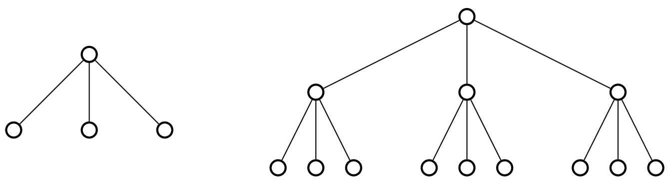
A *rooted tree* is a tree together with a designated vertex r called the *root*. A *complete k-ary tree* of depth d is defined recursively as follows:

- A complete k-ary tree of depth 0 is a single vertex.
- For $d \geq 0$, a complete k-ary tree of depth $d+1$ is obtained by taking k complete k-ary trees $T_1, \dots, T_k$ of depth d, a new root vertex r, and adding edges from r to the roots of $T_1, \dots T_k$.

Here are diagrams of the complete ternary (3-ary) trees of depth 1 and 2:



How many vertices $N(d)$ does a complete k-ary tree of depth d have? When $d \geq 1$, there is one vertex for each of the k subtrees of depth $d-1$, plus the root vertex. This gives the formula

$$N(d) = k \cdot N(d-1) + 1$$

for $d \geq 1$, with the "base case" $N(0) = 1$. Plugging in small values of d, this gives

$$\begin{aligned}
N(1) &= k \cdot N(0) + 1 = k + 1 \\
N(2) &= k \cdot N(1) + 1 = k(k+1) + 1 = k^2 + k + 1 \\
N(3) &= k \cdot N(2) + 1 = k(k^2 + k + 1) + 1 = k^3 + k^2 + k + 1
\end{aligned}$$

and, in general,

$$N(d) = k^d + k^{d-1} + \cdots + 1 \qquad \text{for every } d \geq 0.$$

You can prove the correctness of this formula by induction on d, but we won't bother. Today we are more interested in "understanding" the value of $N(d)$.

1 Geometric sums

We can evaluate a sum of the form

$$S = x^n + x^{d-1} + \cdots + 1$$

for every real number x and positive integer d like this: If we multiply both sides by x, we obtain

$$xS = x^{d+1} + x^d + \cdots + x$$

If we now subtract the first expression from the second one, almost all the right hand sides terms cancel out:

$$xS - S = x^{d+1} - 1$$

which simplifies to $(x-1)S = x^{d+1} - 1$. When $x \neq 1$, we can do a division and obtain the formula

$$x^d + x^{d-1} + \cdots + 1 = \frac{x^{d+1}-1}{x-1} \qquad \text{for every real number } x \neq 1.$$

A sum of this form is called a *geometric sum*.

So the number of vertices in a complete k-ary tree of depth d is $(k^{d+1}-1)/(k-1)$. In particular, for a complete ternary tree, this number is $(3^{d+1}-1)/2$. A complete binary (2-ary) tree of depth d has $2^{d+1}-1$ vertices.

Annuities You have won a prestigious prize and the awarding committee offers you two options for the prize money. Option A is that they pay you $\$50,000$ per year for the rest of your life. Option B is that they pay you $\$800,000$ today. Which one would you choose?

To answer this question it will help to have some idea about how the value of money changes over time. If you keep your money in the bank at no interest then option A will pay off for you in twenty years time. If, on the other hand, you want to throw a lavish party right now then option B would make more sense for you. Now suppose that, as a savvy investor, you are quite confident in making a reliable return of $p = 7\%$ per year. How show this affect your choice?

