Summer Class: Machine Learning and Data Science

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Overview of Machine Learning – History

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- 1943–1960s: Early Neural Models
  - Logical circuits as brain model – McCulloch & Pitts neuron: binary inputs, weighted summation, and thresholding.
  - Perceptron (1958) – Frank Rosenblatt introduces linear classifier trained via weight updates.

Overview of Machine Learning – History

Unfortunately... In 1969, Minsky and Papert mathematically proved that a single-layer perceptron cannot solve problems like XOR (Exclusive or, non-linearly separable). The community was disappointed on neural networks for decades, focusing instead on symbolic AI.

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- 2000s–Today: Optimization Powers Everything
  - Convex Optimization – Well-understood optimization algorithms like LASSO, Ridge Regression, and Logistic Regression train simple, interpretable models.
  - Deep Learning – Backpropagation trains deep nets with millions of parameters.
  - Reinforcement Learning – Optimizing long-term rewards (policy gradients, Q-learning).

Modern AI Achievements

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  - AlexNet: Deep Convolutional Neural Networks. Its dominant victory in the ImageNet competition was the "big bang" for deep learning.

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  - DeepMind's AlphaGo: Deep Neural Networks combined with Monte Carlo Tree Search and Reinforcement Learning. Defeated the world's best Go player. It learns by playing against itself.

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  - 2021: DALL-E: Text-to-Image Generation. Bridged the gap between language and vision. It made generative AI accessible by allowing users to create complex, original images from simple text prompts.

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  - GPT-4 and Many Others: Multimodal Reasoning. These models blur the lines between AI classes. They are fundamentally LLMs but also possess generative and classification capabilities across different data types (text, images, audio, video, code). This convergence is the current state-of-the-art.

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- **Hugging Face: Chatbot Arena LLM Leaderboard**

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Gemini-2.5-Pro-Preview-06-05
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o3-2025-04-16
ChatGPT-4o-latest (2025-03-26)
DeepSeek-R1-0528
Gemini-2.5-Flash-Preview-05-20
Grok-3-Preview-02-24
GPT-4.5-Preview
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(a) Overall

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(b) Math

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(c) Coding

Machine Learning (ML)

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Supervised Learning

Learning with a “supervisor” (labeled data).

Unsupervised Learning

Discovering patterns in unlabeled data.

Semi-supervised Learning (Reinforcement Learning)

Learning through trial and error with rewards/penalties.

Supervised Learning

Algorithm learns from a dataset where each data point is labeled with a known outcome.

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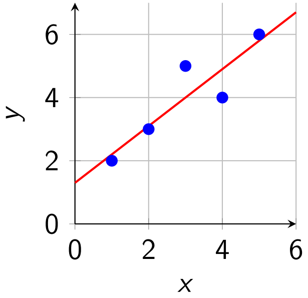
Common Tasks:

- **Classification:** Output is a category (e.g.,"spam" or "not spam", "cat" or "dog").
- **Regression:** Output is a continuous value (e.g., predicting house prices, stock values).

Supervised Learning – Regression and Classification

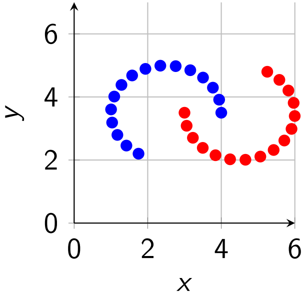
Regression

Goal: Model the relationship between a dependent variable (response) and one or more independent variables (predictors) by fitting an equation.



Classification

Goal: Model decision boundaries that separate input data points to discrete categories or classes in the feature space.



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Common Tasks:

- **Clustering:** Grouping similar data points together.
- **Dimensionality Reduction:** Reducing the number of variables while preserving important information (e.g., Principal Component Analysis).
- **Association Rule Mining:** Discovering potential relationships (e.g., "People who watch *Iron Man 1* **probably** also watch *Iron Man 2*.").

