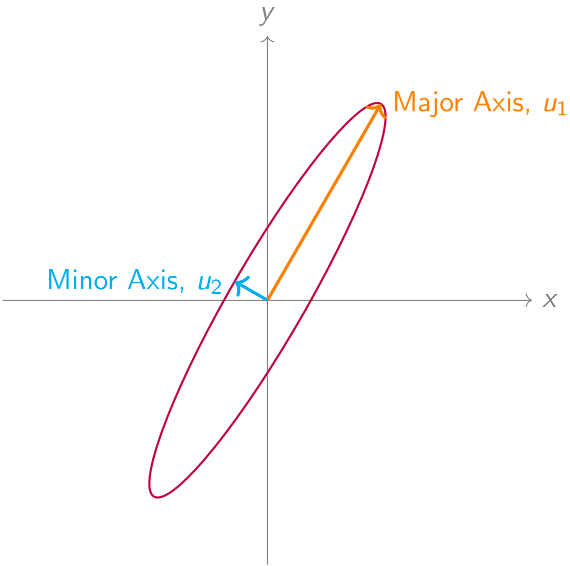


Step 2: Apply Σ

$$x_2 = \Sigma x_1 = \Sigma V^\top x_0$$

It scales everything by 1.5 along the new 30° direction and by 0.5 along the new 120° direction.

Visualizing the Transformation



Step 3: Apply U

$$x_{\text{new}} = Ux_2 = U\Sigma V^\top x_0$$

The columns of the U define the directions of the final ellipse's axes.

Final rotation gives:

$$\boxed{x_{\text{new}} = Ax_0}$$

Full vs Economy (Reduced) SVD

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  - **Case 1: Tall-and-Skinny Matrix** $(m > n)$

$$A_{m\times n} = \underbrace{\begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}}_{U\in\mathbb{C}^{m\times n}} \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}}_{\Sigma\in\mathbb{C}^{n\times n}} \underbrace{\begin{bmatrix} - & v_1^\top & - \\ & \vdots & \\ - & v_n^\top & - \end{bmatrix}}_{V^\top\in\mathbb{C}^{n\times n}}$$

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  - **Case 2: Short-and-Fat Matrix** $(m < n)$

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- Eigenvalues: Σ^2

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- Standard SVD algorithms can be computationally expensive for very large matrices.
- For "Big Data" scenarios where m and n are in the tens of thousands, millions, or more, computing the full SVD becomes impractical in terms of time and memory.
- However, many large real-world matrices are approximately low-rank, or we are only interested in a low-rank approximation (e.g., top k singular values/vectors).

Basic Idea of rSVD

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   - How:
     1. Generate a random matrix $\Omega \in \mathbb{R}^{n \times l}$.
     2. Form the "sketch" or "sample" matrix $Y = A\Omega \in \mathbb{R}^{m \times l}$. Multiplying A by random vectors effectively samples its range.
     3. Compute an orthonormal basis for the columns of Y, typically via QR decomposition: $Y = QR$. The matrix $Q \in \mathbb{R}^{m \times l}$ is our approximate basis.

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   The insight here is that random projections can preserve the essential geometric structure of the original matrix A, particularly its dominant components, allowing us to work with a much smaller sketch Y.

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     3. The final SVD approximation for A is then $A \approx (Q\tilde{U})\Sigma V^T$. Let $U_{rSVD} = Q\tilde{U}$ and then $A \approx U_{rSVD} \Sigma V^T$.

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MATLAB Demonstration.

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where:

- $r = \text{rank}(A)$.
- σ_i is the i-th singular value.
- u_i is the i-th left singular vector (column of U).
- v_i is the i-th right singular vector (column of V).
- Each item $\sigma_i u_i v_i^T$ is an $m \times n$ matrix of rank 1.

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This means any matrix A can be expressed as a weighted sum of r rank-one matrices. The weights are the singular values σ_i. Since $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$, the first few terms in this sum capture the most "significant" parts of the matrix A.

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- A_k is constructed using the k largest singular values and their corresponding singular vectors.
- This process effectively filters out information associated with smaller singular values, which often corresponds to noise or less significant details.

Optimal Truncation – Eckart-Young-Mirsky Theorem

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- Frobenius norm

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- p-norms

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{x : \|x\|_p = 1} \|Ax\|_p, \|A\|_2 = \sigma_1$$

Optimal Truncation – Eckart-Young-Mirsky Theorem

Theorem (Eckart-Young-Mirsky)

For either $\|\cdot\|_2$ or $\|\cdot\|_F$,

$$\|A - A_k\| \leq \|A - B\| \text{ for all rank-k matrices } B.$$

Moreover,

$$\|A - A_k\| = \begin{cases} \sigma_{k+1}, \|\cdot\|_2 \text{ norm} \\ \left(\sum_{i=k+1}^{r} \sigma_i^2\right)^{\frac{1}{2}}, \|\cdot\|_F \text{ norm.} \end{cases}$$

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- MATLAB Demonstration (SVD and rSVD).

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Can we predict the values of the '?' cells?

$$\begin{bmatrix} 5 & ? & 1 & ? \\ ? & 2 & ? & 3 \\ 4 & ? & ? & 5 \\ ? & 1 & 2 & ? \end{bmatrix}$$

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- **Why is Low Rank?**
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  - Think of movie ratings: they are driven by a few latent factors like genre, actors, etc.
- **How does SVD help?**
  - SVD finds the best low-rank approximation of any matrix.
  - It decomposes a matrix A into:

$$A = USV^T$$

  - By keeping only the top k singular values in S and setting others to zero (S_k), we get the best rank-k approximation: $A_k = US_kV^T$.

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4. **Repeat:** Go back to Step 2 and repeat. The guesses for the missing values will converge.

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- **Matrix:** Rows are users, columns are movies, and entries are ratings (1-5 stars).
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	1	2	3	4	5	6
a	5		1	1		2
b		2		4		4
c	4	5		1	1	2
d			3	5	2	
e	2		1		4	4

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Standard Solution: $\beta = (X^T X)^{-1} X^T y$ ($X^T X$ is invertible).

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- If columns are nearly linearly dependent (high multicollinearity), $X^T X$ is ill-conditioned, leading to the numerical instability.

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- The regression coefficients are found by:

$$\beta = X^+ y = V\Sigma^+ U^T y$$

This solution minimizes the least-squares error $\|y - X\beta\|_2^2$.

Linear Regression with SVD

MATLAB Demonstration.

$$\begin{bmatrix} | & | & \cdots & | \\ 1 & x_1 & \cdots & x_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

Summary of What You've Learned Today

- Properties of SVD, U, Σ, V.
- Computation of SVD: Conventional method and randomized SVD.
- Geometric Interpretation of SVD
- Applications
  - Matrix Approximation: Image Compression.
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See you on Wednesday (7/2) for Lecture 3 on
Singular Value Decomposition (PCA, POD, LSA).