To answer this question we'll calculate how much option A is worth in today's money. The 50K that you will be getting in your zeroth year are worth... well, 50K. For next year's 50K you can reason like this. If you had invested x dollars this year, they would be worth $(1+p)x$ dollars next year. So today's value of next year's $50,000$ dollars is the amount x for which $(1+p)x = 50\text{K}$, namely $x = 50\text{K}/(1+p)$. By the same reasoning, the 50K you would be getting in two years' time are worth $50\text{K}/(1+p)^2$ today. Continuing this reasoning, you conclude that the value of option A in today's money is

$$50\text{K} + \frac{50\text{K}}{1+p} + \frac{50\text{K}}{(1+p)^2} + \cdots$$

By the geometric sum formula, the contribution from years zero up to d equals

$$50\text{K} \cdot \frac{1/(1+p)^{d+1} - 1}{1/(1+p) - 1}$$

In the large n limit, the term $1/(1+p)^{d+1}$ vanishes and the value converges to $50\text{K} \cdot (1+p)/p$. For $p = 7\%$, the value of option A is about $\$764,286$. So option B is better.

Here is another way to see the wisdom of option B over option A without evaluating the geometric sum. With a budget of 800K, I can always take out x dollars for spending this year and invest the remaining $800\text{K} - x$ dollars aiming to grow them back to 800K by next year. To do this, I need to choose x so that

$$(1+p) \cdot (800\text{K} - x) = 800\text{K},$$

which solves to

$$x = 800\text{K} \cdot \left(1 - \frac{1}{1+p}\right) = 800\text{K} \cdot \frac{p}{1+p}.$$

When p equals 7% we get that x is about $52,336$ dollars. This improves on the annual 50K in option A.

2 Polynomial sums

In Lecture 3 we proved that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for every integer $n \geq 0$. How did I come up with the expression on the right? Instead of going back to something we already know, let's work a new one together:

$$\text{What is } 1^2 + 2^2 + \cdots + n^2?$$

We have to do a bit of guesswork here. Since the sum $1+2+\cdots+n$ turned out to equal a quadratic polynomial in n, perhaps $1^2 + 2^2 + \cdots + n^2$ might equal a cubic polynomial? Let's make a guess: For all n, there exist real numbers a, b, c, d such that

$$1^2 + 2^2 + \cdots + n^2 = an^3 + bn^2 + cn + d.$$

Suppose our guess was correct. Then what are the numbers a, b, c, d? We can get an idea by evaluating both sides for different values of n:

$$\begin{aligned} 0 &= d && \text{for } n = 0 \\ 1 &= a + b + c + d && \text{for } n = 1 \\ 5 &= 8a + 4b + 2c + d && \text{for } n = 2 \\ 14 &= 27a + 9b + 3c + d && \text{for } n = 3. \end{aligned}$$

I solved this system of equations on the computer and obtained $a = 1/3$, $b = 1/2$, $c = 1/6$, $d = 0$. This suggests the formula

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n$$

for all integers $n \geq 0$. Let us see if we can prove its correctness by induction on n.

We already worked out the base case $n = 0$, so let us do the inductive step. Fix $n \geq 0$ and assume that the equality holds for n. Then

$$1^2 + 2^2 + \cdots + (n+1)^2 = \left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n\right) + (n+1)^2 = \tfrac{1}{3}n^3 + \tfrac{3}{2}n^2 + \tfrac{13}{6}n + 1.$$

This indeed equals $\frac{1}{3}(n+1)^3 + \frac{1}{2}(n+1)^2 + \frac{1}{6}(n+1)$. So we have discovered and proved a new theorem:

Theorem 1. *For every integer $n \geq 0$, $1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.*

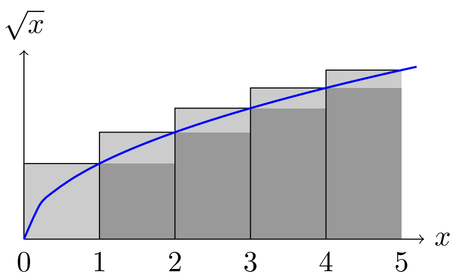
3 Approximating sums

Exact "closed-form" expressions for sums are rather exceptional. Fortunately, we can often obtain very good approximations. One powerful method for approximating sums is the integral method from calculus: It works by comparing the sum with a related integral.