Unsupervised Learning – Clustering

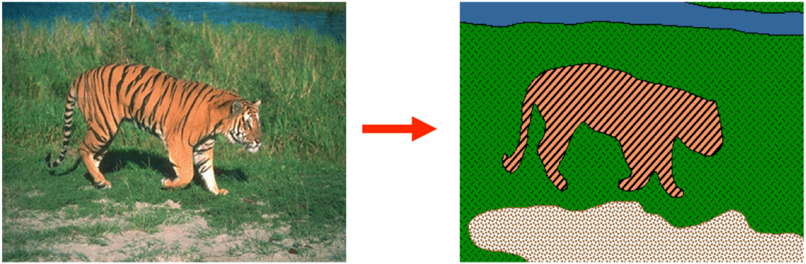

Figure: Image segmentation using K-means algorithm.

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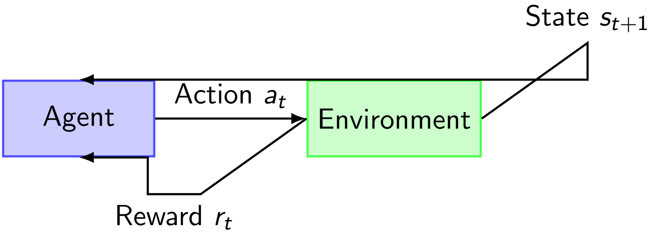

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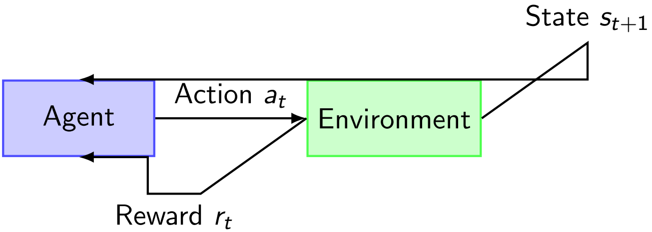

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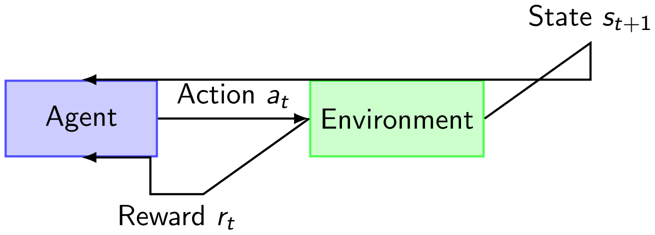

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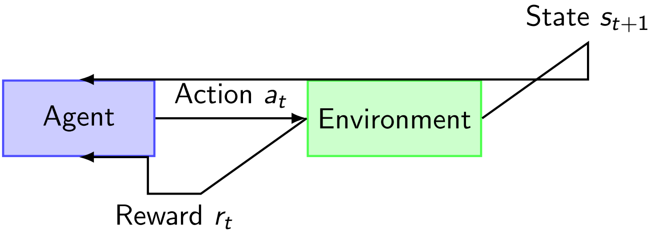

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- The agent updates its strategy (policy).

Classical Machine Learning vs Deep Learning

Aspect	**Classical Machine Learning:** Mathematically predefined models.	**Deep Learning:** Artificial neural networks with multiple layers ("deep"). Inspired by human brain.
Feature Extraction	Hand-crafted features (e.g., SIFT, PCA, statistical measures)	Automatically learned from raw data (via neural network layers)
Algorithm Examples	Decision Trees, SVM, k-Nearest Neighbors (KNN), Logistic Regression	Convolutional Neural Networks (CNN), Recurrent Neural Networks (RNN), Transformers
Data Requirements	Works well with smaller, clean datasets	Requires large volumes of labeled data to perform well
Model Complexity	Shallow, interpretable models with fewer parameters	Deep, multi-layered models with millions of parameters
Computational Power	Low to moderate, runs on CPU	High, typically requires GPUs/TPUs for training

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  - This is where **convex optimization** becomes highly relevant, as many ML loss functions are designed to be convex.

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- **Mathematical Formulation:**

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  - $x \in \mathbb{R}^n$: vector of optimization variables
  - $f_0(x)$: objective function to minimize
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- Unfortunately, most are **intractable**. The exception: convex optimization.

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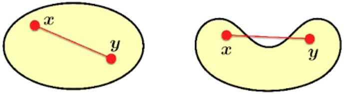

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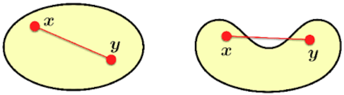

- **Examples:**
  - Euclidean space: $\mathbb{R}^n$
  - Affine subspace: $\{x \in \mathbb{R}^n \mid Ax = b\}$
  - Half-space: $\{x \in \mathbb{R}^n \mid a^T x \leq b\}$
  - Polyhedron: $\{x \in \mathbb{R}^n \mid Ax \leq b\}$
  - Norm ball (general): $\{x \in \mathbb{R}^n \mid \|x - x_c\| \leq r\}$, for any norm $\|\cdot\|$
  - Intersection of Convex Sets

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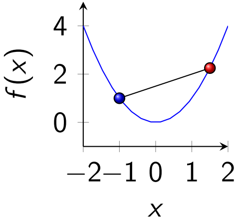
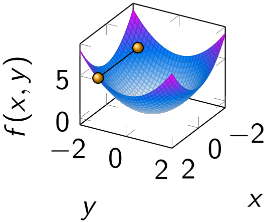
$f(x) = x^2$



$f(x, y) = x^2 + y^2$



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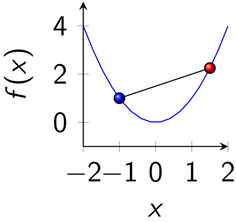

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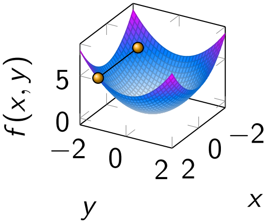

- **Strictly Convex:** Inequality is strict ($<$) for $x \neq y$, $\theta \in [0,1]$. Has at most one global minimum.

Convex Function – Properties

Epigraph Condition

f is convex $\iff$ its epigraph is a convex set.

First-Order Condition if f is continuously differentiable

f is convex $\iff$ $\mathrm{dom}(f)$ is convex and
$f(y) \geq f(x) + \nabla f(x)^T(y - x), \forall x, y \in \mathrm{dom}(f)$.

Second-Order Condition if f is twice continuously differentiable

f is convex $\iff$ $\mathrm{dom}(f)$ is convex and Hessian $\nabla^2 f(x) \succeq 0$.

Convex Function – Examples

- Common Convex Functions:
  - Linear/Affine: $a^T x + b$
  - Quadratic: $x^T Q x + b^T x + c, Q \succeq 0$
  - Norms: $\|x\|_p$
  - Exponential: e^{ax}
  - Negative Logarithm: $-\log(x)$ for $x > 0$

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- Operation Preserving Convexity:
  - Sum of convex functions.
  - Non-negative weighted sum of convex functions.
  - Composition with affine mapping: $f(Ax + b)$ if f is convex.

Convex Function – Optimization

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& h_j(x) = 0, \quad j = 1, \ldots, p
\end{aligned}
$$

- $x \in \mathbb{R}^n$: optimization variables
- $f_0(x)$: convex objective function
- $f_i(x) \le 0$: convex inequality constraint functions
- $h_j(x) = 0$: affine equality constraint functions

Special classes:

- Linear Programming (LP): $f_0(x)$ and $f_i(x)$ are linear (affine).
- Quadratic Programming (QP): $f_0(x)$ is convex quadratic and $f_i(x)$ are linear (affine). Least Squares: $\min \frac{1}{2}\|Ax - b\|_2^2$, s.t., $Rx \preceq d$.

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  - $\alpha_k > 0$: step size (learning rate)
- Challenge: Choose effective d_k and α_k.

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- Core Idea: Iteratively move in the direction opposite to the gradient (steepest descent).

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- Core Idea: For composite problems

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  - Small α: Stay close to the "gradient-updated" point v.
  - Large α: Minimize $g(x)$ (enforcing sparsity, constraints, etc.).

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We approximate $F(x) = f(x) + g(x)$ at x_k by:

$$Q(x; x_k) = f(x_k) + \nabla f(x_k)^T (x - x_k) + \frac{1}{2\alpha} \|x - x_k\|_2^2 + g(x).$$

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