As an example, let us look at the expression:

$$S(n) = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}.$$

For example, $S(5)$ equals the area covered by the shaded rectangles (both light and dark shades) in the following plot:



The area under the rectangles can be lower bounded by the area (i.e., the integral) of the curve $f(x) = \sqrt{x}$ from $x = 0$ to $x = 5$:

$$S(5) \geq \int_0^5 \sqrt{x}\ dx.$$

If we remove the area L covered by the lightly shaded rectangles, the darker shaded area is now dominated by the curve $f(x) = \sqrt{x}$ and so

$$S(5) - L \leq \int_0^5 \sqrt{x}\ dx.$$

The area under L is exactly $\sqrt{5}$: If we stack all of the lightly shaded rectangles on top of one another, we obtain a column of width 1 and height $\sqrt{5}$. Therefore

$$\int_0^5 \sqrt{x}\ dx \leq S(5) \leq \int_0^5 \sqrt{x}\ dx + \sqrt{5}.$$

By the same reasoning, for every integer $n \geq 1$, we have the inequalities

$$\int_0^n \sqrt{x}\ dx \leq S(n) \leq \int_0^n \sqrt{x}\ dx + \sqrt{n}.$$

We can now use rules from calculus to evaluate the integrals: Recalling that $x^{1/2} = \frac{d}{dx}\frac{2}{3}x^{3/2}$, it follows from the fundamental theorem of calculus that

$$\frac{2}{3}n^{3/2} \leq S(n) \leq \frac{2}{3}n^{3/2} + \sqrt{n}.$$

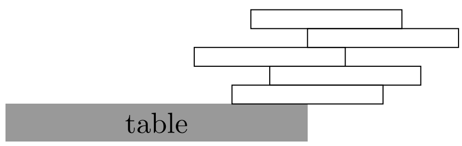
To get a feel for these inequalities, let us plug in a few values of n. (I calculated $S(n)$ by evaluating the sum on the computer.)

n	$\frac{2}{3}n^{3/2}$	$S(n)$	$\frac{2}{3}n^{3/2} + \sqrt{n}$
10	21.082	22.468	24.244
100	666.67	671.46	676.67
1,000	21,081.9	21,097.5	21,113.5
10,000	666,666	666,716	666,766

As n becomes large, the accuracy of these approximations looks quite good.

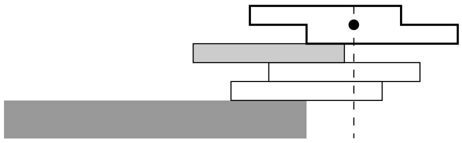
4 Overhang

You have n identical rectangular blocks and you stack them on top of one another at the edge of a table like this:



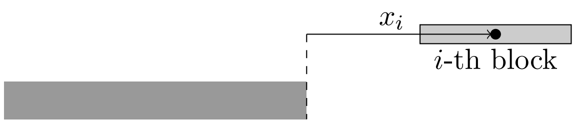
Is this configuration stable, or will it topple over?

In general, a configuration of n blocks is *stable* if for every i between 1 and n, the center of mass of the top i blocks sits over the $(i+1)$st block, where we think of the table as the $(n+1)$st block in the stack. For example, the top stack is not stable because the center of mass of the top two blocks does not sit over the third block:



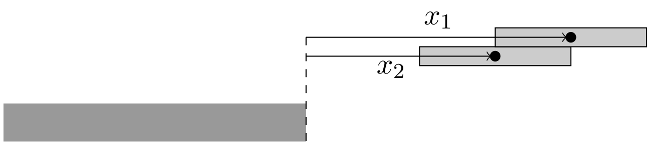
We want to stack our n blocks so that the rightmost block hangs as far over the edge of the table as possible. What should we do? One reasonable strategy is to try to push the top blocks as far as possible away from the table as long as they do not topple over.

We will assume each block has length 2 units and we will use x_i to denote the offset of the center of the i-th block (counting from the top) from the edge of the table:



The offset of a block can be positive, zero, or negative, depending on the position of its center of mass.

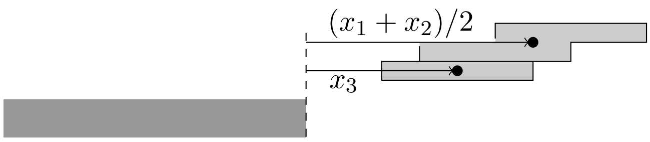
For the top block not to topple, its center of mass must sit over the second block. To move it as far away from the edge of the table as possible, we move its center exactly one unit to the right of the center of the second block:



This forces the offsets x_1 and x_2 to satisfy the equation

$$x_1 = x_2 + 1. \tag{1}$$

Now we move onto the third block. The center of mass of the first two blocks is at offset $(x_1+x_2)/2$ from the edge of the table. To push this as far to the right as possible without toppling over the third block



we must set

$$\frac{x_1 + x_2}{2} = x_3 + 1. \tag{2}$$

Continuing our reasoning in this way, for every i between 1 and n, the offset of the center of mass of the top i blocks is $(x_1 + \cdots + x_i)/i$. To push this as far to the right without toppling over the $(i+1)$st block, we must set

$$\frac{x_1 + x_2 + \cdots + x_i}{i} = x_{i+1} + 1 \quad \text{for all } 1 \le i \le n. \tag{3}$$

Finally, when $i = n+1$, we have reached the table whose offset is zero. Since we are thinking of the table as the $(n+1)$st block, its centre of mass is one unit left to its edge:

$$x_{n+1} = -1. \tag{4}$$

The overhang of the set of blocks is $x_1 + 1$. To figure out what this number is, we need to solve for x_1 in the system of equations (3-4). Let us develop some intuition first. Equation (1) tells us that $x_2 = x_1 - 1$. Plugging in this formula for x_2 into (2), we get that

$$x_3 = x_1 - \frac{1}{2} - 1.$$

Let's do one more step. Equation (3) tells us that $(x_1 + x_2 + x_3)/3 = x_4 + 1$. Plugging in our formulas for x_2 and x_3 in terms of x_1 we get that

$$\frac{x_1 + (x_1 - 1) + (x_1 - \frac{3}{2})}{3} = x_4 + 1$$

from where

$$x_4 = x_1 - \frac{1 + \frac{3}{2}}{3} - 1 = x_1 - \frac{1}{3} - \frac{1}{2} - 1.$$

At this point it is reasonable to guess that x_{i+1} should equal x_1 minus the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{i}.$$

Let us prove that this guess is correct.

Lemma 2. *For all i between 1 and n, $x_i - x_{i+1} = 1/i$.*

Proof. If we multiply both sides of the i-th equation (3) by i we obtain

$$x_1 + x_2 + \cdots + x_{i-1} + x_i = i \cdot (x_{i+1} + 1).$$

Under this scaling the $(i-1)$st equation is

$$x_1 + x_2 + \cdots + x_{i-1} = (i-1) \cdot (x_i + 1).$$

Subtracting the two we obtain that

$$x_i = i(x_{i+1} - 1) - (i-1)(x_i - 1) = ix_{i+1} - (i-1)x_i + 1$$

from where, after moving the variables around, we conclude that

$$x_i = x_{i+1} - \frac{1}{i}.$$

□

It follows immediately from this Lemma that

$$x_1 - x_{n+1} = (x_1 - x_2) + (x_2 - x_3) + \cdots + (x_{n+1} - x_n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Since $x_{n+1} = -1$, the overhang $x_1 + 1$ equals exactly this number, which is called the *n-th harmonic number* and is denoted by $H(n)$. There is no closed-form expression for $H(n)$, but we can obtain an excellent approximation using the integral method. To do this, we compare $H(n)$ with the integral of the function $1/x$: